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Existence Theory for a Langevin Fractional q-Difference Equations in Banach Space

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Abstract In this work, we establish an existence theorem of solutions for a new class of nonlinear Langevin fractional q-difference equation involving Caputo q-derivative in Banach space. Indeed, we will introduce the notion of kuratowski measure of noncompactness and the Mönch's fixed-point theorem, on which; our analysis of the problem will essentially be based. An example is provided to show the applicability of the main result.

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1. Introduction

Fractional calculus and q-calculus in general and fractional differential equations, in particular, are currently well established, looking to the number of papers and books edited and seminars organized on the whole of the world. Fractional calculus belongs to the large area of mathematical analysis. As far as it is known, the notion of q-difference equations goes back to 1910, where it was introduced by Jackson [1]. In the few past decades, the subject has attracted many authors and the q-difference equations appeared as a promising research field, on both applied or theoretical level, see for example [2–8]. For more details on the subject, one can consult [9–19].

One of the simplest ways to understand dynamics of nonequilibrium systems is the theory of Brownian motion and perhaps the most important equation is Langevin equation [20]. Based on the works of M. Gouy [21] and A. Einstein [22] Langevin formulated in 1908, his famous equation witch is widely used by physicians to describe different problems in physics, chemistry and electrical engineering. Generalizations of the Langevin equation have been proposed to describe dynamical processes in a fractal medium [23, 24]. Another way to extend Langevin equation consists on the replacement of ordinary derivative by a fractional derivative [25–29].

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This paper is mainly concerned with existence results for the following Langevin fractional q-difference equation

$$\begin{cases} D_q^{\beta}(D_q^{\alpha}+\lambda)x(t) & = f(t,x(t)), \qquad t \in J=[0,1], 0<\alpha,\beta \leq 1, \\ x(0)=\gamma_1, & x(1)=\gamma_2, \end{cases} \tag{1.1}$$

where D_q is the fractional $q-$derivative of the Caputo type. $f: J \times E \to E$ is a given function satisfying some assumptions that will be specified later and E is a Banach space with norm $\|x\|$, λ is any real number.

We will present the existence results for the problem (1.1) which rely on Mönchs fixed point theorem combined with the technique of Kuratowski measure of noncompactness. We recall that when we analyze a problem involving functional operator, one of the best ways consists on the use of the technique of measure of noncompactness, see for instance [30–39].

The remainder of this article is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts and various lemmas, which are needed later. In Section 3, we give main results of the problem (1.1). At the end, we provide an example illustrating theoretical result.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout the paper.

We denote by $C(J,E)$ the Banach space of all continuous functions from $x: J \to E$ endowed with the norm defined by

$$\|x\|_\infty = \sup\{\|x(t)\|: \quad t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $x: J \to E$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(t)|\, dt.$$

In what follow, we recall some elementary definitions and properties related to fractional q-calculus. For $a \in \mathbb{R}$, we put

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is expressed by

$$(a-b)^{(0)}=1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\left(a-bq^k\right), \quad a,b \in \mathbb{R}, n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right), \quad a,b,\alpha \in \mathbb{R}.$$

Definition 2.1. [40] The q-gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R}-\{0,-1,-2,\ldots\}.$$

The q-gamma function satisfies the classical recurrence relationship

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha).$$

Definition 2.2. [40] For any $\alpha, \beta > 0$, the q-beta function is defined by

$$B_q(\alpha,\beta) = \int_0^1 f^{(\alpha-1)}(1-qf)^{(\beta-1)} d_q f, \quad q \in (0,1)$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Definition 2.3. [40] Let $f : J \to \mathbb{R}$ be a suitable function. We define the q-derivative of order $n \in \mathbb{N}$ of the function f by $D_q^0 f(t) = f(t)$,

$$D_q f(t) := D_q^1 f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t),$$

and

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad t \in I, n \in \{1,2,\ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4. [40] For a given function $f : I_t \to \mathbb{R}$, the expression defined by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f\big(tq^n\big),$$

is called q-integral, provided that the series converges.
We note that $D_q I_q f(t) = f(t)$, while if f is continuous at 0, then

$$I_q D_q f(t) = f(t) - f(0).$$

Definition 2.5. [10] The integral of a function $f : J \to \mathbb{R}$ defined by

$$I_q^0 f(t) = f(t),$$

and

$$I_q^\alpha f(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_q s, \quad t \in J.$$

is called Riemann-Liouville-fractional q-integral of order $\alpha \in \mathbb{R}_+$

Lemma 2.6. [7] *Let $\alpha \in \mathbb{R}_+$ and $\beta \in (-1,\infty)$. One has*

$$I_q^\alpha t^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} t^{\alpha+\beta}, \ \ \beta \in (-1,\infty), \ \alpha \geq 0, \ t > 0$$

In particular, if $f \equiv 1$, then

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}, \quad \textit{for all } t > 0.$$

Definition 2.7. [41] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by

$$^{C}D_q^{\alpha} f(t) = I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f(t), \quad t \in J.$$

We put by convention

$$^{C}D_q^{0} f(t) = f(t).$$

Lemma 2.8. [41] *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$I_q^{\alpha C} D_q^{\alpha} f(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} D_q^k f(0).$$

In particular, if $\alpha \in (0,1)$, then

$$I_q^{\alpha C} D_q^{\alpha} f(t) = f(t) - f(0).$$

Lemma 2.9. [42] *Let u be a function defined on J and suppose that α, β are two real nonegative numbers. Then the following hold:*

$$\begin{aligned} I_q^{\alpha} I_q^{\beta} f(t) =& I_q^{\alpha+\beta} f(t) = I_q^{\beta} I_q^{\alpha} f(t), \\ D_q^{\alpha} I_q^{\alpha} f(t) =& f(t). \end{aligned}$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.10. ([31, 32]) Let E be a Banach space and Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty]$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

From this definition we can directly obtain following facts

(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\mu(B) = \mu(\overline{B})$.

(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

(d) $\mu(A + B) \leq \mu(A) + \mu(B)$

(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.

(f) $\mu(convB) = \mu(B)$.

Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively.

Definition 2.11. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $t \mapsto f(t, u)$ is measurable for each $u \in E$,
(ii) $u \mapsto F(t, u)$ is continuous for almost all $t \in J$.

For a given set V of functions $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Let us now present the fundamental tools on which the proofs of our main results are based ([39]).

Theorem 2.12. (Mönch's fixed point theorem) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$ and let N be a continuous mapping of D into itself. If the implication $V = \overline{conv}N(V)$ or $V = N(V) \cup 0 \Rightarrow \mu(V) = 0$ holds for every subset V of D, then N has a fixed point.*

Lemma 2.13. *Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h\to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\mu(B);\ \text{here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)f(s,y(s))ds : y \in V\right\}\right) \leq \int_J \|G(t,s)\|p(s)\mu(V(s))ds.$$

3. Main Results

For the existence of solutions for the problem (1.1), the following definition and Lemma will be needed.

Definition 3.1. A function $x \in C(J, E)$ is said to be a solution of the problem (1.1) if x satisfies the equation $D_q^\beta(D_q^\alpha + \lambda)x(t) = f(t, x(t))$ on J and the conditions $x(0) = \gamma_1$, $x(1) = \gamma_2$.

Lemma 3.2. *Let $h : J \to E$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^\alpha x(t) + t^\alpha \left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}h(1) + \lambda I_q^\alpha x(1)\right\} + \gamma_1, \tag{3.1}$$

if and only if x is a solution of the fractional boundary-value problem

$$D_q^\beta(D_q^\alpha + \lambda)x(t) = h(t), \qquad t \in J, \tag{3.2}$$

$$x(0) = \gamma_1, \qquad x(1) = \gamma_2. \tag{3.3}$$

Proof. Assume that x satisfies (3.2). Then by applying Lemmas 2.6, 2.8 and 2.9, we can transform the problem (3.2)-(3.3) to an equivalent integral equation

$$x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^\alpha x(t) + c_0\frac{t^\alpha}{\Gamma_q(\alpha+1)} + c_1. \tag{3.4}$$

Applying the boundary conditions (3.3), we get

$$x(0) = c_1,$$
$$x(1) = I_q^{\alpha+\beta}h(1) - \lambda I_q^{\alpha}x(1) + \frac{c_0}{\Gamma_q(\alpha+1)} + c_1.$$

So, we have

$$c_1 = \gamma_1,$$
$$I_q^{\alpha+\beta}h(1) - \lambda I_q^{\alpha}x(1) + \frac{c_0}{\Gamma_q(\alpha+1)} + \gamma_1 = \gamma_2.$$

Consequently

$$c_1 = \gamma_1,$$
$$c_0 = \Gamma_q(\alpha+1)\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}x(1)\right\}.$$

Finally, we obtain

$$x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^{\alpha}x(t) + t^{\alpha}\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}x(1)\right\} + \gamma_1.$$

Which completes the proof. ∎

In the following, we prove existence results, for the boundary value problem (1.1) by using Mönch fixed point theorem, under the following hypotheses.

$(H1)$ $f : J \times E \to E$ satisfies the Caratheodory conditions.

$(H2)$ There exists $P \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$, such that,

$$\|f(t,x)\| \leq P(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E.$$

$(H3)$ For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+} \mu(f(J_{t,h} \times B)) \leq P(t)\mu(B); \quad \textit{here} \quad J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.3. *Assume that conditions (H1)-(H3) hold. Let* $P^* = \sup_{t\in J} P(t)$. *If*

$$P^*M + N < 1, \tag{3.5}$$

with

$$M := \left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\} \quad \textit{and} \quad N := |\lambda|\left\{\frac{2}{\Gamma_q(\alpha+1)}\right\}.$$

Then the problem (1.1) *has at least one solution on J.*

Proof. Using Lemma 3.2, it is sufficient to prove existence of solutions to the integral equation (3.1). For this, we rewrite the problem (1.1) as a fixed point problem. Indeed let us consider the operator $\mathfrak{F} : C(J,E) \to C(J,E)$ defined by

$$\mathfrak{F}x(t) = I_q^{\alpha+\beta}h(t) - \lambda I_q^{\alpha}x(t) + t^{\alpha}\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}h(1) + \lambda I_q^{\alpha}x(1)\right\} + \gamma_1. \tag{3.6}$$

It is obvious that fixed points of the operator $\mathfrak{F}$ are solutions of the problem (1.1). Let

$$R \geq \frac{\gamma_1}{1 - (p^*M + N)}, \tag{3.7}$$

and consider

$$D_R = \{x \in C(J,E) : \|x\| \leq R\}.$$

We can check, without difficulty, that the subset D_R is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step 1: First we show that $\mathfrak{F}$ is continuous:
Let x_n be a sequence such that $x_n \to x$ in $C(J, E)$. Then for each $t \in J$,

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| &\leq I_q^{\alpha+\beta}\|f(s, x_n(s)) - f(s, x(s))\|(t) + |\lambda| I_q^{\alpha}\|x_n(s) - x(s)\|(t) \\
&+ t^{\alpha} I_q^{\alpha+\beta}\|f(s, x_n(s)) - f(s, x(s))\|(1) + t^{\alpha}|\lambda| I_q^{\alpha}\|x_n(s) - x(s)\|(1), \\
&\leq \left\{I_q^{\alpha+\beta}(1)(t) + t^{\alpha} I_q^{\alpha+\beta}(1)(1)\right\}\|f(s, x_n(s)) - f(s, x(s))\| \\
&+ \left\{|\lambda| I_q^{\alpha}(1)(t) + t^{\alpha}|\lambda| I_q^{\alpha}(1)(1)\right\}\|x_n(s) - x(s)\|.
\end{aligned}$$

Thanks to assumption $(H1)$, the sequence $f(t, x_n(t))$ converges uniformly to $f(t, x(t))$. Lebesgue dominated convergence theorem guarantees; that

$$\|\mathfrak{F}(x_n) - \mathfrak{F}(x)\|_{\infty} \to 0 \text{ as } n \to \infty.$$

Then $\mathfrak{F} : D_R \to D_R$ is sequentially continuous.

Step 2: Second, we show that $\mathfrak{F}$ maps D into itself
Take $x \in D$, by (H2), we have, for each $t \in J$ and assume that $\mathfrak{F}x(t) \neq 0$.

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| &\leq I_q^{\alpha+\beta}\|f(s, x(s))\|(t) - \lambda I_q^{\alpha}\|x\|(t) \\
&+ t^{\alpha}\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}\|f(s, x(s))\|(1) + \lambda I_q^{\alpha}\|x\|(1)\right\} + \gamma_1, \\
&\leq I_q^{\alpha+\beta}\|x\|P(s)(t) - \lambda I_q^{\alpha}\|x\|(t) \\
&+ t^{\alpha}\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}\|x\|P(s)(1) + \lambda I_q^{\alpha}\|x\|(1)\right\} + \gamma_1, \\
&\leq P^* R\left\{I_q^{\alpha+\beta}(1)(t) + t^{\alpha} I_q^{\alpha+\beta}(1)(1)\right\} \\
&+ R\left\{|\lambda| I_q^{\alpha}(1)(t) + t^{\alpha}|\lambda| I_q^{\alpha}(1)(1)\right\} + T^{\alpha}(\gamma_2 - \gamma_1) + \gamma_1, \\
&\leq P^* R\left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\} + |\lambda| R\left\{\frac{2}{\Gamma_q(\alpha+1)}\right\} + \gamma_2, \\
&\leq P^* RM + RN + \gamma_2, \\
&\leq R.
\end{aligned}$$

Step 3: we show that $\mathfrak{F}(D_R)$ is equicontinuous
By Step 2, it is obvious that $\mathfrak{F}(D_R) \subset C(J, E)$ is bounded. For the equicontinuity of $\mathfrak{F}(D_R)$, let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in D_R$, so $\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1) \neq 0$. Then,

$$\begin{aligned}
\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\| &\leq I_q^{\alpha+\beta}|f(s, x(s))(t_2) - f(s, x(s))(t_1)| + |\lambda| I_q^{\alpha}(|x(s)|(t_2) - |x(s)|(t_2)) \\
&+ (t_2^{\alpha} - t_2^{\alpha})\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}|f(s, x(s))|(1) + \lambda I_q^{\alpha}|x|(1)\right\}, \\
&\leq P^* R|I_q^{\alpha+\beta}(1)(t_2) - I_q^{\alpha+\beta}(1)(t_1)| + R|\lambda|(I_q^{\alpha}|(1)|(t_2) - I_q^{\alpha}|(1)|(t_1)) \\
&+ (t_2^{\alpha} - t_1^{\alpha})\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}|f(s, x(s))|(1) + \lambda I_q^{\alpha}|x|(1)\right\}, \\
&\leq \frac{R(P^* + |\lambda|)}{\Gamma_q(\alpha+1)}\left\{(t_2^{\alpha} - t_1^{\alpha}) + 2(t_2 - t_1)^{\alpha}\right\} \\
&+ (t_2^{\alpha} - t_1^{\alpha})\left\{\gamma_2 - \gamma_1 - I_q^{\alpha+\beta}|f(s, x(s))|(1) + \lambda I_q^{\alpha}|x|(1)\right\}.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
This means that $\mathfrak{F}(D_R) \subset D_R$.

Finally we show that the implication holds
Let $V \subset D_R$ such that $V = \overline{conv}(\mathfrak{F}(V) \cup \{0\})$. Since V is bounded and equicontinuous, and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J. By assumption $(H2)$ and the properties of the measure μ we have for each $t \in J$.

$$
\begin{aligned}
v(t) &\leq \mu(\mathfrak{F}(V)(t) \cup \{0\})) \leq \mu((\mathfrak{F}V)(t)) \\
&\leq I_q^{\alpha+\beta} P(s)\mu(V(s))(t) + |\lambda| I_q^{\alpha}\mu(V(s))(t) \\
&+ t^{\alpha}\left\{I_q^{\alpha+\beta} P(s)\mu(V(s))(1) + |\lambda| I_q^{\alpha}\mu(V(s))(1)\right\}, \\
&\leq I_q^{\alpha+\beta} P(s)\mu(V(s))(t) + |\lambda| I_q^{\alpha}\mu(V(s))(t) \\
&+ t^{\alpha}\left\{I_q^{\alpha+\beta} P(s)\mu(V(s))(1) + |\lambda| I_q^{\alpha}\mu(V(s))(1)\right\}, \\
&\leq P^*\|v\|\left\{I_q^{\alpha+\beta}(1)(t) + t^{\alpha} I_q^{\alpha+\beta}(1)(1)\right\} \\
&+ \|v\|\left\{|\lambda| I_q^{\alpha}(1)(t) + t^{\alpha}|\lambda| I_q^{\alpha}(1)(1)\right\}, \\
&\leq P^*\|v\|\left\{\frac{2}{\Gamma_q(\alpha+\beta+1)}\right\} + |\lambda|\|v\|\left\{\frac{2}{\Gamma_q(\alpha+1)}\right\}, \\
&\leq P^*\|v\|M + \|v\|N.
\end{aligned}
$$

This means that

$$
\|v\|(1 - p^*M - N) \leq 0.
$$

By (3.5) it follows that $\|v\| = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 2.13, we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (1.1). ∎

4. Example

In this section, we present an example to illustrate the main result.
Let $E = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with the norm

$$
\|x\|_E = \sum_{n=1}^{\infty} |x_n|
$$

Consider the following nonlinear Langevin $\frac{1}{4}$-fractional equation:

$$
\begin{cases}
D_{1/4}^{1/2}\left(D_{1/4}^{1/3} - \frac{5}{27}\right)x(t) = \frac{(\sin t+1)e^{-t}}{24}\left(\frac{x^2(t)}{1+|x(t)|}\right), & t \in J = [0,1], \\
x(0) = \gamma_1 \qquad , x(1) = \gamma_2. &
\end{cases}
\tag{4.1}
$$

Here

$$
\begin{aligned}
&\alpha = 1/2, && \beta = 1/3, && q = 1/4, \\
&\gamma_1 = 3/4, && \gamma_2 = 1/4, && \lambda = 5/27,
\end{aligned}
$$

with

$$
f(t,x) = (((\sin t + 1)e^{-t})/24)(x^2/(1+|x|)).
$$

Clearly, the function f is continuous. For each $x \in E$ and $t \in [0,1]$, we have

$$|f(t,x)| \leq \frac{1}{12}|x|,$$

and

$$p^* = \frac{1}{12}.$$

Hence, the hypothesis (H2) is satisfied with $p^* = \frac{1}{12}$. We shall show that condition (3.5) holds with $J = [0,1]$. Indeed,

$$p^* M + N \simeq 0.6785 < 1.$$

Therefore, we deduce from the conclusion of Theorem (3.3) that the problem (4.1) has a solution on $[0,1]$.

5. Conclusion

We have provided sufficient conditions for the existence of the solutions of a new class of nonlinear Langevin fractional q-difference equations with Dirichlet boundary conditions in Banach space. by using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting an example.

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