Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 129–135

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A Related Fixed Point Theorem for Three Pairs of Mappings on Three Metric Spaces

R. K. Namdeo and B. Fisher

Abstract : A new related fixed point theorem for three pairs of mappings on three complete metric spaces is obtained.

Keywords : Complete metric space; Common fixed point; Related fixed point mappings.
2000 Mathematics Subject Classification : 54H25.

The following related fixed point theorem was for two pairs of mappings on two complete metrc spaces was proved in [3]. See also [1] and [2].

Theorem 1 *Let (X, d) and (Y, ρ) be complete metric spaces. Let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$\rho(BSy, ATy') \le c\frac{f(x,x',y,y')}{h(x,x',y,y')},$$
$$d(SAx, TBx') \le c\frac{g(x,x',y,y')}{h(x,x',y,y')}$$

for all x, x' in X and y, y' in Y for which $h(x,x',y,y') \neq 0$, where

$$\begin{aligned} f(x,x',y,y') &= \max\{d(x,x')\rho(y,y'), d(x,Sy)d(x',Ty'), \\ &\qquad d(x,Ty')d(x',Sy), \rho(y,Bx')\rho(y',Ax)\}, \\ g(x,x',y,y') &= \max\{\rho(Ax,Bx')d(Sy,Ty'), \rho(Ax,BSy)\rho(Bx',ATy'), \\ &\qquad \rho(Ax,ATy')\rho(Bx',BSy), d(Sy,TBx')d(Ty',SAx)\}, \\ h(x,x',y,y') &= \max\{\rho(Ax,Bx'), d(SAx,TBx'), d(Sx,Ty'), \rho(BSy,ATy')\} \end{aligned}$$

and $0 \le c < 1$. If one of the mappings A, B, S and T is continuous, then SA and and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

We now prove a related fixed point theorem for three pairs of mappings on three complete metric spaces.

Theorem 2 *Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces. Let A, B be mappings of X into Y, let C, D be mappings of Y into Z and let E, F be mappings of Z into X satisfying the inequalities*

$$d(ECAx, FDBx') \leq c\frac{f_1(y,y',z.z')}{g_1(x,x')}, \tag{1}$$

$$\rho(BECy, AFDy') \leq c\frac{f_2(z,z',x,x')}{g_2(y,')}, \tag{2}$$

$$\sigma(DBEz, CAFz') \leq c\frac{f_3(x,x',y,y')}{g_3(z,z')} \tag{3}$$

for all x, x' in X; y, y' in Y and z, z' in Z for which $g_1(x,x') \neq 0$; $g_2(y,y') \neq 0$, $g_3(z,z') \neq 0$, where

$$\begin{aligned}
f_1(y,y',z,z') &= \max\{\rho(y,y')d(Ez,Fz'), \sigma(Cy,Dy')\rho(BEz,AFz'), \\
&\qquad d(ECy,FDy')\sigma(DBEz,CAFz')\}, \\
f_2(z,z',x,x') &= \max\{\sigma(z,z')\rho(Ax,Bx'), d(Ez,Fz')\sigma(CAx,DBx'), \\
&\qquad \rho(BEz,AFz')d(ECAx,FDx')\}, \\
f_3(x,x',y,y') &= \max\{d(x,x'), \sigma(Cy,Dy'), \rho(Ax,Bx'), d(ECy,FDy'), \\
&\qquad \sigma(CAx,DBx')\rho(BECy,AFDy')\}, \\
g_1(x,x') &= \max\{d(x,x'), \rho(Ax,Bx'), \sigma(CAx,DBx'), d(ECAx,FDBx')\}, \\
g_2(y,y') &= \max\{\rho(y,y'), \sigma(Cy,Dy'), d(ECy,FDy'), \rho(BECy,AFDy')\}, \\
g_3(z,z') &= \max\{\sigma(z,z'), d(Ez,Fz'), \rho(BEz,AFz'), \sigma(DBEz,CAFz')\}
\end{aligned}$$

and $0 \leq c < 1$. If A and C or B and D are continuous, then ECA and FDB have a unique common fixed point u in X, BEC and AFD have a unique common fixed point v in Y, and DBE and CAF have a unique common fixed point w in Z. Further, $Au = Bu = v, Cv = Dv = w$ and $Ew = Fw = u$.

Proof. Let $x = x_0$ be an arbitrary point in X. We define the sequences $\{x_n\}$ in X, $\{y_n\}$ in Y and $\{z_n\}$ in Z inductively by

$$\begin{aligned}
&Ax_{2n-2} = y_{2n-1}, Cy_{2n-1} = z_{2n-1}, Ez_{2n-1} = x_{2n-1}, \\
&Bx_{2n-1} = y_{2n}, Dy_{2n} = z_{2n}, Fz_{2n} = x_{2n}
\end{aligned}$$

for $n = 1, 2, \ldots$.

We will first of all suppose that for some n,

$$\begin{aligned}
g_1(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), \rho(Ax_{2n}, Bx_{2n-1}), \sigma(CAx_{2n}, DBx_{2n-1}), \\
&\qquad d(ECAx_{2n}, FDBx_{2n-1})\} \\
&= \max\{d(x_{2n}, x_{2n-1}), \rho(y_{2n+1}, y_{2n}), \sigma(z_{2n+1}, z_{2n}), d(x_{2n+1}, x_{2n})\} \\
&= 0.
\end{aligned}$$

Then putting

$$x_{2n-1} = x_{2n} = x_{2n+1} = u, \quad y_{2n} = y_{2n+1} = v, \quad z_{2n} = z_{2n+1} = w,$$

we see that

$$ECAu = FDBu = u = Ew = Fw, \quad AFDv = v = Au = Bu,$$
$$CAFw = w = Cv = Dv,$$

from which it follows that

$$BECv = v, \quad DBEw = w.$$

Similarly, $g_1(x_{2n}, x_{2n+1}) = 0$ for some n implies that there exist points u in X, v in V and w in Z such that

$$ECAu = FDBu = u = Ew = Fw, \quad BECv = AFDv = v = Au = Bu$$
$$DBEw = CAFw = w = CvDv. \tag{4}$$

Similarly, if one $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$, $g_3(z_{2n+1}, z_{2n})$ is equal to zero for some n, then equations (4) follow.

We will therefore suppose that $g_1(x_{2n-1}, x_{2n})$, $g_1(x_{2n}, x_{2n+1})$, $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$ and $g_3(z_{2n+1}, z_{2n})$ are all non-zero for all n.

We have

$$f_1(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) = \max\{\rho(y_{2n-1}, y_{2n})d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n})\sigma(z_{2n}, z_{2n+1})\}, \tag{5}$$

$$f_2(z_{2n-1}, zz_{2n}, x_{2n}, x_{2n-1}) = \max\{\sigma(z_{2n-1}, z_{2n})\rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n})\sigma(z_{2n}, z_{2n+1}), \rho(y_{2n}, y_{2n+1})d(x_{2n}, x_{2n+1}), \tag{6}$$

$$f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n}), \rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \sigma(z_{2n}, z_{2n+1})\rho(y_{2n}, y_{2n+1})\}, \tag{7}$$

$$g_1(x_{2n}, x_{2n-1}) = \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1}), \sigma(z_{2n}, z_{2n+1}), d(x_{2n}, x_{2n+1}), \tag{8}$$

$$g_2(y_{2n-1}, y_{2n}) = \max\{\rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n}), d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1}), \tag{9}$$

$$g_3(z_{2n-1}, z_{2n}) = \max\{\sigma(z_{2n-1}, z_{2n}), d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1}), \sigma(z_{2n}, z_{2n+1}). \tag{10}$$

Applying inequality (1), we get

$$d(x_{2n+1}, x_{2n}) = d(ECAx_{2n}, FDBx_{2n-1}) \le c\frac{f_1(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})}{g_1(x_{2n}, x_{2n-1})} \tag{11}$$

and it now follows from (5), (8) and (11) that

$$d(x_{2n}, x_{2n+1}) \leq c\max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n})\}. \tag{12}$$

Applying inequality (2), we get

$$\begin{aligned}\rho(y_{2n}, y_{2n+}) &= \rho(BECy_{2n-1}, AFDy_{2n}) \\ &\leq c\frac{f_2(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1})}{g_2(y_{2n-1}, y_{2n})}\end{aligned} \tag{13}$$

and it now follows from (6), (9) and (13) that

$$\rho(y_{2n}, y_{2n+1}) \leq c\max\{d(x_{2n}, x_{2n+1}), \sigma(z_{2n}, z_{2n+1})\}. \tag{14}$$

Applying inequality (3), we get

$$\begin{aligned}\sigma(z_{2n}, z_{2n+1}) &= \sigma(DBEz_{2n-1}, CAFz_{2n}) \\ &\leq c\frac{f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{g_3(z_{2n-1}, z_{2n})}\end{aligned} \tag{15}$$

and it now follows from (7), (10) and (15) that

$$\sigma(z_{2n}, z_{2n+1}) \leq c\max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\}. \tag{16}$$

Using inequalities (12), (14) and (16) we now get

$$\begin{aligned}\rho(y_{2n}, y_{2n+1}) &\leq c\max\{cd(x_{2n-1}, x_{2n}, c\rho(y_{2n-1}, y_{2n}), c\sigma(z_{2n-1}, z_{2n})\} \\ &\leq c\max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n})\}.\end{aligned} \tag{17}$$

On applying inequality (1) again, we get

$$\begin{aligned}d(x_{2n-1}, x_{2n}) &= d(ECAx_{2n-2}, FDBx_{2n-1}) \\ &\leq c\frac{f_1(y_{2n-1}, y_{2n-2}, z_{2n-1}, z_{2n-2})}{g_1(x_{2n-2}, x_{2n-1})}\end{aligned}$$

from which it follows that

$$d(x_{2n-1}, x_{2n}) \leq c\max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\} \tag{18}$$

and similarly on using inequalities (2) and (3), we get

$$\rho(y_{2n-1}, y_{2n}) \leq c\max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\}, \tag{19}$$

$$\sigma(z_{2n-1}, z_{2n}) \leq c\max\{d(x_{2n-2}, x_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\}. \tag{20}$$

On using inequalities (18), (19) and (20), we get

$$\rho(y_{2n-1}, y_{2n}) \leq c\max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\}. \tag{21}$$

It now follows from inequalities (12) and (18) that

$$\begin{aligned} d(x_n, x_{n+1}) &\le c \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n)\} \\ &\le k^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}. \end{aligned} \tag{22}$$

Similarly, on using inequalities (17), (21), (16) and (20), we get

$$\rho(y_n, y_{n+1}) \le k^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}, \tag{23}$$

$$\sigma(z_n, z_{n+1}) \le k^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}. \tag{24}$$

Since $c < 1$, it follows from inequalites (22), (23) and (24) that $\{x_n\}$ is a Cauchy sequence in X with a limit u, $\{y_n\}$ is a Cauchy sequence in Y with a limit v and $\{z_n\}$ is a Cauchy sequence in Z with a limit w.

Now suppose that A and C are continuous. Then

$$v = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ax_{2n} = Au, \quad w = \lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} Cy_{2n-1} = Cv \tag{25}$$

and hence

$$\lim_{n\to\infty} f_1(v, y_{2n}, w, z_{2n}) = d(Ew, u)\sigma(DBEw, u), \tag{26}$$

$$\lim_{n\to\infty} f_2(w, z_{2n}, u, x_{2n-1}) = \rho(BEw, v)d(Ew, u), \tag{27}$$

$$\lim_{n\to\infty} f_3(v, y_{2n}, w, z_{2n}) = 0, \tag{28}$$

$$\lim_{n\to\infty} g_1(u, x_{2n-1}) = d(Ew, u), \tag{29}$$

$$\lim_{n\to\infty} g_2(v, v_n) = \max\{d(Ew, u)\rho(BEw, v)\}, \tag{30}$$

$$\lim_{n\to\infty} g_3(w, z_{2n}) = \max\{d(Ew, u), \rho(BEw, w), \sigma(DBEw, w). \tag{31}$$

If $\lim_{n\to\infty} g_1(u, x_{2n-1}) = 0$, then $Ew = u$ and $ECAu = u$.

If it were possible that

$$\lim_{n\to\infty} g_1(u, x_{2n-1}) = d(Ew, u) \neq 0,$$

then on applying inequality (1) and equations (25), (26) and (29), we get

$$d(Ew, u) = \lim_{n\to\infty} d(ECAu, FDBx_{2n-1}) \le c\sigma(DBEw, w). \tag{32}$$

On using inequality (3) and equations (28) and (31), we get

$$\sigma(DBEw, w) = \lim_{n\to\infty} \sigma(DBEw, CAFz_{2n}) = 0$$

which implies that $DBEw = w$ and hence from (32) we must have $Ew = u$.

On using inequality (2) and equations (25), (27) and (30), we have

$$\rho(BEw, v) = \lim_{n\to\infty} \rho(BECv, AFDy_{2n}) \le cd(Ew, u) = 0$$

which implies that

$$Bu = v, \quad Dv = w, \quad ECAu = u, \quad BECv = v.$$

Now suppose that $Fw \neq u$. On applying inequality (1), we have

$$\begin{aligned} d(u, Fw) &= \lim_{n\to\infty} d(ECAx_{2n}, FDBu) \\ &\leq c\frac{\lim_{n\to\infty} f_1(y_{2n-1}, v, z_{2n-1}, w)}{\lim_{n\to\infty} g_1(x_{2n}, u)} \\ &= c\sigma(w, CAFw). \end{aligned} \tag{33}$$

Applying inequality (3), we now have

$$\begin{aligned} \sigma(w, CAFw) &= \lim_{n\to\infty} \sigma(DBEz_{2n-1}, CAFw) \\ &\leq c\frac{\lim_{n\to\infty} f_3(x_{2n}, u, y_{2n-1}, v)}{\lim_{n\to\infty} g_3(z_{2n-1}, w)} \\ &= 0. \end{aligned}$$

This implies that $w = CAFw$ and hence from (33), we must have $Fw = u$. Equations (4) follows.

Equations (4) follow similarly if B and D are continuous.

To prove the uniqueness, let ECA and FDB have a second distinct fixed point u'. Then, using inequalities (1), (2) and (3) respectively, we have

$$d(u, u') = d(ECAu, FDBu') \leq c\frac{f_1(Au, Bu', CAu, DBu)}{g_1(u, u')}$$

which implies that

$$d(u, u') \leq c\max\{\rho(v, Au'), \rho(v, Bu'), \sigma(w, CAu')\}, \tag{34}$$

$$\rho(v, Au') = \rho(BECAu, AFDBu') \leq c\frac{f_2(CAu, DBu', u, u')}{g_2(Au, Bu')}$$

which implies that

$$\rho(v, Au') \leq c\max\{d(u, u'), \rho(v, Bu')\} \tag{35}$$

and

$$\sigma(w, CAu') = \sigma(DBECAu, CAFDBu') \leq c\frac{f_3(u, u', Au, Bu')}{g_3(CAu, DBu')}$$

which implies that

$$\sigma(w, CAu') \leq c\max\{d(u, u'), \rho(v, Au'), \rho(v, Bu')\}. \tag{36}$$

On applying inequality (2) again, we have

$$\rho(Bu', v) = \rho(BECAu', AFDBu) \le c\frac{f_2(CAu', DBu, u', u)}{g_2(Au, Bu')}$$

which implies that

$$\rho(v, Bu') \le c\max\{d(u, u'), \rho(v, Au')\}. \quad (37)$$

It now follows from (34) to (37) that

$$d(u, u') \le c\max\{\rho(v, Au'), \rho(v, Bu')\} \quad (38)$$

and then (35), (37) and (38) imply that $u = u'$, proving the uniqueness of u.

We can prove similarly that v is the unique common fixed point of BEC and AFD and w is the unique common fixed point of DBE and CAF. □

References

[1] B. Fisher and P. P. Murthy, Related fixed point theorems for two pairs of mappings on two metric spaces, *Kyungpook Math. J.*, **37**(1997), 343–347.

[2] R. K. Namdeo, S. Jain and B. Fisher, A related fixed point theorem for two pairs of mappings on two complete metric spaces, *Hacettepe J. Math. Stat.*, **32**(2003), 7–11.

[3] R. K. Namdeo and B. Fisher, A related fixed point theorem for two pairs of mappings on two complete metric spaces, *Nonlinear Analysis Forum*, **8**(1)(2003), 23–27.

(Received 25 May 2005)

R. K. Namdeo
Dr. H.S. Gour University,
Sagar M.P. 470003, India.
e-mail: rkn_math@yahoo.com

Brian Fisher
Department Of Mathematics,
University of Leicester,
Leicester, LE1 7RH, England.
email:fbr@le.ac.uk