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Characterization of Digraphs of Right (Left) Zero Unions of Groups

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Abstract: We characterize directed graphs which are Cayley graphs of certain completely regular semigroups. We specify these semigroups as so called left or right zero unions of groups and prove that they can be composed only of copies of the same group.

Keywords: Cayley graph, digraph, completely regular semigroup, right regular orthogroup, right zero union of groups.

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The investigation and characterization of digraphs which are Cayley graphs of certain algebraic structures have a long history, documented for example in Maschke's Theorem from 1896 about groups of genus zero, that is, groups G which possess a system of generating elements A, such that the Cayley graph $C(G, A)$ is planar. A modern presentation can be found in [5].

For a groupoid G and some subset $A \subseteq G$ one defines the *Cayley graph* $C(G, A)$ as follows: G is the vertex set and (u, v), $u, v \in G$, is an arc in $C(G, A)$ if there exists an element $a \in A$ such that $v = ua$. We see that here the Cayley graph is defined by right translations which is quite usual. However, the reflection of algebraic properties of G may depend strongly on this decision, as we will also see in the sequel.

In [3] Cayley graphs which represent groupoids, quasigroups, loops or groups are described, for terminolgy see for example [2]. We use some of these results and study Cayley graphs which represent certain completely regular semigroups, which we call right (left) zero unions of groups. They have some interesting algebraic properties on the lines investigated in [4]. They also generalize right or left zero semigroups.

Any book on graph theory, for example [1], will provide terminolgy which may be used here without definition. We use the term *strong subgraph* instead of the also common term induced subgraph.

1 Basic Knowledge

Definition 1.1 A digraph (V,E) with vertex set V and edge set E is called a *regular digraph* if there exists a non-negative integer n such that indegree and outdegree of every vertex is n. In addition, if every vertex $v \in V$ has a loop or non of them has a loop, then (V,E) is called a *normal regular digraph.*

Definition 1.2 Let (V_1,E_1) and (V_2,E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u,v) \in E_1$ implies $(\varphi(u),\varphi(v)) \in E_2$. We write $\varphi : (V_1,E_1) \to (V_2,E_2)$. A digraph homomorphism $\varphi : (V,E) \to (V,E)$ is called a *digraph endomorphism.*
If $\varphi : (V_1,E_1) \to (V_2,E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism.* And a digraph isomorphism $\varphi : (V,E) \to (V,E)$ is called a *digraph automorphism.*

Definition 1.3 A digraph (V,E) is called a *groupoid (group, semigroup, etc.) digraph* or *digraph of a groupoid, etc.* if there exits a groupoid G and $A \subseteq G$ such that (V,E) isomorphic to the Cayley graph $C(G,A)$.
We speak about *G-groupoid digraphs*, if we want to consider various subsets $A \subseteq G$ and the respective Cayley graphs $C(G,A)$.

Lemma 1.4 *Let G_1, G_2 be groupoids, $A \subseteq G_1$, and $\theta : G_1 \to G_2$ a groupoid isomorphism. Then $\theta : C(G_1,A) \to C(G_2,\theta(A))$ is a digraph isomorphism.*

Proof. It is clear that θ is a bijection. Now
(x,y) is an arc in $C(G_1,A) \Leftrightarrow \exists a \in A, y = xa$
$\Leftrightarrow \exists a \in A, \theta(y) = \theta(x)\theta(a)$
$\Leftrightarrow \exists a \in A, (\theta(x),\theta(y))$ is an arc in $C(G_2,\theta(A))$.
Therefore θ is a digraph isomorphism. □

Theorem 1.5 (3) *A digraph (V,E) is a groupoid digraph if and only if $E = \emptyset$ or for every vertex $u \in V$ there exists a vertex $v \in V$ such that $(u,v) \in E$.*

Theorem 1.6 (3) *A digraph (V,E) with n vertices is a group digraph if and only if the group $Aut(V,E)$ of all digraph automorphisms of (V,E) contains an n-element subgroup $\triangle$ such that for any two vertices $u,v \in V$ there exists $\delta \in \triangle$ such that $\delta(u) = v$. In this case (V,E) is a normal regular digraph.*

The following normal regular digraph is not a group digraph.

2 Right (Left) Zero Union of Groups

In this section we consider a class of completely regular semigroups, i. e., of unions of groups.
Note that the proof of the following lemma remains valid if $(G_1; \circ_1)$ and $(G_2; \circ_2)$ are monoids.

Lemma 2.1 *Let $(G_1; \circ_1), (G_2; \circ_2)$ be groups. Define a binary operation $*$ on the set $G_1 \cup G_2$ by:*

$$\begin{aligned} x_i * y_i &= x_i \circ_i y_i \quad if \quad x_i, y_i \in G_i, i = 1, 2, \\ x_1 * y_2 &= \theta_2(x_1) \circ_2 y_2 \quad if \quad x_1 \in G_1, y_2 \in G_2, \\ x_2 * y_1 &= \theta_1(x_2) \circ_1 y_1 \quad if \quad x_2 \in G_2, y_1 \in G_1, \end{aligned}$$

where $\theta_2 : G_1 \to G_2, \theta_1 : G_2 \to G_1$ are group homomorphisms. Then $(G_1 \cup G_2; *)$ is a semigroup if and only if $\theta_1 = \theta_2^{-1}$.

Proof. Necessity. By hypothesis $(G_1 \cup G_2; *)$ is associative. Denote by e_i the identity of $G_i, i = 1, 2$. For an arbitrary $x_1 \in G_1$ we compute $(x_1 * e_2) * e_1 = (\theta_2(x_1) \circ_2 e_2) * e_1 = \theta_1((\theta_2(x_1) \circ_2 e_2)) \circ_1 e_1 = \theta_1((\theta_2(x_1)) \circ_1 \theta_1(e_2)) \circ_1 e_1 = \theta_1\theta_2(x_1)$ and $x_1 * (e_2 * e_1) = x_1 \circ_1 (\theta_1(e_2) \circ_1 e_1) = x_1$. Equality of the two expressions for all $x_1 \in G_1$ shows that θ_1 is left inverse to θ_2. Similarly we obtain that θ_1 is right inverse to θ_2 if we start computation with $(x_2 * e_1) * e_2$ for an arbitrary $x_2 \in G_2$. Sufficiency is obvious. □

The construction in the previous lemma can also be transformed from "right" to "left". It generalizes to arbitrary unions of isomorphic groups and even to unions of isomorphic semigroups. It is clear that after this lemma we can restrict our attention to unions of copies of one group or semigroup. Moreover the multiplication of elements from different copies as given in the lemma makes clear that the semigroup constructed is nothing else but the direct product of the group with the two element right zero semigroup R_2. Similarly, if we consider the left variant we have to take the direct product with the two element left zero semigroup L_2. Recall that the multiplication in the right zero semigroup $R_k = \{1, \ldots, k\}$ is defined by $ij = j$ for all $i, j \in R_k$. Recall moreover that the direct product $R \times S$ of two semigroups R and S is the cartesian product of their elements with componentwise multiplication.

Definition 2.2 Let G be a group and R_k for $k \in \mathbb{N}$, $k \geq 2$, the k-element right zero semigroup. We call the semigroup $G \times R_k$ the *right zero union of the groups* G or shortly an *RZUG over* G.
Correspondingly, if L_k for $k \in \mathbb{N}$, $k \geq 2$, is the k-element left zero semigroup, we call the semigroup $G \times L_k$ the *left zero union of the groups* G or shortly an *LZUG over* G.

We note that an RZUG is a right regular orthogroup and an LZUG is a left regular orthogroup. For the terminology compare [4].

3 RZUG Digraphs

Now we can prove the main result. For a digraph of a right zero union of groups we will use the term RZUG digraph and, similarly we will use LZUG digraph.

Theorem 3.1 *Let (V, E) be an RZUG digraph of the group G with the digraph isomorphism $\varphi : C(G \times R_k, A) \to (V, E)$ for $k \in \mathbb{N}$ and $A \subseteq G \times R_k$, $R_k = \{1, \dots, k\}$ the k-element right zero semigroup, $k \geq 2$. If we denote the image in (V, E) of a vertex $(u, i) \in G \times R_k$ under φ by u_i then (V, E) is the vertex disjoint union of k strong G-group subgraphs $(V_1, E_1), \dots, (V_k, E_k)$ such that*

$$\forall i, j \in \{1, \dots, k\}[(u_i, v_i) \in E_i \Leftrightarrow (u_j, v_i) \in E].$$

Proof. For each $i \in \{1, \dots, k\}$ set $V_i := \varphi(G \times \{i\}), A_i = A \cap (G \times \{i\}), E_i = E \cap (V_i \times V_i)$. Now the restriction of φ to $C(G \times \{i\}, A_i)$ induces a digraph isomorphism

$$\varphi_i : C(G, p_1(A_i)) \to (V_i, E_i)$$

where p_1 denotes the first projection from the cartesian product.
We have to show that $\forall i \forall j[(u_i, v_i) \in E_i \Leftrightarrow (u_j, v_i) \in E]$.
Take $i \in \{1, \dots, k\}, u_i, v_i \in V_i$ with $(u_i, v_i) \in E_i$. Then $(\varphi_i^{-1}(u_i), \varphi_i^{-1}(v_i))$ is an arc in $C(G, p_1(A_i))$, i. e., there exists an element $a \in p_1(A_i)$ with $ua = v$. Then we have $(u, i)(a, i) = (v, i)$, and consequently, for all $j \in \{1, \dots, k\}$ we get $(u, j)(a, i) = (v, i)$ by definition of the multiplication in $G \times R_k$, i. e., $((u, j), (v, i)) \in E(C(G \times R_k, A))$. Hence $(u_j, v_i) \in E$ for all $j \in \{1, \dots, k\}$.
Conversely, take $i, j \in \{1, \dots, k\}, u_j \in V_j, v_i \in V_i$ with $(u_j, v_i) \in E$.
Then $((u, j), (v, i))$ is an arc in $C(G \times R_k, A)$, i. e., there exists $a' \in A$ such that $(v, i) = (u, j)a'$ where $a' = (a, i)$ for some $a \in p_1(A_i)$, from the definition of multiplication in $G \times R_k$. Then clearly, $(v, i) = (u, i)(a, i)$, i. e., $((u, i), (v, i))$ is an arc in $C(G \times R_k, A)$. Now we have that $(a, i) \in A_i, v = ua, a \in p_1(A_i)$ and thus $(u_i, v_i) \in E_i$. □

Theorem 3.2 *Let (V, E) be a digraph. If it is the vertex disjoint union of k strong G-group subgraphs $(V_1, E_1), \dots, (V_k, E_k)$ for some group G, $V = \bigcup_{i=1}^{k} V_i, k \geq 2$, with* *digraph isomorphisms* $\varphi_i : C(G, A_i) \to (V_i, E_i)$ *for each* $i \in \{1, \ldots, k\}$, $A_i \subseteq G$ *and if*

$$\forall i, j \in \{1, \ldots, k\}[(u_i, v_i) \in E_i \Leftrightarrow (u_j, v_i) \in E]$$

then (V, E) *is an RZUG digraph of* $G \times R_k$.

Proof. Set $A = \bigcup_{i=1}^{k}(A_i \times \{i\})$ and let $\varphi : C(G \times R_k, A) \to (V, E)$ be such that its restriction to $C(G \times \{i\}, A_i \times \{i\})$ is induced by $\varphi_i : C(G, A_i) \to (V_i, E_i)$. It is clear that φ is a well defined bijection. We will show that φ and φ^{-1} are digraph homomorphisms.
Take $(u, i), (v, j) \in G \times R_k$ such that $((u, i), (v, j))$ is an arc in $C(G \times R_k, A)$ with $a' \in A$ such that $(v, j) = (u, i)a'$. From the definition of the multiplication in $G \times R_k$ we see that $a' = (a, j)$ where $a \in A_j$. Then $v = ua$, i. e., (u, v) is an arc in $C(G, A_j)$ and hence $(u_j, v_j) \in E_j$. From the condition we then get that $(u_i, v_j) \in E$ for all $i \in \{1, \ldots, k\}$. This shows that φ is a digraph homomorphism. Consider now $\varphi^{-1} := (V, E) \to C(G \times R_k, A)$. Take $u_j \in V_j, v_i \in V_i$ with $(u_j, v_i) \in E$. Then $(u_i, v_i) \in E_i$ by the condition. But then $(\varphi_i^{-1}(u_i), \varphi_i^{-1}(v_i))$ is an arc in $C(G, A_i)$. Consequently there exists $a \in A_i$ with $v = ua$. By definition of the multiplication in $G \times R_k$ we get for all $j \in \{1, \ldots, k\}$ that $(v, i) = (u, j)(a, i)$. Therefore for all $j \in \{1, \ldots, k\}$ we have that $((u, j), (v, i))$ is an arc in $C(G \times R_k, A)$. This shows that φ^{-1} is also a digraph homomorphism. □

Corollary 3.3 *A digraph* (V, E) *is an RZUG digraph if and only if there exists* $k \in \mathbb{N}$, $k \geq 2$, *such that* (V, E) *is the vertex disjoint union of* k *isomorphic strong* G*-group subgraphs* $(V_1, E_1), \ldots, (V_k, E_k)$ *such that* $V = \bigcup_{i=1}^{k} V_i$ *and*

$$\forall i, j \in \{1, \ldots, k\}[(u_i, v_i) \in E_i \Leftrightarrow (u_j, v_i) \in E].$$

Remark It is clear that for the less interesting situation where $A = \emptyset$ we have the following: A digraph (V, E) is an RZUG digraph if and only if (V, E) consists of k totally disconnected isomorphic strong G-group subgraphs $(V_1, E_1), \ldots, (V_k, E_k)$ such that $V = \bigcup_{i=1}^{k} V_i$ for some group G.

The following RZUG digraphs $C(\mathbb{Z}_3 \times R_3, A)$ where $\mathbb{Z}_3 = \{0, 1, 2\}$ denotes the three element group make the structure quite lucid. It becomes also clear that for $A = \{a_1, a_2\}$ we have to form the edge sum of the respective graphs with one element sets $A = \{a_1\}$ and $A = \{a_2\}$ and so on.

$C(\mathbb{Z}_3 \times R_3, A)$ with $A = \{(0, 3)\}, A = \{(1, 2)\}, A = \{(2, 1)\}$

$\lim s_{i=1}^{k} V_i$

4 LZUG Digraphs

Now we prove the corresponding result for a digraph of a left zero union of groups. Again G denotes a group and L_k now denotes the k-element left zero semigroup.

Theorem 4.1 *Let (V,E) be an LZUG digraph of G with the digraph isomorphism $\varphi : C(G\times L_k, A) \to (V,E)$ for $k \in \mathbb{N}$, $k \geq 2$, and $A \subseteq G\times L_k$. Set $u_i = \varphi((u,i)) \in (V,E)$ for $(u,i) \in G\times L_k$. Then (V,E) is the vertex disjoint union of k strong G-group subgraphs $(V_1,E_1),\ldots,(V_k,E_k)$ with $V_i = \varphi(G\times\{i\}), E_i = E\cap(V_i\times V_i)$ and edges such that*

$$\forall i \in \{1,\ldots,k\}[(u_i,v_i)\in E_i \Leftrightarrow \exists l \in \{1,\ldots,k\}(u_l,v_l)\in E_l]$$

and

$$(u_i,v_j)\in E \Rightarrow i=j \text{ and } (u_i,v_i)\in E_i.$$

Proof. For each $i \in \{1,\ldots,k\}$ set $A_i = A\cap(G\times\{i\})$. Now the restriction of φ to $C(G\times\{i\},A_i)$ induces a digraph isomorphism

$$\varphi_i : C(G,p_1(A_i)) \to (V_i,E_i)$$

where p_1 denotes the first projection from the cartesian product.
We have to show the two conditions. Necessity in the first condition is trivial. So suppose that $(u_l,v_l)\in E_l$, then $(\varphi_l^{-1}(u_l),\varphi_l^{-1}(v_l))$ is an arc in $C(G,p_1(A_i))$, i. e., there exists an element $a\in p_1(A_l)$ with $ua = v$. Then we have $(u,l)(a,l) = (v,l)$, and consequently, for all $i\in\{1,\ldots,k\}$ we get $(u,i)(a,l) = (v,i)$ by definition of the multiplication in $G\times L_k$, i. e., $((u,i),(v,i))\in E_i$.
To show the second condition take $(u_i,v_j)\in E$ then $(\varphi^{-1}(u_i),\varphi^{-1}(v_j)) = ((u,i),(v,j))$ is an arc in $C(G\times L_k,A)$ i. e., there exists $(a,l)\in A$ such that $(v,j) = (u,i)(a,l)$. From the definition of multiplication in $G\times L_k$ we get $i=j$ and $ua=v$ and thus $(u_i,v_i)\in E_i$. □

Theorem 4.2 *Let (V,E) be a digraph. If it is the vertex disjoint union of k strong G-group subgraphs $(V_1,E_1),\ldots,(V_k,E_k)$ for some group G, $V = \bigcup_{i=1}^{k} V_i, k\geq 2$, with digraph isomorphisms $\varphi_i : C(G,A_i)\to(V_i,E_i)$ for each $i\in\{1,\ldots,k\}$, $A_i\subseteq G$ such that $\varphi_i(u) = u_i$ for all $u\in G$ and if*

$$\forall i \in \{1,\ldots,k\}[(u_i,v_i)\in E_i \Leftrightarrow \exists l \in \{1,\ldots,k\}(u_l,v_l)\in E_l]$$

and

$$(u_i,v_j)\in E \Rightarrow i=j \text{ and } (u_i,v_i)\in E_i$$

then (V,E) is an LZUG digraph over G, i. e., there exists a digraph isomorphism $\varphi : C(G\times L_k,A)\to(V,E)$ for $A = \bigcup_{i=1}^{k}(A_i\times\{i\})\subseteq G\times L_k$.

Proof. Define $\varphi((u,i)) = \varphi_i(u) = u_i$ for $(u,i) \in G \times L_k$. Clearly, φ is a well defined bijection. We will show that φ and φ^{-1} are digraph homomorphisms.
Take $(u,i),(v,j) \in G \times L_k$ such that $((u,i),(v,j))$ is an arc in $C(G \times L_k, A)$ with $a' = (a,l) \in A$ such that $(v,j) = (u,i)(a,l)$. From the definition of the multiplication in $G \times L_k$ we get that $i = j$ and $ua = v$. But then $((u,l),(v,l))$ is an arc in $C(G, A_l)$. And as φ_l is a digraph isomorphism we have $(u_l, v_l) \in E_l$. Now the first condition implies that $(u_i, v_i) = (\varphi((u,i)), \varphi((v,i)) \in E_i \subseteq E$. This shows that φ is a digraph homomorphism.
Take $u_j \in V_j, v_i \in V_i$ with $(u_j, v_i) \in E$. Then $i = j$ by the second condition and thus $(u_i, v_i) \in E_i \subseteq E$. Since φ_i is a digraph isomorphism we get $(\varphi_i^{-1}(u_i), \varphi_i^{-1}(v_i))$ is an arc in $C(G, A_i)$. Consequently there exists $a \in A_i$ with $v = ua$. By definition of the multiplication in $G \times L_k$ we get $(v,i) = (u,i)(a,i)$. Thus $((u,i),(v,i)) = (\varphi^{-1}(u_i), \varphi^{-1}(v_i))$ is an arc in $C(G \times L_k, A)$. This shows that φ^{-1} is also a digraph homomorphism. □

Corollary 4.3 *A digraph (V,E) is an LZUG digraph if and only if there exist $k \in \mathbb{N}$, $k \geq 2$, such that (V,E) is the vertex disjoint union of k isomorphic strong G-group subgraphs $(V_1,E_1),\ldots,(V_k,E_k)$ for some group G, $V = \bigcup_{i=1}^{k} V_i$, and*

$$\forall i \in \{1,\ldots,k\}[(u_i,v_i) \in E_i \Leftrightarrow \exists l \in \{1,\ldots,k\}(u_l,v_l) \in E_l]$$

and

$$(u_i, v_j) \in E \Rightarrow i = j \text{ and } (u_i, v_i) \in E_i.$$

The following LZUG digraphs illustrate the result for LZUG digraphs isomorphic to $C(\mathbb{Z}_3 \times L_3, A)$.

$$C(\mathbb{Z}_3 \times L_3, A) \text{ with } A = \{(0,3)\}, A = \{(1,2)\}, A = \{(2,1)\}$$

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