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Strong Convergence Theorem of Bregman Algorithm for Solving Variational Inequalities in Banach Spaces

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Abstract Due to the significance of the variational inequality which related to solve various problems in other branches of sciences and engineering, in this paper, we introduce a new algorithm for finding solution of this problem by using Bregman method in real reflexive Banach spaces. Under some mild conditions, the convergence result is exactly proved. Our result improved and extended some previous results in the literature.

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1. Introduction

Throughout this paper, let E be a real reflexive Banach space with the topological dual E^* and the function $f : E \to (-\infty, \infty)$ be a proper, lower semicontinuous and convex with the effective domain $\mathrm{dom} f = \{x \in E : f(x) < \infty\}$. Given any $x \in \mathrm{int}(\mathrm{dom} f)$, the direction derivative of f at x in the direction of $y \in E$ is denoted by $f'(x, y)$, that is,

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1.1}$$

Recall the differentiable function that will be used in this paper, the function f is called *Gâteaux differentiable* at x if the limit as $t \to 0$ in (1.1) exists at any $y \in E$. The function f is said to be *Frèchet differentiable* at x if the limit (1.1) is attained uniformly in $\|y\| = 1$ and $x \in E$. It is easily claimed that every *Frèchet differentiable* function is *Gâteaux differentiable* and if f is *Frèchet differentiable*, then it is continuous but if f is *Gâteaux differentiable*, then it is not necessary that f is continuous [1].

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The *Variational Inequalities (VI)* problem of Fichera [2] and Stampacchia [3] is the problem which aim to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \ \forall y \in C, \tag{1.2}$$

where E is a real Banach space with its dual space E^*, the norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let C be a nonempty, closed and convex subset of E, $A : E \to E^*$ is a given operator. The solution set of VI is represented by *VI(C,A)*. This is a fundamental problem in optimization theory and captures various applications such as mathematical programming, partial differential equations, optimal control and so on.

Several surveys dedicated to solve various problems in other branches of sciences and engineering and their applications have appeared (see [4–11]).

Due to its significance in various fields, there were many researchers tried to make the method improvement for solving *VI*, the one classical method is the projection method given by the following iterative algorithm,

$$x_{n+1} = \Pi_C(x_n - \lambda_n A x_n). \tag{1.3}$$

In 1976, Korpelevich [12] proposed the *Extragradient Method (EM)* for solving *VI* with monotone and Lipshitz continuous mapping A in the finite dimension Euclidean space. *EM* is a two-projection process and is defined by

$$\begin{cases} x_0 \in C, \\ y_n = \Pi_C(x_n - \alpha_n A x_n), \\ x_{n+1} = \Pi_C(x_n - \alpha_n A y_n), \end{cases} \tag{1.4}$$

where $n \geq 1$ and $\alpha_n \in \left(0, \frac{1}{L}\right)$. The *EM* is one of the most well known method which attracted all researchers to improve and applied this method to the real-world problems. By the way, in the *EM* (1.4), the calculation of two projections onto the feasible set C is required which might not be easy and may affect the efficiency of the method.

Later, in 2000, Tseng [13] introduced $Tseng's$ *extragradient method* for solving monotone variational inequalities in a real Hilbert space.

$$\begin{cases} y_n = \Pi_C(x_n - \alpha A x_n), \\ x_{n+1} = y_n - \alpha(A y_n - A x_n), \end{cases} \tag{1.5}$$

where A is monotone and L-Lipschitz continuous from C into H and $\alpha \in \left(0, \frac{1}{L}\right)$. It was proved that this method converges weakly to a point in $VI(C, A)$. Note that this modified method to decomposition in convex programming and monotone variational inequalities.

In 2018, Yang et al. [14] presented the following modification of the subgradient extragradient method with adjustment step size for solving monotone variational inequalities.

$$\begin{cases} y_n = \Pi_C(x_n - \alpha_n A x_n), \\ x_{n+1} = \Pi_{T_n}(x_n - \alpha_n A y_n), \\ T_n = \{x \in H : \langle x_n - \alpha_n A x_n - y_n, x - y_n \rangle \leq 0\}, \end{cases} \tag{1.6}$$

where $\alpha_n > 0$, $\mu \in (0, 1)$ and α_n is adaptive updated as follows:

$$\alpha_{n+1} = \begin{cases} \min\left\{\mu \dfrac{\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle A x_n - A y_n, x_{n+1} - y_n \rangle}, \alpha_n\right\} & \text{if } \langle A x_n - A y_n, x_{n+1} - y_n \rangle > 0, \\ \alpha_n & \text{otherwise.} \end{cases}$$

The weak convergence of the algorithm was established without the knowledge of the Lipschitz constant of the mapping.

In this paper, motivated and inspired by Tseng [13] and Yang et al. [14], we introduce the following algorithm

$$\begin{cases} y_n = \Pi_C^f(\nabla f^{-1}(\nabla f(x_n) - \gamma_n A x_n)), \\ z_n = \nabla f^{-1}(\nabla f(y_n) - \gamma_n(A y_n - A x_n)), \\ x_{n+1} = \nabla f^{-1}(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)), \end{cases} \tag{1.7}$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\mu \dfrac{\|x_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A x_n - A y_n, z_n - y_n\rangle}, \gamma_n\right\} & \text{if } \langle A x_n - A y_n, z_n - y_n\rangle > 0, \\ \gamma_n & \text{otherwise.} \end{cases}$$

we prove that $\{x_n\}$ converges strongly to the point $\Pi_C^f x$ under some suitable condition imposed on the parameters.

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be use in the proof.

Let E be a reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E. The Legendre function $f : E \to (-\infty, \infty]$ is defined in Bauschke et al. (see [15]). The function f is *Legendre function* if and only if it satisfies the following two conditions:

(L1) $\text{int}(\text{dom}f) \neq \emptyset$ and f is *Gâteaux differentiable* with $\text{dom}\nabla f = \text{int}(\text{dom}f)$;

(L2) $\text{int}(\text{dom}f^*) \neq \emptyset$ and f^* is *Gâteaux differentiable* with $\text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$.

Since E is reflexive Banach space, we always obtain $(\partial f)^{-1} = \partial f^*$ (see [16, p. 83]). This, by (L1) and (L2), implies the following facts:
(i) ∇f is a bijection with $\nabla f = (\nabla f^*)^{-1}$ (see [17, Theorem 5.10]);
(ii) $\text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$ and $\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom}f)$ (see [18], p.123),
where $\text{ran}\nabla f$ denotes the range of ∇f.

Definition 2.1. ([19, 20]) Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle,$$

is call the Bregman distance with respect to f.
By the definition, we know the following two important properties: the two point identity, for any $x, y \in \text{int}(\text{dom}f)$

$$D_f(x, y) + D_f(y, x) = \langle x - y, \nabla f(x) - \nabla f(y)\rangle,$$

the three point identity [21] for any $x \in \text{dom}f$ and $y, z \in \text{int}(\text{dom}f)$,

$$D_f(x, y) = D_f(x, z) - D_f(z, y) + \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{2.1}$$

Definition 2.2. ([22, Proposition 2.1]) If $f : E \to (-\infty, +\infty]$ is uniformly *Frèchet* differentiable and bounded on subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Definition 2.3. (Bregman [19]) Let $f : E \to (-\infty, +\infty]$ be convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int}(\text{dom}f)$ onto a nonemtry closed convex set $C \subset \text{int}(\text{dom}f)$ is the unique vector $\Pi_C(x) \in C$ satisfying

$$D_f(\Pi_C^f(x), x) := \inf\{D_f(x, y) : x \in C\}.$$

it is know from [23] :

$$z = \Pi_C^f(x) \iff \langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \ \forall y \in C. \tag{2.2}$$

We also know the following equivalence

$$D_f(y, \Pi_C^f(x)) + D_f(\Pi_C^f(x), x) \leq D_f(y, x), \ \forall y \in C, x \in \text{int}(\text{dom}f). \tag{2.3}$$

A convex and differentiable function f is strongly convex if there exists a constant $\sigma > 0$ such that

$$f(x) > f(y) + \langle \nabla f(y), x - y\rangle + \frac{\sigma}{2}\|x - y\|^2, \ \forall x \in \text{dom}f, \text{ and } y \in \text{int}(\text{dom}f).$$

From the definition of Bregman distance, we have

$$D_f(x, y) \geq \frac{\sigma}{2}\|x - y\|^2. \tag{2.4}$$

Following [20, 24], the function $V_f : E \times E \longrightarrow [0, +\infty)$ associated with f defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \ \forall x \in E, x^* \in E^*.$$

V_f is non-negative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality, it is easy to see that

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \tag{2.5}$$

for all $x \in E$ and $x^*, y^* \in E^*$. In addition, if $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function, then $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is proper weak lower semicontinuous and convex function. Hence, V_f is convex in the second variable. Thus, for all $z \in E$

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $x_i \subset E$ and $t_i \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$.

Lemma 2.4. ([15]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_0, x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.5. ([15, Proposition 2.2]) *If $x \in \text{int}(\text{dom}f)$, then the following statement are equivalent:*

(i) The function f is totally convex at x,

(ii) For any sequence $\{x_n\} \subset \text{dom}f$,

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \| y_n - x \| = 0.$$

We say that the function f is sequentially consistent ([23, p.9]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\text{int}(\text{dom}f)$ and $\text{dom}f$, respectively such that the first one is bounded and

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{2.6}$$

Lemma 2.6. ([25, Lemma 2.1.2]) *The function $f : E \to (-\infty, +\infty]$ is sequentially consistent if and only if the function f is totally convex on bounded subsets of E.*

Moreover, if f is a Lagendre function, *Frèchet* differentiable and bounded on bounded subsets of E, then for any two sequences $\{x_n\}$ and $\{y_n\}$ in dom$f$ and int(dom$f$), we have

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \| y_n - x_n \| = 0 \Rightarrow \lim_{n\to\infty} \| \nabla f y_n - \nabla f x_n \| = 0.$$

Lemma 2.7. ([26]) *Assume that $\{s_n\}$ is a sequence of nonnegative real number such that*

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n b_n, \ \forall n \geq 1,$$

where $\{\beta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) $\limsup_{n\to\infty} b_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. ([25]) *The function f is totally convax on bounded subsets if and only if it is sequentially consistent.*

Lemma 2.9. ([27]) *Consider the VI(1.2). If the mapping $h : [0, 1] \to E^*$ defined as $h(t) = A(tx + (1 - t)y)$ is continuous for all $x, y \in C$ (i.e., h is hemicontinuous), then $M(C, A) \subset VI(C, A)$. Moreover, if A is pseudo-monotone, then $VI(C, A)$ is closed, convex and $VI(C, A) = M(C, A)$.*

3. Main Result

In this section, we discuss a strong convergence of Bregman projection algorithms for solving pseudo-monotone variational inequalities. Let C be a nonempty, closed and convex subset of a reflexive Banach space E. The function $f : E \to \mathbb{R} \cup \{+\infty\}$ is strongly coercive Legendre function which is bounded, uniformly *Frèchet* differentiable and totally convex on bounded subsets of E and its gradient ∇f is weak-weak continuous, $x_n \rightharpoonup x$ implies that $\nabla f(x_n) \rightharpoonup \nabla f(x)$. The mapping $A : E \to E^*$ is pseudo-monotone, i.e., for all $x, y \in E$, $\langle Ax, y - x\rangle \geq 0$ implies $\langle Ay, y - x\rangle \geq 0$ and Lipschitz continuous with a constant $L > 0$. The solution set of VIs is nonemty, that is, $VI(C, A) \neq \emptyset$. Now, we propose a new projection algorithm for solving VIs of psudo-monotone mappings.

Algorithm 1:

Given $\gamma_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.4). Let $x_1, u \in E$ be arbitrary. Set $n = 1$

Step 1. Compute

$$y_n = \Pi^f_C(\nabla f^{-1}(\nabla f(x_n) - \gamma_n Ax_n)).$$

If $x_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIs. Else, do **Step 2.**

Step 2. Compute

$$z_n = \nabla f^{-1}(\nabla f(y_n) - \gamma_n(Ay_n - Ax_n)),$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\mu\dfrac{\|x_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Ax_n - Ay_n, z_n - y_n\rangle}, \gamma_n\right\} & \text{if } \langle Ax_n - Ay_n, z_n - y_n\rangle > 0, \\ \gamma_n & \text{otherwise.} \end{cases}$$

Step 3. Compute

$$x_{n+1} = \nabla f^{-1}(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)).$$

Set $n := n+1$ and go to **Step 1.**

Theorem 3.1. *The sequence* $\{x_n\}$ *generated by* **Algorithm 1** *converges strongly to a point* $\Pi^f_{VIs}x$*, provided that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $z \in VI(C, A)$, then

$$\begin{aligned} D_f(z, y_n) &= D_f(z, \Pi^f_C(\nabla f^{-1}(\nabla f(x_n) - \gamma_n Ax_n))) \\ &\leq D_f(z, \nabla f^{-1}(\nabla f(x_n) - \gamma_n Ax_n)) \\ &= f(z) - f(y_n) - \langle \nabla f(x_n) - \gamma_n Ax_n, z - y_n\rangle \\ &= f(z) - f(y_n) - \langle \nabla f(x_n), z - y_n\rangle + \gamma_n\langle Ax_n, z - y_n\rangle \\ &= f(z) - f(x_n) - \langle \nabla f(x_n), z - x_n\rangle + \langle \nabla f(x_n), z - x_n\rangle \\ &\quad + f(x_n) - f(y_n) - \langle \nabla f(x_n), z - y_n\rangle + \gamma_n\langle Ax_n, z - y_n\rangle \\ &= f(z) - f(x_n) - \langle \nabla f(x_n), z - x_n\rangle - f(y_n) + f(x_n) \\ &\quad + \langle \nabla f(x_n), y_n - x_n\rangle + \gamma_n\langle Ax_n, z - y_n\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) + \gamma_n\langle z - y_n, Ax_n\rangle. \end{aligned} \tag{3.1}$$

By the definition of Bregman distance, we have

$$\begin{aligned} D_f(z, z_n) &= D_f(z, \nabla f^{-1}(\nabla f(y_n) - \gamma_n(Ay_n - Ax_n)) \\ &= f(z) - f(z_n) - \langle \nabla f(y_n) - \gamma_n(Ay_n - Ax_n), z - z_n\rangle \\ &= f(z) - f(z_n) - \langle \nabla f(y_n), z - z_n\rangle + \gamma_n\langle Ay_n - Ax_n, z - z_n\rangle \\ &= f(z) - f(y_n) - \langle \nabla f(y_n), z - y_n\rangle + \langle \nabla f(y_n), z - y_n\rangle \\ &\quad + f(y_n) - f(z_n) - \langle \nabla f(y_n), z - z_n\rangle + \gamma_n\langle Ay_n - Ax_n, z - z_n\rangle \\ &= f(z) - f(y_n) - \langle \nabla f(y_n), z - y_n\rangle - f(z_n) + f(y_n) \\ &\quad + \langle \nabla f(y_n), z_n - y_n\rangle + \gamma_n\langle Ay_n - Ax_n, z - z_n\rangle \\ &= D_f(z, y_n) - D_f(z_n, y_n) + \gamma_n\langle Ay_n - Ax_n, z - z_n\rangle. \end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), we get

$$\begin{aligned} D_f(z, z_n) &\leq D_f(z, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \gamma_n\langle Ax_n, z - y_n\rangle \\ &\quad + \gamma_n\langle Ay_n - Ax_n, z - z_n\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \gamma_n\langle Ax_n, z - y_n\rangle \\ &\quad + \gamma_n\langle Ay_n, z - z_n\rangle - \gamma_n\langle Ax_n, z - z_n\rangle \end{aligned}$$

$$\begin{aligned}
&= D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) + \gamma_n\langle Ax_n, z_n - y_n\rangle \\
&\quad +\gamma_n\langle Ay_n, z - z_n\rangle \\
&= D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) + \gamma_n\langle Ax_n, z_n - y_n\rangle \\
&\quad -\gamma_n\langle Ay_n, y_n - z\rangle + \gamma_n\langle Ay_n, y_n - z_n\rangle \\
&= D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) + \gamma_n\langle Ax_n - Ay_n, z_n - y_n\rangle \\
&\quad -\gamma_n\langle Ay_n, y_n - z\rangle.
\end{aligned}$$

Since A is pseudo-monotone and $z \in VI(C,A)$, we have

$$\langle Ay_n, y_n - z\rangle \geq 0.$$

By the definition of γ_{n+1}, we have

$$\begin{aligned}
D_f(z,z_n) &\leq D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) + \gamma_n\langle Ax_n - Ay_n, z_n - y_n\rangle \\
&= D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) \\
&\quad + \frac{\gamma_n}{\gamma_{n+1}}\gamma_{n+1}\langle Ax_n - Ay_n, z_n - y_n\rangle \\
&\leq D_f(z,x_n) - D_f(y_n,x_n) - D_f(z_n,y_n) \\
&\quad + \frac{\mu}{2}\frac{\gamma_n}{\gamma_{n+1}}(\|x_n - y_n\|^2 + \|z_n - y_n\|^2) \\
&= D_f(z,x_n) - D_f(y_n,x_n) + \frac{\mu}{2}\frac{\gamma_n}{\gamma_{n+1}}\|x_n - y_n\|^2 - D_f(z_n,y_n) \\
&\quad + \frac{\mu}{2}\frac{\gamma_n}{\gamma_{n+1}}\|z_n - y_n\|^2.
\end{aligned}$$

Using (2.4), we have

$$\begin{aligned}
D_f(z,z_n) &\leq D_f(z,x_n) - \left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right) D_f(y_n,x_n) \\
&\quad - \left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right) D_f(z_n,y_n).
\end{aligned} \tag{3.3}$$

By **Algorithm 1**, we note that

$$\begin{aligned}
D_f(z,x_{n+1}) &\leq D_f(z, \nabla f^{-1}(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\
&\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,z_n) \\
&\leq \max\{D_f(z,u), D_f(z,x_n)\} \\
&\;\;\vdots \\
&\leq \max\{D_f(z,u), D_f(z,x_1)\}.
\end{aligned} \tag{3.4}$$

Hence, $\{D_f(z,x_n)\}$ is bounded. Using [28] we obtain that $\{x_n\}$ is also bounded. Consequently, we see that $\{\nabla f(x_n)\}, \{z_n\}, \{y_n\}$ are bounded.

From (2.5), we obtain

$$\begin{aligned} D_f(z, x_{n+1}) &= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\ &\leq V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n) - \alpha_n(\nabla f(u) - \nabla f(z)) \\ &\quad + \langle \alpha_n(\nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n) + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq \alpha_n V_f(z, \nabla f(z)) + (1-\alpha_n) V_f(z, \nabla f(z_n) \\ &\quad + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &= (1-\alpha_n) D_f(z, z_n) + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq (1-\alpha_n) D_f(z, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle. \end{aligned} \tag{3.5}$$

Afterward, we show that the sequence $\{x_n\}$ generated by **Algorithm 1** converges strongly to an element in $VI(C, A)$.

Case I: Let $z \in VI(C, A)$. Suppose that exists $n_0 \in \mathbb{N}$ such that $\{D_f(z, f(x_n))\}$ is monotonically non-increasing for $n \geq n_0$. Since $\{D_f(z, f(x_n))\}$ is bounded, $\{D_f(z, f(x_n))\}$ converges and therefore

$$D_f(z, f(x_n)) - D_f(z, f(x_{n+1})) \to 0 \text{ as } n \to \infty.$$

We consider the following inequality

$$\begin{aligned} D_f(z, x_{n+1}) &\leq D_f(z, \nabla f^{-1}(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\ &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) \\ &\leq \alpha_n D_f(z, u) + (1-\alpha_n)[D_f(z, x_n) - \left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right) D_f(y_n, x_n) \\ &\quad - \left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right) D_f(z_n, y_n)]. \end{aligned}$$

This implied that

$$\begin{aligned} (1-\alpha_n)\left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right)[D_f(y_n, x_n) + D_f(z_n, y_n)] &\leq \alpha_n[D_f(z, u) - D_f(z, x_n)] \\ &\quad + D_f(z, x_n) - D_f(z, x_{n+1}). \end{aligned}$$

From (3.5) and $\alpha_n \to 0$, we get

$$D_f(z, x_{n+1}) \leq D_f(z, x_n).$$

This implies that

$$\lim_{n\to\infty}(D_f(z, x_{n+1}) - D_f(z, x_n)) = 0. \tag{3.6}$$

Moreover, we get

$$\lim_{n\to\infty}\left(1 - \frac{\mu}{\sigma}\frac{\gamma_n}{\gamma_{n+1}}\right)[D_f(y_n, x_n) + D_f(z_n, y_n)] = 0.$$

Hence

$$\lim_{n\to\infty} D_f(y_n, x_n) = \lim_{n\to\infty} D_f(z_n, y_n) = 0.$$

We obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.7}$$

We consider the following inequality

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\|.$$

From (3.7), we get

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Furthermore,

$$\begin{aligned} D_f(z_n, x_{n+1}) &\leq D_f(z_n, \nabla f^{-1}(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\ &\leq \alpha_n D_f(z_n, u) + (1-\alpha_n) D_f(z_n, z_n) \\ &\to 0,\ n\to\infty. \end{aligned}$$

Therefore

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0.$$

Consider

$$\|x_n - x_{n+1}\| \leq \|x_n - z_n\| + \|z_n - x_{n+1}\|.$$

We obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0.$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{z}$. We now show that $\tilde{z} \in VI(C, A)$. From

$$y_{n_k} = \Pi_C(\nabla f^{-1}(\nabla f(x_{n_k}) - \gamma_{n_k} A x_{n_k})),$$

it follows from (2.2) that

$$\langle \nabla f(x_{n_k}) - \gamma_{n_k} A x_{n_k} - \nabla f(y_{n_k}), x - y_{n_k}\rangle \leq 0,\ \forall x \in C, \tag{3.8}$$

which implies

$$\langle \nabla f(x_{n_k}) - \nabla f(y_{n_k}), x - y_{n_k}\rangle \leq \gamma_{n_k}\langle A x_{n_k}, x - y_{n_k}\rangle,\ \forall x \in C,$$

or equivalently

$$\left\langle \frac{\nabla f(x_{n_k}) - \nabla f(y_{n_k})}{\gamma_{n_k}}, x - y_{n_k}\right\rangle + \langle A x_{n_k}, y_{n_k} - x_{n_k}\rangle \leq \langle A x_{n_k}, x - x_{n_k}\rangle,\quad \forall x \in C. \tag{3.9}$$

Since f is uniformly $Frèchet$ differentiable,∇f is uniformly continuous on bounded subsets of E and so $\|\nabla f(x_{n_k}) - \nabla f(y_{n_k})\| \to 0$ as $k \to \infty$. Form (3.9) with the fact that $\lim_{k\to\infty \gamma_{n_k}} = \gamma > 0$ and $\{A x_{n_k}\}$ is bounded, we can show that

$$\liminf_{k\to\infty}\langle A x_{n_k}, x - x_{n_k}\rangle \geq 0,\ \forall x \in C. \tag{3.10}$$

Let $\{\epsilon_k\}$ be a sequence in $(0,1)$ such that $\{\epsilon_k\}$ as $k\to\infty$. For any $k \geq 1$, there exists a smallest number $N \in \mathbb{N}$ satisfying

$$\langle A x_{n_k}, x - x_{n_k}\rangle + \epsilon_k \geq 0,\ \forall k \in N.$$

This implies that

$$\langle Ax_{n_k}, x + \epsilon_k y_{n_k} - x_{n_k}\rangle \geq 0, \ \forall k \in N,$$

for some $w_{n_k} \in E$ satisfying $\langle Ax_{n_k}, w_{n_k}\rangle = 1$ (since $Ax_{n_k} \neq 0$). Since A is pseudo-monotone, we obtain

$$\langle A(x + \epsilon_k w_{n_k}), x + \epsilon_k w_{n_k} - x_{n_k}\rangle \geq 0, \ \forall k \in N.$$

Thus

$$\langle Ax, x - x_{n_k}\rangle \geq \langle Ax - A(x + \epsilon_k w_{n_k}), x + \epsilon_k w_{n_k} - x_{n_k}\rangle - \epsilon_k\langle Ax, w_{n_k}\rangle, \ \forall k \in N. \quad (3.11)$$

Since $\epsilon_k \to 0$ and A is continuous. Thus, we have

$$\liminf_{k\to\infty}\langle Ax, x - x_{n_k}\rangle \geq 0, \ \forall x \in C.$$

Hence

$$\langle Ax, x - \tilde{z}\rangle = \lim_{k\to\infty}\langle Ax, x - x_{n_k}\rangle \geq 0, \ \forall x \in C.$$

From Lemma 2.9, we obtain $\tilde{z} \in VI(C, A)$. Next, we show that $\{x_n\}$ convergence strongly to z.
Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n_k+1} - z\rangle = \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n_k+1} - z\rangle. \quad (3.12)$$

On the other hand, since $\| x_{n_k+1} - x_{n_k} \| \to 0$ and $x_{n_k} \to \tilde{z}$ as $k \to \infty$ we have from (2.5) and (3.12)

$$\begin{aligned}\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n_k+1} - z\rangle &= \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n_k+1} - z\rangle \\ &= \langle \nabla f(u) - \nabla f(z), \tilde{z} - z\rangle \\ &\leq 0. \quad (3.13)\end{aligned}$$

By Lemma 2.7 and (3.13), we can conclude that $\lim_{n\to\infty} D_f(z, x_n) = 0$. Therefore, by Lemma 2.5, x_n convergs strongly to z. The proof is completed.

Case II: Suppose that $D_f(z, \nabla f(x_n))$ is not monotonically decreasing. Let $\varphi : \mathbb{N} \to \mathbb{N}$ for all $n \geq n_0$ be defined by

$$\varphi_n = \max\{k \in \mathbb{N} : \varphi_k \leq \varphi_{k+1}\}.$$

Obviously, φ is nondecreasing, $\varphi(n) \to \infty$ as $n \to \infty$ and

$$0 \leq D_f(z, x_{\varphi(n)}) \leq D_f(z, x_{\varphi(n)+1}), \ \forall n \geq n_0.$$

Following a similar argument to *Case I*, we get

$$\| x_{\varphi(n)} - y_{\varphi(n)} \| \to 0, \ \| x_{\varphi(n)+1} - x_{\varphi(n)} \| \to 0,$$

as $n \to \infty$ and $\Omega_w(x_{\varphi(n)}) \subset VI(C, A)$, where $\Omega_w(x_{\varphi(n)})$ is the weak subsequential limit of $\{x_{\varphi(n)}\}$. We can show that

$$\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_{\varphi(n)+1} - z\rangle \leq 0.$$

From (3.8), we have

$$D_f(z, x_{\varphi(n)+1}) \leq (1 - \gamma_{\varphi(n)})D_f(z, x_{\varphi(n)}) + \gamma_{\varphi(n)}\langle \nabla f(u) - \nabla f(z)), x_{\varphi(n)+1} - z\rangle.$$

Since $D_f(z, x_{\varphi(n)}) \leq D_f(z, x_{\varphi(n)+1})$, we get

$$\begin{aligned} 0 &\leq D_f(z, x_{\varphi(n)+1}) - D_f(z, x_{\varphi(n)}) \\ &\leq (1-\gamma_{\varphi(n)})D_f(z, x_{\varphi(n)}) + \gamma_{\varphi(n)}\langle \nabla f(u) - \nabla f(z), x_{\varphi(n)+1} - z\rangle - D_f(z, x_{\varphi(n)}), \end{aligned}$$

therefore, from (3.13), we have

$$D_f(z, x_{\varphi(n)}) \leq \langle \nabla f(u) - \nabla f(z), x_{\varphi(n)+1} - z\rangle \to 0 \text{ as } n \to \infty.$$

Consequencely, we obtain, for all $n \geq n_0$,

$$0 \leq D_f(z, x_{\varphi(n)}) \leq \max\{D_f(z, x_{\varphi(n)}), D_f(z, x_{\varphi(n)+1})\} = D_f(z, x_{\varphi(n)+1}).$$

Thus

$$D_f(z, x_n) \to 0 \text{ as } n \to \infty.$$

Hence, from (2.6)

$$\lim_{n\to\infty} \| x_n - z \| = 0.$$

We concluded that $\{x_n\}$ converge strongly to z. This completes the proof. ∎

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