



# Fixed Point and Coincidence Point Theorems on Banach Spaces over Topological Semifields and Their Applications

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**Abstract :** We establish some common fixed point and coincident point theorems for a quadruple of self-mappings on a Banach space  $X$  over a topological semifield. Our first result extends the main result of Pathak et al. [8] to a general class of mappings in which we have dropped the requirement of pairwise commutativity of mappings by imposing certain restrictions on parameters. Our second result deals with the existence of coincidence points for a quadruple of self-mappings under different conditions than in Theorem 1 of [8]. We also discuss an application of our main result to solve certain non-linear function equations in Banach space over a topological semifield..

**Keywords :** Banach space; Common fixed point; Coincidence point; Topological semifield.

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## 1 Introduction

In 1960, M. Antonovskii, V. Boltyanskii and T. Sarymsakow [1] introduced the notion of a topological semifield.

**Definition 1.1** If a semifield  $R$ ; i.e., a division ring  $R$  that has the structure of a topological group such that  $(x, y) \rightarrow x + y$  (sum) and  $(x, y) \rightarrow x.y$  (product) are both continuous mappings of  $R \times R \rightarrow R$ , then  $R$  is called a *topological ring*. If a topological ring  $F$  is a field (not necessarily commutative) such that  $x \rightarrow x^{-1}$  (inverse element) is a continuous mapping of  $F^* = F - \{0\}$  into  $F^*$  then  $F$  is called a *topological semifield*.

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Let  $E$  be a topological semifield and let all its positive elements be in  $K$ . For any two elements  $x$  and  $y$  of  $E$ , we put  $x \ll y$  if  $y - x$  is in  $\bar{K}$ . Every topological semifield  $E$  contains a sub semifield which is isomorphic to the field of real numbers  $\mathbb{R}$ , (see [1]). This subsemifield is called the *axis* of  $E$ . Each topological semifield can be regarded as a topological linear space over the field  $\mathbb{R}$  by identifying the axis and  $\mathbb{R}$ .

Let  $d$  be a mapping of  $X \times X \rightarrow E$ , satisfying the usual axioms for a metric. The ordered triple  $(X, d, E)$  is called the *metric space over the topological semifield*, see [1], [2], [5] and [6].

Let  $X$  be a linear space over the field  $\mathbb{R}$ . If there exists a mapping  $\|\cdot\| : X \rightarrow E$  satisfying the usual axioms for a norm, then the ordered triple  $(X, \|\cdot\|, E)$  is called a *feeble normed space over the topological semifield*, see [1], [4] and [6].

## 2 Main Results

Now we use the following definitions:

**Definition 2.1** A subset  $S$  of a linear topological space  $E$  is said to be *sequentially complete* if and only if each Cauchy sequence in  $S$  converges to a point in  $S$ .

**Definition 2.2** Let  $(X, \|\cdot\|, E)$  be a feeble normed space over a topological semifield  $E$  and let  $d(x, y) = \|x - y\|$  for all  $x, y$  in  $X$ . A space  $(X, \|\cdot\|, E)$  is called a *Banach space over a topological semifield  $E$*  if  $(X, d, E)$  is a sequentially complete metric space over the topological semifield  $E$ .

**Definition 2.3** A point  $u$  of  $X$  is said to be *coincidence point* of a pair of mappings  $(A, S)$  if there exists  $t \in X$  such that  $St = Tu$ .

**Definition 2.4** [3] Two self mappings  $A$  and  $S$  of  $X$  is said to be *weakly compatible* if there exist a point  $u \in X$  such that  $ASu = SAu$  whenever  $Au = Su$ .

Pathak et.al.[8] gave the following result.

**Theorem A.**[8] *Let  $X$  be a Banach space over a topological semifield  $E$ . Let  $A, B, S$  and  $T$  be four continuous self mappings of  $X$  which commute with each other and satisfy the following conditions:*

$$p\|Sx - Ty\|^m + \|Sx - Ax\|^m \ll q\|Ty - By\|^m,$$

$$p\|Sx - Ty\|^m + \|Ty - By\|^m \ll q\|Sx - Ax\|^m$$

for all  $x, y \in X$ , where  $p, m > 0$ , and  $0 < q < 1$ . Then the sequence  $\{y_n\}$  defined recursively by:

$$y_{2n+1} = Sx_{2n+1} = (1-t)Tx_{2n} + tBx_{2n},$$

$$y_{2n+2} = Tx_{2n+2} = (1-t)Sx_{2n+1} + tAx_{2n+1},$$

where  $x_0$  is a point in  $X$ ,  $0 < t < 1$  and  $0 \leq q - pt^m < 1$ , converges to the unique common fixed point of  $A, B, S$  and  $T$  in  $X$ .

We note here that Theorem A should have contained the extra condition that the range of  $S$  contained the range of  $(1 - t)T + tB$  and the range of  $T$  contained the range of  $(1 - t)S + tA$ .

We now prove our first theorem in which we have dropped the condition of “pairwise commutativity of mappings” of Theorem A by restricting the parameters  $p$  and  $q$  as follows:

**Theorem 2.5** *Let  $X$  be a Banach space over a topological semifield  $E$ . Let  $A, B, S$  and  $T$  be four continuous self mappings of  $X$ , satisfying the following conditions:*

$$p\|Sx - Ty\|^m + \|Sx - Ax\|^m \ll q\|Ty - By\|^m, \tag{2.1}$$

$$p\|Sx - Ty\|^m + \|Ty - By\|^m \ll q\|Sx - Ax\|^m \tag{2.2}$$

for all  $x, y \in X$ , where  $p, m > 0$ , and  $q > p$ . Suppose that the range of  $S$  contains the range of  $(1 - t)T + tB$  and the range of  $T$  contains the range of  $(1 - t)S + tA$ , where  $0 < t < 1$  and  $0 < q - p < q - pt^m < q < 1$ . Then  $A, B, S$  and  $T$  have a unique common fixed point.

**Proof.** Let  $x_0$  be an arbitrary point in  $X$  and put  $y_1 = (1 - t)Tx_0 + tBx_0$ . Since the the range of  $S$  contains the range of  $(1 - t)T + tB$ , there exists a point  $x_1$  in  $X$  such that  $Sx_1 = y_1$ . Now put  $y_2 = (1 - t)Sx_0 + tAx_1$ . Since the the range of  $T$  contains the range of  $(1 - t)S + tA$ , there exists a point  $x_2$  in  $X$  such that  $Tx_2 = y_2$ . More generally, having chosen the point  $x_{2n}$  in  $X$ , we choose a point  $x_{2n+1}$ , such that

$$Sx_{2n+1} = y_{2n+1} = (1 - t)Tx_{2n} + tBx_{2n} \tag{2.3}$$

and then choose a point  $x_{2n+2}$ , such that

$$Tx_{2n+2} = y_{2n+2} = (1 - t)Sx_{2n+1} + tAx_{2n+1} \tag{2.4}$$

for  $n = 1, 2, \dots$

From (2.3) and (2.4), we obtain

$$\|Sx_{2n+1} - Tx_{2n}\| = t\|Bx_{2n} - Tx_{2n}\|, \tag{2.5}$$

$$\|Tx_{2n+2} - Sx_{2n+1}\| = t\|Ax_{2n+1} - Sx_{2n+1}\|. \tag{2.6}$$

If we put  $x = x_{2n+1}$  and  $y = x_{2n}$  in (2.1), we have from (2.5) and (2.6)

$$p\|Sx_{2n+1} - Tx_{2n}\|^m + \|Sx_{2n+1} - Ax_{2n+1}\|^m \ll q\|Tx_{2n} - Bx_{2n}\|^m.$$

Then from (2.5) and (2.6), we have

$$\begin{aligned} p\|Sx_{2n+1} - Tx_{2n}\|^m + t^{-m}\|Tx_{2n+2} - Sx_{2n+1}\|^m \\ \ll qt^{-m}\|Sx_{2n+1} - Tx_{2n}\|^m \end{aligned}$$

and then

$$\|Tx_{2n+2} - Sx_{2n+1}\|^m \ll (q - pt^m)\|Sx_{2n+1} - Tx_{2n}\|^m. \quad (2.7)$$

Similarly putting  $x = x_{2n+1}$  and  $y = x_{2n+2}$  in (2.2) and using (2.5) and (2.6), we get

$$\begin{aligned} p\|Sx_{2n+1} - Tx_{2n+2}\|^m + \|Tx_{2n+2} - Bx_{2n+2}\|^m \\ \ll q\|Sx_{2n+1} - Ax_{2n+1}\|^m, \\ p\|Sx_{2n+1} - Tx_{2n+2}\|^m + t^{-m}\|Sx_{2n+3} - Tx_{2n+2}\|^m \\ \ll qt^{-m}\|Tx_{2n+2} - Sx_{2n+1}\|^m \\ \|Sx_{2n+3} - Tx_{2n+2}\|^m \ll (q - pt^m)\|Tx_{2n+2} - Sx_{2n+1}\|^m. \end{aligned} \quad (2.8)$$

From relations (2.7) and (2.8), we have

$$\|Sx_{2n+3} - Tx_{2n+2}\|^m \ll (q - pt^m)^2\|Sx_{2n+1} - Tx_{2n}\|^m.$$

Hence

$$\|Sx_{2n+3} - Tx_{2n+2}\| \ll [(q - pt^m)^2]^{1/m}\|Sx_{2n+1} - Tx_{2n}\| \quad (2.9)$$

which implies that

$$\|Sx_{2n+1} - Tx_{2n}\| \ll [(q - pt^m)^2]^{n/m}\|Sx_1 - Tx_0\|$$

and since  $0 < q - pt^m < 1$ , it follows that  $\{y_n\}$  is a Cauchy sequence.

Since  $X$  is complete, it then follows that the sequence  $\{y_n\}$  converges to a point  $u$  in  $X$ . Using (2.3) and (2.4), we see that  $\{Sx_{2n+1}\}$  and  $\{Tx_{2n}\}$  also converge to  $u$ . Further, in view of (2.5) and (2.6),  $\{Bx_{2n}\}$  and  $\{Ax_{2n+1}\}$  also converges to  $u$ .

Putting  $x = Ax_{2n+1}$  and  $y = Bx_{2n}$  in (2.1) and (2.2), we have

$$\begin{aligned} p\|SAx_{2n+1} - TBx_{2n}\|^m + \|SAx_{2n+1} - AAx_{2n+1}\|^m \\ \ll q\|TBx_{2n} - BBx_{2n}\|^m \end{aligned} \quad (2.10)$$

$$\begin{aligned} p\|SAx_{2n+1} - TBx_{2n}\|^m + \|TBx_{2n} - BBx_{2n}\|^m \\ \ll q\|SAx_{2n+1} - AAx_{2n+1}\|^m. \end{aligned} \quad (2.11)$$

Letting  $n \rightarrow \infty$  in (2.10) and (2.11) we have,

$$p\|Su - Tu\|^m + \|Su - Au\|^m \ll q\|Tu - Bu\|^m, \quad (2.12)$$

$$p\|Su - Tu\|^m + \|Tu - Bu\|^m \ll q\|Su - Au\|^m, \quad (2.13)$$

since  $A, B, S$  and  $T$  are continuous mappings and  $\|\cdot\|$  is a continuous function.

With the help of (2.12) and (2.13) we will now show that  $u$  is a coincidence point of the pairs  $(A, S)$  and of  $(B, T)$ .

Suppose, if possible, that  $Au \neq Su$ . Since  $p, q > 0$ , we have from (2.1), (2.2), (2.12) and (2.13) that

$$\begin{aligned} \|Su - Au\|^m &<< \|Su - Au\|^m + p\|Su - Tu\|^m \\ &<< q\|Tu - Bu\|^m \\ &<< q\|Tu - Bu\|^m + qp\|Su - Tu\|^m, \\ &<< q^2\|Su - Au\|^m, \end{aligned}$$

which is a contradiction. Thus we have

$$Su = Au. \tag{2.14}$$

We now show that  $Bu = Tu$ . If not, then again using (2.1), (2.2), (2.12) and (2.13) we have

$$\begin{aligned} \|Tu - Bu\|^m &<< \|Tu - Bu\|^m + p\|Su - Tu\|^m \\ &<< q\|Su - Au\|^m \\ &<< q\|Su - Au\|^m + qp\|Su - Tu\|^m \\ &<< q^2\|Tu - Bu\|^m, \end{aligned}$$

which is a contradiction. Thus we have

$$Tu = Bu. \tag{2.15}$$

Using (2.14) and (2.15) in (2.12) and (2.13) we have

$$p\|Su - Tu\|^m << 0$$

which gives

$$Su = Tu. \tag{2.16}$$

From (2.14), (2.15) and (2.16), we see that  $u$  is a coincidence point of  $A$ ,  $B$   $S$  and  $T$  and we put

$$Au = Su = Tu = Bu = w. \tag{2.17}$$

*Claim I:* We claim that  $w$  is a common fixed point of  $B$  and  $T$ . To show this, put  $x = u$  and  $y = w$  in conditions (2.1) and (2.2). We then have

$$\begin{aligned} p\|Su - Tw\|^m + \|Su - Au\|^m &<< q\|Tw - Bw\|^m, \\ p\|Su - Tw\|^m + \|Tw - Bw\|^m &<< q\|Su - Au\|^m. \end{aligned}$$

Using (2.17), it follows that

$$p\|w - Tw\|^m << q\|Tw - Bw\|^m, \tag{2.18}$$

$$p\|w - Tw\|^m + \|Tw - Bw\|^m << 0 \tag{2.19}$$

so that from (2.18) we have

$$p\|w - Tw\|^m \ll q\|Tw - Bw\|^m \ll \|Tw - Bw\|^m,$$

as  $q < 1$ , and from (2.19) we have

$$\|Tw - Bw\|^m \ll \|Tw - Bw\|^m + p\|w - Tw\|^m \ll 0$$

as  $p > 0$ .

Thus  $Tw = Bw$  and putting  $Tw = Bw$  in (2.18) we have  $Tw = w$ . Therefore  $w$  is a common fixed point of  $B$  and  $T$ .

*Claim II:* We now claim that  $w$  is a common fixed point of  $A$  and  $S$ . To show this, put  $x = w$  and  $y = u$  in conditions (2.1) and (2.2). We then have

$$p\|Sw - Tu\|^m + \|Sw - Aw\|^m \ll q\|Tu - Bu\|^m \ll 0$$

on using (2.17). It follows that

$$Sw = Aw = Tu = w.$$

Therefore  $w$  is a common fixed point of  $A$  and  $S$ .

From claims I and II we see that  $w$  is a common fixed point of  $A, B, S$  and  $T$ .

We now prove the uniqueness of  $w$ . If  $w'$  is a second common fixed point of  $A, B, S$  and  $T$ , then on putting  $x = w$  and  $y = w'$  in (2.1) we have

$$p\|Sw - Tw'\|^m + \|Sw - Aw\|^m \ll q\|Tw' - Bw'\|^m$$

that is

$$p\|w - w'\|^m + \|w - w\|^m \ll q\|w' - w'\|^m$$

or,

$$p\|w - w'\|^m \ll 0.$$

Hence  $w = w'$ , proving the uniqueness of common fixed point of  $A, B, S$  and  $T$ . This completes the proof of the theorem.  $\square$

**Remark 2.6** (i) When  $A, B, S$  and  $T$  all commute with each other and  $p, m > 0$ ;  $0 < t < 1$ ;  $0 < q < 1$  and  $0 < q - pt^m < 1$  then our Theorem 2.1 reduces to Theorem 2.1 of Pathak et. al. [8].

(ii) When  $S = T$  and all the mappings  $A, B$  and  $S$  commute with each other, then our Theorem 2.1 reduces to Theorem 1 of Pathak et. al. [7].

**Theorem 2.7** Let  $X$  be a Banach space over a topological semifield  $E$  and let  $K$  be a set of positive elements of  $X$ . Let  $A, B, S$  and  $T$  be four continuous self mappings of  $X$ , satisfying the following relations

$$\|Ax - Sx\| \ll p\|Sx - Ty\|, \quad (2.20)$$

$$\|By - Ty\| \ll p\|Ax - Sx\| \quad (2.21)$$

for all  $x, y \in X$ , where  $0 < p < 1$ . Suppose that the range of  $S$  contains the range of  $(1 - t)T + tB$  and the range of  $T$  contains the range of  $(1 - t)S + tA$ , where  $0 < t < 1$ . Then the sequence  $\{y_n\}$  defined recursively by

$$\begin{aligned} y_{2n+1} &= Sx_{2n+1} = (1 - t)Tx_{2n} + tBx_{2n}, \\ y_{2n+2} &= Tx_{2n+2} = (1 - t)Sx_{2n+1} + tAx_{2n+1}, \end{aligned}$$

where  $x_0$  is a point in  $X$ , converges to a point  $u \in X$ .

Further, if the range of  $T$  contains the range of  $S$  and if  $(A, S)$  is a weakly compatible pair of mappings, then  $Au = w$  is a coincidence point of  $A, B, S$  and  $T$ .

**Proof.** From (2.3) and (2.4), we obtain

$$\begin{aligned} \|Sx_{2n+1} - Tx_{2n}\| &= t\|Bx_{2n} - Tx_{2n}\|, \\ \|Tx_{2n+2} - Sx_{2n+1}\| &= t\|Ax_{2n+1} - Sx_{2n+1}\|. \end{aligned}$$

Putting  $x = x_{2n+1}$  and  $y = x_{2n}$  in (2.20), we have

$$\|Ax_{2n+1} - Sx_{2n+1}\| \ll p\|Sx_{2n+1} - Tx_{2n}\|$$

and using (2.6), it follows that

$$\|Sx_{2n+1} - Tx_{2n+2}\| \ll pt\|Sx_{2n+1} - Tx_{2n}\|. \tag{2.22}$$

Similarly, putting  $x = x_{2n+1}$  and  $y = x_{2n+2}$  in (2.21) we have

$$\|Bx_{2n+2} - Tx_{2n+2}\| \ll p\|Ax_{2n+1} - Sx_{2n+1}\|$$

and using (2.5), we get

$$\|Sx_{2n+3} - Tx_{2n+2}\| \ll pt\|Sx_{2n+1} - Tx_{2n+2}\|. \tag{2.23}$$

From (2.22) and (2.23) we have

$$\|Sx_{2n+3} - Tx_{2n+2}\| \ll p^2t\|Sx_{2n+1} - Tx_{2n}\|$$

for  $n = 0, 1, 2, \dots$  and it follows that

$$\|Sx_{2n+1} - Tx_{2n}\| \ll (p^2t)^n\|Sx_1 - Tx_0\|.$$

Since  $0 < p^2t < 1$ , it follows that  $\{y_n\}$  is a Cauchy sequence.

Now, since  $X$  is complete, it follows that the sequence  $\{y_n\}$  converges to a point  $u$  in  $X$ . Using (2.3) and (2.4), we see that  $\{Sx_{2n+1}\}$  and  $\{Tx_{2n}\}$  also converge to  $u$ . Further, in view of (2.5) and (2.6),  $\{Bx_{2n}\}$  and  $\{Ax_{2n+1}\}$  also converge to  $u$ .

Using the continuity of  $A, B, S$  and  $T$ , we now see that

$$\begin{aligned} \lim_{n \rightarrow \infty} SAx_{2n+1} &= Su, & \lim_{n \rightarrow \infty} ASx_{2n+1} &= Au, \\ \lim_{n \rightarrow \infty} BTx_{2n} &= Bu, & \lim_{n \rightarrow \infty} TBx_{2n} &= Tu. \end{aligned}$$

Since the range of  $T$  contains the range of  $S$ , there exists  $v \in X$  such that  $Su = Tv = w$ , say. Putting  $x = u$  and  $y = v$  conditions (2.20) and (2.21) yield

$$\|Au - Su\| \ll p\|Su - Tv\|,$$

$$\|Bv - Tv\| \ll p\|Au - Su\|$$

that is,

$$Su = Tv = w = Au = Bv. \quad (2.24)$$

This shows that  $u$  is a coincidence point of  $(A, S)$  and  $v$  is coincidence point of  $(B, T)$ .

We now show that  $w$  is a coincidence point of both pairs  $(A, S)$  and  $(B, T)$ . Weak compatibility of  $(A, S)$  implies that  $ASu = SAu$  since  $Au = Su = w$ . That is,

$$Aw = Sw. \quad (2.25)$$

Putting  $x = y = w$ , the conditions (2.20) and (2.21) yield

$$\|Aw - Sw\| \ll p\|Sw - Tw\|,$$

$$\|Bw - Tw\| \ll q\|Aw - Sw\|.$$

Thus we have

$$Bw = Tw \quad (2.26)$$

and we conclude that

$$Aw = Sw = Bw = Tw.$$

Therefore,  $w$  is a coincidence point of  $A, B, S$  and  $T$ . This completes the proof of the theorem.  $\square$

**Remark 2.8** In Theorem 2.7, weak compatibility of the pair of mappings  $(A, S)$  and  $(B, T)$  do not necessarily imply  $Aw = Sw$  and  $Bw = Tw$ , respectively; i.e.,  $w$  is not necessarily a coincidence point of  $A, B, S$  and  $T$ .

### 3 An Application

Now we prove an application of Theorem 2.5 for certain non-linear function equations in Banach space  $X$  over a topological semifield  $E$  as follows:

**Theorem 3.1** *Let  $X$  be a Banach space over a topological semifield  $E$  and let  $A, B, S$  and  $T$  be four continuous self mappings on  $X$  satisfying conditions (2.1)*



and (2.2) of Theorem 2.5. Let  $\{f_n\}$  and  $\{g_n\}$  be sequences of elements in  $X$  and let  $w_n$  be the unique solution of the system of equations

$$u - Su = f_n, \tag{3.1}$$

$$u - Tu = g_n, \tag{3.2}$$

and let  $\{w_n\}$  be a sequence of solutions of the system of equations

$$Su - Au = 0, \quad Tu - Bu = 0 \tag{3.3}$$

satisfying

$$\|w_i - w_j\| \ll \|w_i - Sw_i\| + \|Sw_i - Tw_j\| + \|Tw_j - w_j\| \tag{3.4}$$

for  $i, j = 1, 2, \dots$ . If

$$\lim_{n \rightarrow \infty} \|f_n\| = \lim_{n \rightarrow \infty} \|g_n\| = 0,$$

then the sequence  $\{w_n\}$  converges to the solution of the equations

$$u = Su = Au = Bu = Tu. \tag{3.5}$$

**Proof.** By hypothesis, for all  $n$ , we have

$$\|Sw_n - Aw_n\| = 0, \quad \|Tw_n - Bw_n\| = 0.$$

Suppose that

$$\|w_n - Sw_n\| \neq 0, \quad \|w_n - Tw_n\| \neq 0,$$

then by (2.2) and (3.4), we have for  $r > n$ ,

$$\begin{aligned} \|w_n - w_r\| &<< \|w_n - Sw_n\| + \|Sw_n - Tw_r\| + \|Tw_r - w_r\| \\ &<< \|f_n\| + p^{-\frac{1}{m}} \{q\|Sw_n - Aw_n\|^m - \|Tw_r - Bw_r\|^m\}^{\frac{1}{m}} + \|g_r\|, \end{aligned}$$

or,

$$\|w_n - w_r\| \ll \|f_n\| + \|g_r\|. \tag{3.6}$$

Similarly, by (2.1) and (3.4) we have for  $r > n$ ,

$$\|w_n - w_r\| \ll \|f_n\| + p^{-\frac{1}{m}} \{q\|Tw_r - Bw_r\|^m - \|Sw_n - Aw_n\|^m\}^{\frac{1}{m}} + \|g_r\|,$$

or,

$$\|w_n - w_r\| \ll \|f_n\| + \|g_r\|. \tag{3.7}$$

Proceeding to the limit as  $n \rightarrow \infty$  we obtain from (3.6) and (3.7) that  $\|w_n - w_r\| \rightarrow 0$ . This implies that  $\{w_n\}$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, it follows that the sequence  $\{w_n\}$  converges to a point  $w$  in  $X$ .

Since  $A$ ,  $B$ ,  $S$  and  $T$  are continuous, it follows from (3.1) that

$$\begin{aligned}\|w - Sw\| &= \lim_{n \rightarrow \infty} \|w_n - Sw_n\| = \lim_{n \rightarrow \infty} \|f_n\| = 0, \\ \|w - Tw\| &= \lim_{n \rightarrow \infty} \|w_n - Tw_n\| = \lim_{n \rightarrow \infty} \|g_n\| = 0, \\ \|Sw - Aw\| &= \lim_{n \rightarrow \infty} \|Sw_n - Aw_n\| = 0, \\ \|Tw - Bw\| &= \lim_{n \rightarrow \infty} \|Tw_n - Bw_n\| = 0.\end{aligned}$$

This implies that  $w = Aw = Bw = Sw = Tw$ , completing the proof of the theorem.  $\square$

## 4 An Example

Finally, we give an example to validate our coincidence Theorem 2.7 as follows:

**Example 4.1** Let  $E = \mathbb{R}$  be a topological semifield and  $X = \mathbb{R}$  be its subsemifield over the field of real number under ordinary addition and multiplication. Suppose  $K = \mathbb{R}^+ = (0, \infty)$  be the set of all positive elements of  $E$ , so that  $\bar{K} = [0, \infty)$ .

Let us define a norm  $\|\cdot\|$  in  $\mathbb{R}$  by

$$\|x - y\| = |x - y| \quad \forall x, y \in \mathbb{R}$$

and four continuous self-mappings on  $\mathbb{R}$  by

$$Ax = \frac{x}{5}, \quad Sx = \frac{x}{4}, \quad Bx = \frac{x}{3}, \quad Tx = \frac{x}{2} : \quad \forall x, y \in \mathbb{R}.$$

Since each topological subsemifield is isomorphic to the field of real numbers  $\mathbb{R}$ , it is obvious that  $\mathbb{R}$  is isomorphic to  $\mathbb{R}$  itself. For instance, if we choose a mapping  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\phi(x) = \frac{x}{2}$  for all  $x \in \mathbb{R}$  then  $\phi$  is topologically isomorphic from  $\mathbb{R}$  to  $\mathbb{R}$ , as  $\phi$  is one-one, onto, linear and both  $\phi$  and  $\phi^{-1}$  are continuous in their respective domains.

Here, we observe that

- (i)  $A^{-1}x = 5x$ ,  $S^{-1}x = 4x$ ,  $B^{-1}x = 3x$  and  $T^{-1}x = 2x$ .
- (ii)  $AX = SX = BX = TX = \mathbb{R}$ , so that the range of  $S$  contains the range of  $(1-t)T + tB$  and the range of  $T$  contains the range of  $(1-t)S + tA$ , where  $0 < t < 1$  and  $SX = TX$ .
- (iii)  $(\mathbb{R}, \|\cdot\|)$  is a Banach space.
- (iv) Let us test the convergence of  $\{x_n\}$  as defined in Theorem 2.7. That is,

$$\begin{aligned}y_{2n+1} &= Sx_{2n+1} = (1-t)Tx_{2n} + tBx_{2n}, \\ y_{2n+2} &= Tx_{2n+2} = (1-t)Sx_{2n+1} + tAx_{2n+1}.\end{aligned}$$

Thus for an arbitrary  $x_0 \in \mathbb{R}$  we obtain

$$Sx_1 = (1 - t)Tx_0 + tBx_0 = \frac{1}{2}x_0\left(1 - \frac{t}{3}\right),$$

so that

$$x_1 = S^{-1}\left\{\frac{1}{2}x_0\left(1 - \frac{t}{3}\right)\right\} = 2x_0\left(1 - \frac{t}{3}\right)$$

and

$$Tx_2 = (1 - t)Sx_1 + tAx_1 = \frac{1}{4}x_1\left(1 - \frac{t}{5}\right)$$

so that

$$x_2 = T^{-1}\left\{\frac{1}{4}x_1\left(1 - \frac{t}{5}\right)\right\} = \frac{1}{2}x_1\left(1 - \frac{t}{5}\right) = x_0\left(1 - \frac{t}{3}\right)\left(1 - \frac{t}{5}\right).$$

More generally, we have

$$x_{2n+1} = 2x_{2n}\left(1 - \frac{t}{3}\right) = 2x_0\left(1 - \frac{t}{3}\right)^{n+1}\left(1 - \frac{t}{5}\right)^n,$$

$$x_{2n+2} = \frac{1}{2}x_{2n+1}\left(1 - \frac{t}{5}\right) = x_0\left(1 - \frac{t}{3}\right)^{n+1}\left(1 - \frac{t}{5}\right)^{n+1}$$

for  $n = 1, 2, \dots$

Since  $0 < t < 1$ , it follows that  $\lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} x_{2n+2} = 0$  and so  $\lim_{n \rightarrow \infty} x_n = 0$ .

(v) We have thus proved that the sequence  $\{x_n\}$  converges to  $u = 0$  and  $A0 = S0$ . Since  $AS0 = SA0$ , the pair  $(A, S)$ , is weak compatible.

(vi) Let us test the relations (2.20) and (2.21) for all  $x, y \in X$ .

Since

$$\|Ax - Sx\| = \frac{|x|}{20}, \|By - Ty\| = \frac{|y|}{6}, \|Sx - Ty\| = \frac{|x - 2y|}{4}$$

we have from (2.20)

$$\frac{|x|}{20} \ll \frac{p|x - 2y|}{4} \ll \frac{p}{4}(|x| + 2|y|),$$

since  $K = \mathbb{R}^+$ , and this implies that

$$\left(\frac{1}{20} - \frac{p}{4}\right)|x| \ll \frac{p|y|}{2}. \tag{4.1}$$

Similarly, from (2.21) we have

$$\frac{|y|}{6} \ll \frac{p|x|}{20}. \tag{4.2}$$

Using (4.2) in (4.1) we obtain,

$$\left(\frac{1}{20} - \frac{p}{4}\right)|x| \ll \frac{p|y|}{2} \ll \frac{3p^2|x|}{20} < \frac{3p|x|}{20}.$$

That is,

$$\left(\frac{1}{20} - \frac{p}{4}\right) < \frac{3p}{20}.$$

This implies

$$\frac{1}{8} < p. \quad (4.3)$$

Thus conditions (2.20) and (2.21) are satisfied for all  $x, y \in X$  and for  $\frac{1}{8} < p < 1$ .

(vii) Lastly we see that, for  $w = 0$  we have

$$A0 = S0 = B0 = T0.$$

Thus  $w = 0$  is a coincidence point of mappings  $A, B, S$  and  $T$ . This verifies Theorem 2.2.

**An Open Question.** To what extent we can mute continuity requirement of quadruple of self-mappings in Theorems 2.1 and 2.2 ?

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