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On the Diophantine Equations $q^x + p(2q+1)^y = z^2$ and $q^x + p(4q+1)^y = z^2$

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Abstract In this paper, by using basic concepts of number theory, we present some conditions of the non-existence of non-negative integer solutions (x, y, z) for the Diophantine equations $q^x + p(2q+1)^y = z^2$ and $q^x + p(4q+1)^y = z^2$, where p and q are prime numbers.

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1. Introduction

There are a lot of studies about the exponential Diophantine equations in the form $a^x + b^y = z^2$, where a and b are positive integers. Many researchers investigated the non-existence of solutions for the Diophantine equations, when a and b are fixed positive integers, see [1–5]. Later, it has been generalized by adding some conditions for a and b. In 2018, Gupta, Kumar and Kishan [6] proved that the Diophantine equation $p^x + (p+6)^y = z^2$ has no non-negative integer solution, where p and $p+6$ are prime numbers with $p \equiv 1 \pmod 6$. In 2019, Mina and Bacani [7] presented results that guarantee the non-existence of positive integer solutions of the Diophantine equation $p^x + q^y = z^{2n}$, where p, q and n are positive integers. In the same year, Kumar, Gupta and Kishan [8] showed that if p and $p+12$ are primes with $p \equiv 1 \pmod 6$, then the Diophantine equation $p^x + (p+12)^y = z^2$ has no non-negative integer solution. In 2020, Dokchan and Pakapongpun [9] proved that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution, where a is a positive integer with $a \equiv 5 \pmod{42}$. In 2021, Thongnak, Kaewong and Chuayjan [10] determined that the Diophantine equation $(p+2)^x + (5p+6)^y = z^2$ is not solvable in non-negative integers, where $p, p+2$ and $5p+6$ are prime numbers. Dokchan and Pakapongpun [11] proved that the Diophantine equation $p^x + (p+20)^y = z^2$ has no positive integer solution, where p and $p+20$ are primes. In [12], Tadee found some conditions of the non-existence of non-negative integer solutions for the Diophantine

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equation $p^x + (p+14)^y = z^2$, where p and $p+14$ are prime numbers. In 2022, Alabbood [13] proved that the Diophantine equation $p^x + (p+\lambda+1)^y = z^2$ has no positive integer solution, where $p > 3, p+\lambda+1$ and λ are prime numbers with λ being a non Sophie Germain prime such that $\lambda \equiv 3 \pmod 4$.

Next, some of mathematical researchers have added the coefficients of the equation. For example, in 2019, Laipaporn, Wananiyakul and Khachorncharoenkul [14] proved that the Diophantine equation $3^x + p \cdot 5^y = z^2$ has no non-negative integer solution, where p is a prime number with $p \equiv 5, 17 \pmod{24}$. In 2022, Tangjai, Chaeoueng and Phumchaichot [15] showed that the Diophantine equation $7^x + 5 \cdot p^y = z^2$ has no non-negative integer solution, where p is an odd prime number with $p \equiv 1, 2, 4 \pmod 7$. In the same year, Thongnak, Chuayjan and Kaewong [16] solved the Diophantine equation $11 \cdot 3^x + 11^y = z^2$. Recently, Tadee [17] proved that if n is a positive integer with $n \equiv 3 \pmod 4$, then the Diophantine equation $(n+2)^x + 2 \cdot n^y = z^2$ has no non-negative integer solution. Moreover, Porto, Buosi and Ferreira [18] studied the Diophantine equation $p \cdot 3^x + p^y = z^2$, where p is a prime number. Motivated by the above papers, we will find the conditions of the non-existence of non-negative integer solutions (x, y, z) for the Diophantine equations

$$q^x + p(2q+1)^y = z^2 \tag{1.1}$$

and

$$q^x + p(4q+1)^y = z^2, \tag{1.2}$$

where p and q are prime numbers. In order to find such conditions, we will use properties of UFD in $\mathbb{Z}$, congruence and the Legendre symbol. Moreover, the research results of Tadee and Siraworakun [19] also play an important role in proving this, which will be discussed in the next section.

2. Preliminaries

In this section, we recall the definition of the Legendre symbol. Moreover, we present its some properties, which have been proven by Tadee and Siraworakun [19].

Definition 2.1. [20] Let a be an integer and p be an odd prime. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod p \text{ is solvable} \\ 0 & \text{if } p \mid a \\ -1 & \text{if } x^2 \equiv a \pmod p \text{ is not solvable} \end{cases}.$$

Theorem 2.2. [19] *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q.

Theorem 2.3. [19] *Let p and q be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 3q + r^{S_1}q + r^{S_1},\ -3q + r^{S_2}q + r^{S_2} \pmod{4q} \\ -1 & \text{if } p \equiv 3q + r^{S_2}q + r^{S_2},\ -3q + r^{S_1}q + r^{S_1} \pmod{4q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q.

3. The Diophantine Equation $q^x + p(2q+1)^y = z^2$

Lemma 3.1. *Let p, q and $2q+1$ be distinct odd prime numbers.*

1. *If $x = 0$ and (1.1) has a non-negative integer solution, then $p \equiv 3, 2q-1 \pmod{2q}$.*
2. *If $x > 0$ is even and (1.1) has a non-negative integer solution, then $p \equiv 1 \pmod{2q}$.*

Proof. Since x is even, we get $x = 2k$ for some non-negative integer k. From (1.1), we have $(z-q^k)(z+q^k) = p(2q+1)^y$, which implies that $p \mid (z+q^k)$ or $p \mid (z-q^k)$.

Case 1. $p \mid (z+q^k)$. Since p and $2q+1$ are distinct prime numbers, there exists a non-negative integer u such that $z - q^k = (2q+1)^u$ and $z + q^k = p(2q+1)^{y-u}$. Therefore, we get $2q^k = p(2q+1)^{y-u} - (2q+1)^u$. If $k = 0$, then $2 \equiv p - 1 \pmod{2q}$. Consequently, $p \equiv 3 \pmod{2q}$. If $k > 0$, then $p \equiv 1 \pmod{q}$. Since p and q are odd prime numbers, we have $p \equiv 1 \pmod{2q}$.

Case 2. $p \mid (z-q^k)$. Since p and $2q+1$ are distinct prime numbers, there exists a non-negative integer v such that $z - q^k = p(2q+1)^v$ and $z + q^k = (2q+1)^{y-v}$. It follows that $2q^k = (2q+1)^{y-v} - p(2q+1)^v$. If $k = 0$, then $2 \equiv 1 - p \pmod{2q}$, which shows that $p \equiv 2q-1 \pmod{2q}$. If $k > 0$, then $p \equiv 1 \pmod{q}$ and so $p \equiv 1 \pmod{2q}$. ■

Theorem 3.2. *Let p, q and $2q+1$ be distinct odd prime numbers with $p \equiv 1 \pmod{4}$. Then the Diophantine equation (1.1) has no non-negative integer solution if $p \not\equiv 3, 2q-1 \pmod{2q}$ and $q \equiv p + r^{S_2}p + r^{S_2} \pmod{2p}$, where $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.1) is true. Since $q \equiv p + r^{S_2}p + r^{S_2} \pmod{2p}$, where $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p, we have $\left(\frac{p}{q}\right) = -1$, by Theorem 2.2. Since $p \not\equiv 3, 2q-1 \pmod{2q}$, we get $x > 0$, by Lemma 3.1 (1). Taking modulo q to equation (1.1), we obtain $p \equiv z^2 \pmod{q}$. Thus $\left(\frac{p}{q}\right) = 1$, which is a contradiction. ■

Example 3.3. The Diophantine equation $23^x + 5 \cdot 47^y = z^2$ has no non-negative integer solution.

Lemma 3.4. *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod{4}$. If $p \equiv 1, 2q-1 \pmod{2q}$, then $\left(\frac{q}{p}\right) = 1$.*

Proof. Since $q \equiv 1 \pmod{4}$ and $p \equiv 1, -1 \pmod{2q}$, we get $\left(\frac{p}{q}\right) = 1$ and so

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} = 1.$$

■

Theorem 3.5. *Let p, q and $2q+1$ be distinct odd prime numbers with $q \equiv 1 \pmod{4}$. Then the Diophantine equation (1.1) has no non-negative integer solution if $p \not\equiv 3 \pmod{2q}$ and $p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.1) is true. Since $p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q, we have $\left(\frac{q}{p}\right) = -1$, by Theorem 2.2. Since $p \not\equiv 3 \pmod{2q}$, it implies that x is odd, by Lemma 3.1 and Lemma 3.4. From (1.1), we obtain $q^x \equiv z^2 \pmod{p}$. Then $\left(\frac{q^x}{p}\right) = \left(\frac{q}{p}\right)^x = 1$ and so $\left(\frac{q}{p}\right) = 1$, a contradiction. ■

Example 3.6. The Diophantine equation $5^x + p \cdot 11^y = z^2$ has no non-negative integer solution, where p is a prime number with $p \equiv 7 \pmod{10}$.

Example 3.7. The Diophantine equation $29^x + p \cdot 59^y = z^2$ has no non-negative integer solution, where p is a prime number with $p \equiv 11, 15, 17, 19, 21, 27, 31, 37, 39, 41, 43, 47, 55 \pmod{58}$.

4. The Diophantine Equation $q^x + p(4q+1)^y = z^2$

Theorem 4.1. *Let p and q be positive integers with $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$. Then the Diophantine equation (1.2) has no non-negative integer solution.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.2) is true. Since $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$, we have $q^x + p(4q+1)^y \equiv 2 \pmod 4$. From (1.2), it follows that $z^2 \equiv 2 \pmod 4$, which contradicts the fact that $z^2 \equiv 0, 1 \pmod 4$. ■

Remark 4.2. By Theorem 4.1, we can see that p and q do not have to be distinct prime numbers. For example, the Diophantine equation $9^x + 9 \cdot 37^y = z^2$ has no non-negative integer solution.

Lemma 4.3. *Let p, q and $4q+1$ be distinct odd prime numbers.*

1. *If $x = 0$ and (1.2) has a non-negative integer solution, then $p \equiv 3, 4q-1 \pmod{4q}$.*
2. *If $x > 0$ is even and (1.2) has a non-negative integer solution, then $p \equiv 1, 2q+1 \pmod{4q}$.*

Proof. To prove in the same way as Lemma 3.1. ■

Theorem 4.4. *Let p, q and $4q+1$ be distinct odd prime numbers with $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$. Then the Diophantine equation (1.2) has no non-negative integer solution if $p \not\equiv 3, 2q+1, 4q-1 \pmod{4q}$.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.2) is true. Since $p \not\equiv 1, 3, 2q+1, 4q-1 \pmod{4q}$, it implies that x is odd, by Lemma 4.3. Since $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$, we have $q^x + p(4q+1)^y \equiv (-1)^x - 1 \pmod 4$. From (1.2), we get $z^2 \equiv (-1)^x - 1 \equiv 2 \pmod 4$, which is impossible since $z^2 \equiv 0, 1 \pmod 4$. ■

Example 4.5. The Diophantine equation $43^x + 7 \cdot 173^y = z^2$ has no non-negative integer solution.

Theorem 4.6. *Let p, q and $4q+1$ be distinct odd prime numbers with $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$. Then the Diophantine equation (1.2) has no non-negative integer solution if $p \equiv 3q + r^{S_2}q + r^{S_2} \pmod{4q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.2) is true. Since $p \equiv 3q + r^{S_2}q + r^{S_2} \pmod{4q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q, we have $\left(\frac{q}{p}\right) = -1$ by Theorem 2.3. Since $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$, we get $q^x + p(4q+1)^y \equiv (-1)^x + 1 \pmod 4$. From (1.2), we obtain $(-1)^x + 1 \equiv z^2 \pmod 4$. Since $z^2 \equiv 0, 1 \pmod 4$, it implies that x is odd. From (1.2), we have $q^x \equiv z^2 \pmod p$. Therefore $\left(\frac{q^x}{p}\right) = \left(\frac{q}{p}\right)^x = 1$ and so $\left(\frac{q}{p}\right) = 1$, a contradiction. ■

Example 4.7. The Diophantine equation $3^x + p \cdot 13^y = z^2$ has no non-negative integer solution, where p is a prime number with $p \equiv 5 \pmod{12}$.

Example 4.8. The Diophantine equation $7^x + p \cdot 29^y = z^2$ has no non-negative integer solution, where p is a prime number with $p \equiv 5, 13, 17 \pmod{28}$.

Theorem 4.9. *Let p, q and $4q+1$ be distinct odd prime numbers with $p \equiv 3 \pmod 4$. Then the Diophantine equation (1.2) has no non-negative integer solution if $p \not\equiv 3, 4q-1 \pmod{4q}$ and $q \equiv 3p + r^{S_2}p + r^{S_2}, -3p + r^{S_1}p + r^{S_1} \pmod{4p}$, where $S_1 \in \{2, 4, 6, \ldots, p-1\}$, $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p.*

Proof. Assume that there exist non-negative integers x, y, z such that (1.2) is true. Since $q \equiv 3p + r^{S_2}p + r^{S_2}, -3p + r^{S_1}p + r^{S_1} \pmod{4p}$, where $S_1 \in \{2, 4, 6, \ldots, p-1\}$, $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p, we have $\left(\frac{p}{q}\right) = -1$, by Theorem 2.3. Since $p \not\equiv 3, 4q-1 \pmod{4q}$, we get $x > 0$, by Lemma 4.3 (1). From (1.2), it follows that $p \equiv z^2 \pmod q$. Then $\left(\frac{p}{q}\right) = 1$, a contradiction. ■

Example 4.10. The Diophantine equation $13^x + 7 \cdot 53^y = z^2$ has no non-negative integer solution.

5. Conclusions

By using the properties of UFD in $\mathbb{Z}$, congruence and the Legendre symbol, we have shown that the Diophantine equation $q^x + p(2q+1)^y = z^2$ has no non-negative integer solution, where p, q and $2q+1$ are distinct odd prime numbers, when it satisfies one of the following cases: case 1 $p \equiv 1 \pmod 4$, $p \not\equiv 3, 2q-1 \pmod{2q}$ and $q \equiv p + r^{S_2}p + r^{S_2} \pmod{2p}$, where $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p or case 2 $q \equiv 1 \pmod 4$, $p \not\equiv 3 \pmod{2q}$ and $p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q. Lastly, we have shown that the Diophantine equation $q^x + p(4q+1)^y = z^2$ has no non-negative integer solution, where p, q and $4q+1$ are distinct odd prime numbers, when it satisfies one of the following cases: case 1 $p \equiv 3 \pmod 4$, $q \equiv 3 \pmod 4$ and $p \not\equiv 3, 2q+1, 4q-1 \pmod{4q}$ or case 2 $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, $p \equiv 3q + r^{S_2}q + r^{S_2} \pmod{4q}$, where $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q or case 3 $p \equiv 3 \pmod 4$, $p \not\equiv 3, 4q-1 \pmod{4q}$ and $q \equiv 3p + r^{S_2}p + r^{S_2}$, $-3p + r^{S_1}p + r^{S_1} \pmod{4p}$, where $S_1 \in \{2, 4, 6, \ldots, p-1\}$, $S_2 \in \{1, 3, 5, \ldots, p-2\}$ and r is a primitive root modulo p.

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