Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 105–113

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Inversion of Matrices over Boolean Semirings

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Abstract : It is well-known that a square matrix A over a commutative ring R with identity is invertible over R if and only if $\det A$ is a multiplicatively invertible element of R. As a consequence, we have that a square matrix A over a Boolean ring R with identity 1 is invertible over R if and only if $\det^+ A + \det^- A = 1$ where $\det^+ A$ and $\det^- A$ are the positive determinant and the negative determinant of A, respectively. This result is generalized to Boolean semirings with identity. By a *Boolean semiring* we mean a commutative semiring S with zero in which $x^2 = x$ for all $x \in S$. By making use of Reutenauer and Sraubing's work in 1984, we show that an $n \times n$ matrix A over a Boolean semiring S with identity 1 is invertible over S if and only if $\det^+ A + \det^- A = 1$ and $2A_{ij}A_{ik} = 0$ $[2A_{ji}A_{ki} = 0]$ for all $i, j, k \in \{1, \ldots, n\}$ such that $j \neq k$.

Keywords : Boolean semiring, invertible matrix.
2000 Mathematics Subject Classification : 16Y60, 15A09.

1 Introduction

A *semiring* is a triple $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ for all $x, y, z \in S$. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ $[x \cdot y = y \cdot x]$ for all $x, y, z \in S$. We call $(S, +, \cdot)$ *commutative* if $(S, +, \cdot)$ is both additively and multiplicatively commutative. An element $0 \in S$ is called a *zero* of $(S, +, \cdot)$ if $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. By an *identity* of a semiring $(S, +, \cdot)$ we mean an element $1 \in S$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$. Notice that both a zero and an identity of a semiring are unique. An element x of a semiring S with zero 0 [identity 1] is said to be *additively* [*multiplicatively*] *invertible* in S if there is an element $y \in S$ such that $x + y = y + x = 0$ $[xy = yx = 1]$. Such an element $y \in S$ is obviously unique.

Recall that a ring R is called a *Boolean ring* if $x^2 = x$ for all $x \in R$. Then every Boolean ring is commutative and $-x = x$, that is, $2x = 0$ for all $x \in R$([2], p.120). If R is a Boolean ring with identity 1 and $x, y \in R$ are such that $xy = 1$, then

$$x = x1 = x(xy) = x^2y = xy = 1.$$

This shows that 1 is the only multiplicatively invertible element of a Boolean ring with identity 1.

Example 1.1. ([2], p. 120) If X is a set, $\mathcal{P}(X)$ is the power set of X,

$$A + B = (A \setminus B) \cup (B \setminus A) \text{ and } A \cdot B = A \cap B \text{ for all } A, B \in \mathcal{P}(X).$$

Then $(\mathcal{P}(X), +, \cdot)$ is a Boolean ring having $\emptyset$ and X as its zero and identity, respectively. We can see that X is the only multiplicatively invertible element of $(\mathcal{P}(X), +, \cdot)$.

By a *Boolean semiring* we mean a commutative semiring S with zero in which $x^2 = x$ for all $x \in S$. Then every Boolean ring is a Boolean semiring. In fact, Boolean semirings are a generalization of Boolean rings.

Example 1.2. Let X be a nonempty set. Define

$$A \oplus B = A \cup B \text{ and } A \cdot B = A \cap B \text{ for all } A, B \in \mathcal{P}(X).$$

Then $(\mathcal{P}(X), \oplus, \cdot)$ is clearly a Boolean semiring having $\emptyset$ and X as its zero and identity, respectively. We can see that $\emptyset$ is the only additively invertible element of $(\mathcal{P}(X), \oplus, \cdot)$. Then $(\mathcal{P}(X), \oplus, \cdot)$ is not a Boolean ring. Also, $A \oplus A = A$ for all $A \in \mathcal{P}(X)$.

Example 1.3. Let $S = \{0\} \cup \left[\frac{1}{2}, 1\right]$ and define

$$\begin{array}{rcll} x \oplus 0 &=& 0 \oplus x = x & \text{for all } x \in S, \\ x \oplus y &=& \frac{1}{2} & \text{for all } x, y \in \left[\frac{1}{2}, 1\right], \\ x \circ y &=& \min\{x, y\} & \text{for all } x, y \in S. \end{array}$$

It is straightforward to show that $(S, \oplus, \circ)$ is a Boolean semiring with zero 0 and identity 1. Moreover, 0 is the only additively invertible element of the semiring $(S, \oplus, \circ)$ and for $x \in S$, $x \oplus x = x$ if and only if either $x = 0$ or $x = \frac{1}{2}$.

Let S be a commutative semiring with zero 0 and identity $1 \neq 0$, n a positive integer and $M_n(S)$ the set of all $n \times n$ matrices over S. Then under usual matrix addition and matrix multiplication, $M_n(S)$ is an additively commutative semiring. The $n \times n$ zero matrix and the $n \times n$ identity matrix over S are the zero and the identity of $M_n(S)$, respectively. If $n > 1$, then $M_n(S)$ is not multiplicatively commutative. For $A \in M_n(S)$ and $i, j \in \{1, \ldots, n\}$, let A_{ij} be the entry of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column. The transpose of A will be denoted by A^t, that is, $A^t_{ij} = A_{ji}$ for all $i, j \in \{1, \ldots, n\}$. Then for all $A, B \in M_n(S)$, $(A^t)^t = A$, $(A + B)^t = A^t + B^t$ and $(AB)^t = B^t A^t$. A matrix $A \in M_n(S)$ is called *invertible* over S if $AB = BA = I_n$ for some $B \in M_n(S)$ where I_n is the $n \times n$ identity matrix over S. Notice that B is unique. Also, for $A \in M_n(S)$, A is invertible over S if and only if A^t is invertible over S. In 1963, Rutherford [4] characterized invertible matrices over a Boolean algebra of 2 elements.

Let $\mathcal{S}_n$ be the symmetric group of degree $n \geq 2$, $\mathcal{A}_n$ the alternating group of degree n and $\mathcal{B}_n = \mathcal{S}_n \setminus \mathcal{A}_n$, that is,

$$\mathcal{A}_n = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ is an even permutation}\},$$
$$\mathcal{B}_n = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ is an odd permutation}\}.$$

If S is a commutative semiring with zero and identity and n a positive integer greater than 1, then for $A \in M_n(S)$, the *positive determinant* and the *negative determinant* of A are defined respectively by

$$\det{}^+ A = \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right),$$
$$\det{}^- A = \sum_{\sigma \in \mathcal{B}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right).$$

If S is a commutative ring with identity, then for $A \in M_n(S)$, $\det A = \det^+ A - \det^- A$. Hence if S is a Boolean ring with identity, then $\det A = \det^+ A + \det^- A$ for all $A \in M_n(S)$.

We can see that

$$\mathcal{A}_n = \{\sigma^{-1} \mid \sigma \in \mathcal{A}_n\} \text{ and } \mathcal{B}_n = \{\sigma^{-1} \mid \sigma \in \mathcal{B}_n\},$$

$\det^+ I_n = 1$ and $\det^- I_n = 0$ and for $A \in M_n(S)$,

$$\begin{aligned}
\det{}^+ \left(A^t\right) &= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A^t_{i\sigma(i)} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma(i),i} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),i} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),\sigma(\sigma^{-1}(i))} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right) \quad \text{since } \begin{array}{c}\{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\} \\ = \{1, \ldots, n\}\end{array} \\
&= \det{}^+ A.
\end{aligned}$$

It can be shown similarly that $\det^-(A^t) = \det^- A$.

In 1985, Reutenauer and Straubing [3] gave the following significant results.

Theorem 1.4. ([3]) *Let S be a commutative semiring with zero and identity and n a positive integer ≥ 2. If $A, B \in M_n(S)$, then there is an element $r \in S$ such that*

$$\det^+(AB) = (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r,$$
$$\det^-(AB) = (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r.$$

Theorem 1.5. ([3]) *Let S be a commutative semiring with zero and identity and n a positive integer. For $A, B \in M_n(S)$, if $AB = I_n$, then $BA = I_n$.*

It is well-known that for a square matrix A over a field F, A is invertible over F if and only if $\det A \neq 0$. The following known theorem is a generalization of this fact.

Theorem 1.6. ([1], p.160) *Let R be a commutative ring with identity. A square matrix A over R is invertible over R if and only if $\det A$ is a multiplicatively invertible element of R.*

By the properties of a Boolean ring with identity mentioned above, the following result is a direct consequence of Theorem 1.6.

Corollary 1.7. *Let R be a Boolean ring with identity 1 and n a positive integer ≥ 2. An $n \times n$ matrix A over R is invertible over R if and only if $\det^+ A + \det^- A = 1$.*

The purpose of this research is to generalize Corollary 1.7 to Boolean semirings with identity 1. We show that for a positive integer $n \geq 2$, an $n \times n$ matrix over a Boolean semiring with identity 1 is invertible if and only if
(i) $\det^+ A + \det^- A = 1$ and
(ii) $2A_{ij}A_{ik} = 0$ for all $i, j, k \in \{1, \ldots, n\}$ such that $j \neq k$.

The condition (ii) may be replaced by
(ii)´ $2A_{ji}A_{ki} = 0$ for all $i, j, k \in \{1, \ldots, n\}$ such that $j \neq k$.

2 Invertible Matrices over Boolean Semirings

For a set X, $|X|$ denotes the cardinality of X.

In the remainder of this paper, let n be a positive integer greater than 1. Recall that $|\mathcal{S}_n| = n!$, $|\mathcal{A}_n| = \frac{n!}{2}$, $|\mathcal{B}_n| = \frac{n!}{2}$ and $\sigma\mathcal{A}_n = \mathcal{B}_n$ for all $\sigma \in \mathcal{B}_n$.

The following lemma is needed.

Lemma 2.1. *For distinct $i, j \in \{1, 2, \ldots, n\}$, $\{(\sigma(i)\ \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\} = \mathcal{B}_n$.*

Proof. Let $i, j \in \{1, \ldots, n\}$ be distinct. If $\sigma \in \mathcal{A}_n$, then $(\sigma(i)\ \sigma(j)) \in \mathcal{B}_n$, so $\{(\sigma(i)\ \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\} \subseteq \mathcal{B}_n$. Assume that $\sigma_1, \sigma_2 \in \mathcal{A}_n$ and $\sigma_1 \neq \sigma_2$.

Case 1 : $(\sigma_1(i)\ \sigma_1(j)) = (\sigma_2(i)\ \sigma_2(j))$. By the cancellation property of $\mathcal{S}_n$, we have $(\sigma_1(i)\ \sigma_1(j))\,\sigma_1 \neq (\sigma_2(i)\ \sigma_2(j))\,\sigma_2$.

Case 2 : $(\sigma_1(i)\ \sigma_1(j)) \neq (\sigma_2(i)\ \sigma_2(j))$. Then $\{\sigma_1(i), \sigma_1(j)\} \neq \{\sigma_2(i), \sigma_2(j)\}$. We may assume without loss of generality that $\sigma_1(i) \notin \{\sigma_2(i), \sigma_2(j)\}$. Then $\sigma_1(i) \neq \sigma_2(i)$, so

$$(\sigma_1(i)\ \sigma_1(j))\,\sigma_1(j) = \sigma_1(i) \neq \sigma_2(i) = (\sigma_2(i)\ \sigma_2(j))\,\sigma_2(j).$$

This implies that $(\sigma_1(i)\ \sigma_1(j))\,\sigma_1 \neq (\sigma_2(i)\ \sigma_2(j))\,\sigma_2$.

This shows that $|\{(\sigma(i)\ \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\}| = |\mathcal{A}_n| = |\mathcal{B}_n|$. But since $\{(\sigma(i)\ \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\} \subseteq \mathcal{B}_n$, the equality holds, as desired. □

The following general properties of Boolean semirings are needed.

Lemma 2.2. *Let S be a Boolean semiring. The following statements hold.*
(i) *For all $x \in S, 2x = 4x$.*
(ii) *If $x \in S$ is an additively invertible element of S, then $2x = 0$.*
(iii) *If S has an identity 1, then 1 is the only multiplicatively invertible element of S.*

Proof. (i) If $x \in S$, then $2x = x+x = (x+x)^2 = x^2+x^2+x^2+x^2 = x+x+x+x = 4x$.

(ii) Let $x, y \in S$ be such that $x+y = 0$. Then $2x+2y = 0$. Since $4x = 2x$ by (i), we have

$$2x = 2x + 0 = 2x + 2x + 2y = 4x + 2y = 2x + 2y = 0.$$

(iii) The same proof is given for Boolean rings in Section 1. □

Lemma 2.3. *Let S be a commutative semiring with zero 0 and identity 1. For $A \in M_n(S)$, if A is invertible over S, then $A_{ij}A_{ik}$ is additively invertible in S for $i, j, k \in \{1, \ldots, n\}$ such that $j \neq k$.*

Proof. It is clear that if $a_1, \ldots, a_m \in S$ are additively invertible in S, then so is $c_1a_1 + \cdots + c_ma_m$ for all $c_1, \ldots, c_m \in S$.

Let $B \in M_n(S)$ be such that $AB = BA = I_n$. Then for distinct $p, q \in \{1, \ldots, n\}$,

$$0 = (I_n)_{pq} = (BA)_{pq} = \sum_{l=1}^{n} B_{pl}A_{lq}$$

which implies that $B_{pl}A_{lq}$ is additively invertible in S for all $p, q, l \in \{1, \ldots, n\}$ such that $p \neq q$. Let $i, j, k \in \{1, \ldots, n\}$ be such that $j \neq k$. Then

$$\begin{aligned} A_{ij}A_{ik} &= A_{ij}A_{ik}(AB)_{ii} \\ &= A_{ij}A_{ik}\Big(\sum_{l=1}^{n} A_{il}B_{li}\Big) \\ &= A_{ik}^2(B_{ki}A_{ij}) + \sum_{\substack{l=1 \\ l\neq k}}^{n} A_{ij}A_{il}(B_{li}A_{ik}). \end{aligned}$$

But $B_{ki}A_{ij}, B_{1i}A_{ik}, \ldots, B_{k-1,i}A_{ik}, B_{k+1,i}A_{ik}, \ldots, B_{ni}A_{ik}$ are additively invertible in S, so it follows that $A_{ij}A_{ik}$ is additively invertible in S.

□

Theorem 2.4. *Let S be a Boolean semiring with identity 1 and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) $\det^+ A + \det^- A = 1$ *and*

(ii) $2A_{ij}A_{ik} = 0$ *for all $i, j, k \in \{1, \ldots, n\}$ such that $j \neq k$.*

Proof. Assume that A is invertible over S. Then $AB = BA = I_n$ for some $B \in M_n(S)$. By Theorem 1.4, there is an element $r \in S$ such that

$$\begin{aligned} \det^+(AB) &= (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r, \\ \det^-(AB) &= (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r. \end{aligned}$$

Since $\det^+(AB) = \det^+(I_n) = 1$ and $\det^-(AB) = \det^-(I_n) = 0$, it follows that

$$(\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r = 1, \tag{1}$$

$$(\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r = 0. \tag{2}$$

Then (1)+(2) gives

$$(\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + 2r = 1$$

which implies that

$$(\det^+ A + \det^- A)(\det^+ B + \det^- B) + 2r = 1. \tag{3}$$

From (2), we have that r is an additively invertible element of S, so by Lemma 2.2 (ii), $2r = 0$. It follows from (3) that

$$(\det^+ A + \det^- A)(\det^+ B + \det^- B) = 1. \tag{4}$$

Lemma 2.2 (iii) and (4) yield $\det^+ A + \det^- A = 1$. Thus (i) holds. Since A is invertible over S, (ii) is obtained by Lemma 2.2(ii) and Lemma 2.3.

Conversely, assume that (i) and (ii) hold. Define $B \in M_n(S)$ by

$$B_{ij} = \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma(j)=i}} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \right)$$

for all $i, j \in \{1, \ldots, n\}$. Claim that $AB = I_n$. If $i, j \in \{1, \ldots, n\}$, then

$$\begin{aligned}
(AB)_{ij} &= \sum_{t=1}^{n} A_{it} B_{tj} \\
&= \sum_{t=1}^{n} A_{it} \left(\sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma(j)=t}} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \right) \right) \\
&= \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma(j)=1}} A_{i1} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \right) + \cdots + \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma(j)=n}} A_{in} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \right). \qquad (5)
\end{aligned}$$

It is clear that $\mathcal{S}_n = \{\sigma \in \mathcal{S}_n \mid \sigma(j) = 1\} \cup \{\sigma \in \mathcal{S}_n \mid \sigma(j) = 2\} \cup \ldots \cup \{\sigma \in \mathcal{S}_n \mid \sigma(j) = n\}$ which is a disjoint union. Then (5) gives

$$(AB)_{ij} = \sum_{\sigma \in \mathcal{S}_n} A_{i\sigma(j)} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \right). \qquad (6)$$

Case 1 : $i = j$. Then from (6), we have

$$\begin{aligned}
(AB)_{ij} &= \sum_{\sigma \in \mathcal{S}_n} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i}}^{n} A_{k\sigma(k)} \right) \\
&= \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{k=1}^{n} A_{k\sigma(k)} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{k=1}^{n} A_{k\sigma(k)} \right) + \sum_{\sigma \in \mathcal{B}_n} \left(\prod_{k=1}^{n} A_{k\sigma(k)} \right) \\
&= \det{}^+ A + \det{}^- A = 1.
\end{aligned}$$

Case 2 : $i \neq j$ and $n = 2$. Then either $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$. Note that $\mathcal{S}_2 = \{(1), (1\ 2)\}$. It follows from (6) and (ii) that

$$\begin{aligned}
(AB)_{12} &= A_{12}A_{11} + A_{11}A_{12} = 2A_{11}A_{12} = 0, \\
(AB)_{21} &= A_{21}A_{22} + A_{22}A_{21} = 2A_{21}A_{22} = 0.
\end{aligned}$$

Case 3 : $i \neq j$ and $n > 2$. It follows from (6) that

$$\begin{aligned}(AB)_{ij} &= \sum_{\sigma \in \mathcal{S}_n} A_{i\sigma(j)} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\sigma(k)} \right) \\ &= \sum_{\sigma \in \mathcal{A}_n} A_{i\sigma(j)} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\sigma(k)} \right) + \sum_{\sigma \in \mathcal{B}_n} A_{i\sigma(j)} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\sigma(k)} \right).\end{aligned} \tag{7}$$

For each $\sigma \in \mathcal{A}_n$ let $\bar{\sigma} = (\sigma(i)\ \sigma(j))\sigma$. By Lemma 2.1 and (7), we have

$$(AB)_{ij} = \sum_{\sigma \in \mathcal{A}_n} \left(A_{i\sigma(j)} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\sigma(k)} \right) + A_{i\bar{\sigma}(j)} A_{i\bar{\sigma}(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\bar{\sigma}(k)} \right) \right). \tag{8}$$

But for every $\sigma \in \mathcal{A}_n$, $\bar{\sigma}(i) = (\sigma(i)\ \sigma(j))\sigma(i) = \sigma(j), \bar{\sigma}(j) = (\sigma(i)\ \sigma(j))\sigma(j) = \sigma(i)$ and for $k \in \{1, \ldots, n\} \setminus \{i, j\}$, $\bar{\sigma}(k) = (\sigma(i)\ \sigma(j))\sigma(k) = \sigma(k)$, so it follows from (8) and (ii) that

$$(AB)_{ij} = \sum_{\sigma \in \mathcal{A}_n} 2A_{i\sigma(j)} A_{i\sigma(i)} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} A_{k\sigma(k)} \right) = 0.$$

This proves that $AB = I_n$. By Theorem 1.5, $BA = I_n$. Hence A is invertible over S. □

As mentioned previously, $\det^+ A^t = \det^+ A$, $\det^- A^t = \det^- A$ and A is invertible over S if and only if A^t is invertible over S. Then as a consequence of Theorem 2.4, we have

Corollary 2.5. *Let S be a Boolean semiring with identity 1 and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) $\det^+ A + \det^- A = 1$ *and*

(ii) $2A_{ji}A_{ki} = 0$ *for all* $i, j, k \in \{1, \ldots, n\}$ *such that* $j \neq k$.

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(Received 5 June 2009)

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