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The Ornstein–Uhlenbeck Process and Variance Gamma Process: Parameter Estimation and Simulations

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Abstract The Variance Gamma (VG) model has been increasingly used as an alternative to the standard geometric Brownian motion (GBM) model in modelling asset prices. We consider a $\Gamma(\alpha,\theta)$ Ornstein-Uhlenbeck process and build a continuous sample path Variance-Gamma (VG) model with five parameters $(\mu,\delta,\sigma,\alpha,\theta)$: location (μ), symmetry (δ), volatility (σ), and shape (α) and scale (θ). The simulations of the five parameters Variance-Gamma (VG) Process are performed after fitting the VG model to the underlying distribution of the daily SPY ETF return.

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1. Introduction

Stochastic volatility (SV) models extend the standard geometric Brownian motion (GBM) model, where the observed volatility is modelled as a stochastic process. In a stochastic volatility framework [1], the constant volatility (σ) in a standard geometric Brownian motion (GBM) model is replaced by a deterministic function of a stochastic process $(\sigma(Y_t))$ where Y_t represents the solution of a stochastic differential equation (SDE). This implies that the stochastic volatility model has two sources of randomness, which can be either correlated or not. In the literature, we have two main SV models: Diffusion-based SV models and non-Gaussian Ornstein-Uhlenbeck based SV models. In the popular diffusion-based SV models, Y_t follows a Feller's square root process [2] or a Log-normal process [3] and the deterministic function is a squared root of the stochastic process $(\sigma(Y_t) = \sqrt{Y_t})$. The non-Gaussian Ornstein-Uhlenbeck-based SV models have been introduced and thoroughly studied in [4–7]. The SV models with Ornstein-Uhlenbeck type processes are

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mathematically tractable, have many appealing features and important implications in option pricing [8].
The paper builds a five-parameter Variance-Gamma (VG) process with a continuous sample path using $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process type. The estimated parameters are used to perform the simulations of the $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process and the Variance-Gamma (VG) process. See [9, 10], for the methodology and detailed results on the estimation of the five-parameter Variance-Gamma (VG) model.
The remainder of this paper is organized as follows. In the next section, we consider a $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck type process and build a five-parameter Gamma Variance process. In the third section, we present the parameter estimation results and proceed with the simulation of the $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process and the Variance-Gamma (VG) process.

2. Variance - Gamma Process: Stochastic Volatility Model

2.1. $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is a diffusion process introduced by Ornstein and Uhlenbeck [11] to model the stochastic behavior of the velocity of a particle undergoing Brownian motion. The Ornstein-Uhlenbeck diffusion $\sigma^2 = \{\sigma^2(t), t \geq 0\}$ is the solution of the Langevin Stochastic Differential Equation (SDE) (2.1)

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dB(\lambda t). \tag{2.1}$$

where $\lambda > 0$ and $B = \{B_t, t \geq 0\}$ is a Brownian motion. In recent years, the Ornstein-Uhlenbeck process has been used in finance to capture important distributional deviations from Gaussianity and to model dependence structures. The extension of the Ornstein-Uhlenbeck processes was obtained by replacing the Brownian motion in (2.1) by $z(t)$, a background driving Lévy process (BDLP) [4, 6]. The SDE (2.1) becomes

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t) \qquad \lambda > 0. \tag{2.2}$$

where the process $z(t) = \{z(t), t \geq 0, z(0) = 0\}$ is a subordinator; a process with non-negative, independent and stationary increments, which implies $\sigma^2(t) \geq 0$.

Lemma 2.1. *The general form of the stationary process $\sigma^2(t)$, solution of (2.2) is given by :*

$$\sigma^2(t) = -\int_0^{+\infty} e^{-s} dz(\lambda t - s) \qquad \lambda > 0. \tag{2.3}$$

Proof.
For $\lambda > 0$

$$\sigma^2(t) = -\int_0^{+\infty} e^{-s} dz(\lambda t - s) = -\int_0^{+\infty} e^{-\lambda s} dz(\lambda(t-s)) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dz(\lambda s). \tag{2.4}$$

By using the variable changing method, we can have different expressions of (2.3).

$$\sigma^2(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dz(\lambda s) \qquad \Longrightarrow \qquad d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t) \tag{2.5}$$

■

(2.3) can be written as follows:

$$\sigma^2(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dz(\lambda s) = e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)} dz(\lambda s) \tag{2.6}$$

with

$$\sigma^2(0) = \int_{-\infty}^{0} e^{\lambda s} dz(\lambda s).$$

Theorem 2.2.
We consider a Lévy process $z(t) = \sum_{k=1}^{N(t)} \xi_k$, which is generated by a compound Poisson process: $N(t)$ is a Poisson process with instantaneous rate α, and ξ_k follows an exponential distribution with rate θ.
Then the stationary marginal distribution of $\sigma^2(t)$ is a $\Gamma(\alpha, \theta)$ Gamma distribution.

Proof.

$$u \geq 0$$

$$\sigma^2(t+u) = \int_{-\infty}^{t+u} e^{-\lambda(t+u-s)} dz(\lambda s) = e^{-\lambda u}\sigma^2(t) + \int_0^u e^{-\lambda(u-s)} dz(\lambda s). \tag{2.7}$$

The stationary solution $\sigma^2(t)$ of (2.2) can be written as in (2.7). Because of the stationarity, we have

$$\vartheta(\xi) = \vartheta(\xi e^{-\lambda u})\Phi(u, \xi). \tag{2.8}$$

$\vartheta(\xi)$ is the characteristic function of the stationary distribution of $\sigma^2(t)$ and $\Phi(u, \xi)$ is the characteristic function of $\int_0^u e^{-\lambda(u-s)} dz(\lambda s)$. We have $0 \leq e^{-\lambda u} \leq 1$ for $u \geq 0$, and the relation (2.7) shows that $\sigma^2(t)$ is self-decomposable.
$z(t)$ is a compound Poisson process with the function characteristic.

$$\begin{aligned} \rho(\xi) &= \int_0^{\infty} (e^{i\xi x} - 1)\alpha f(x) dx = \frac{i\xi\alpha}{\theta - i\xi} \\ g(\xi) = E(e^{i\xi z(1)}) &= exp\left\{\int_0^{\infty} (e^{i\xi x} - 1)\alpha f(x) dx\right\} = exp(\rho(\xi)). \end{aligned} \tag{2.9}$$

It was shown in [4] that $\Phi(u, \xi)$ can be expressed as follows

$$\Phi(u, \xi) = exp\left\{\lambda \int_0^u \rho(\xi e^{-\lambda(u-s)}) ds\right\} = exp\left\{\int_{\xi e^{-\lambda u}}^{\xi} \frac{\rho(w)}{w} dw\right\}. \tag{2.10}$$

By replacing, $\frac{\rho(w)}{w} = \frac{i\alpha}{\theta - iw}$, we have

$$\Phi(u, \xi) = \left(\frac{\theta - i\xi e^{-\lambda u}}{\theta - i\xi}\right)^{\alpha}. \tag{2.11}$$

$\vartheta(\xi)$ is continuous at zero, and we have:

$$\vartheta(\xi) = \lim_{u \to \infty} \vartheta(\xi e^{-\lambda u})\Phi(u, \xi) = \left(\frac{1}{1 - i\frac{1}{\theta}\xi}\right)^{\alpha} = \left(1 - i\theta^{-1}\xi\right)^{-\alpha}. \tag{2.12}$$

From (2.12), $\vartheta(\xi)$ is the characteristic function of a Gamma distribution; and the stationary marginal distribution of $\sigma^2(t)$ is a $\Gamma(\alpha, \theta)$ Gamma distribution. ∎

We can integrate the stationary non-negative process $\sigma^2(t)$.

$$\sigma^{2*}(t) = \int_0^t \sigma^2(s)ds. \tag{2.13}$$

By integration by part method, (2.13) becomes

$$\begin{aligned}\sigma^{2*}(t) &= \lambda^{-1}\sigma^2(0)(1-e^{-\lambda t}) + \lambda^{-1}\int_0^t \left(1-e^{-\lambda(t-s)}\right)dz(\lambda s)\\ &= \lambda^{-1}\left(-\sigma^2(t) + z(\lambda t) + \sigma^2(0)\right).\end{aligned} \tag{2.14}$$

It results from (2.14) that the process $\sigma^{2*}(t)$ is continuous as $z(\lambda t)$ and $\sigma^2(t)$ co-break [7]. In addition, the shape of $\sigma^{2*}(t)$ is determined by $z(\lambda t)$. In fact, $\sigma^{2*}(t)$ and $z(\lambda t)$ co-integrate. The co-integration can be shown by transforming the equation (2.14) into (2.15).

$$\lambda\sigma^{2*}(t) - z(\lambda t) = -\sigma^2(t) + \sigma^2(0). \tag{2.15}$$

$\lambda\sigma^{2*}(t) - z(\lambda t)$ is a stationary process.

2.2. Variance - Gamma Process: Semi-Martingale

Let $Y^* = \{Y_t^*\}$, a stochastic process used to model the log of an asset price.

$$A_t = \beta t + \delta\sigma^{2*}(t) \tag{2.16}$$

$$M_t = \sigma\int_0^t \sigma(s)dW(s) \tag{2.17}$$

$$Y_t^* = A_t + M_t = \beta t + \delta\sigma^{2*}(t) + \sigma\int_0^t \sigma(s)dW(s) \tag{2.18}$$

where β and δ are the drift parameters, t represents the continuous time clock, and $W(t)$ is the standard Brownian motion and independent of $\sigma^2(t)$. The mean process A_t is a predictable process with locally bounded variation. In fact, A_t is continuous and differentiable because of $\sigma^{2*}(t)$.

$$\sigma(t) = \sqrt{\sigma^2(t)} \qquad \sigma^{2*}(t) = \int_0^t \sigma^2(s)ds. \tag{2.19}$$

$\sigma(t)$ is the spot or instantaneous volatility, and $\sigma^{2*}(t)$ is the chronometer or the integrated variance of the process.
M_t is a local martingale. The derivative of M_t in (2.18) can be written as a Stochastic Differential Equation (SDE) (2.20).

$$dM_t = \sigma\sigma(t)dW(t). \tag{2.20}$$

Y_t^* is a special semi-martingale [13] and the decomposition $Y_t^* = A_t + M_t$ is unique.

3. Variance - Gamma Process : Simulations

3.1. Parameter Estimation based on daily SPY ETF price

The stochastic process in (2.18) is the solution of the following Stochastic Differential Equation (SDE):

$$dY_t^* = (\beta + \delta\sigma^2(t))dt + \sigma\sigma(t)dW(t). \tag{3.1}$$

Given an interval of length Δ, we define σ_n^2 and Y_n over the interval $[(n-1)\Delta;\, n\Delta]$.

$$\begin{aligned} Y_n &= \int_{(n-1)\Delta}^{n\Delta} dY_s^* = Y_{n\Delta}^* - Y_{(n-1)\Delta}^* \\ \sigma_n^2 &= \int_{(n-1)\Delta}^{n\Delta} d\sigma^{2*}(s) = \sigma_{n\Delta}^{2*} - \sigma_{(n-1)\Delta}^{2*}. \end{aligned} \tag{3.2}$$

The volatility component can be transformed into a normally distributed variable $X(1) \stackrel{\mathrm{d}}{=} N(0,1)$. Here, $N(0,1)$ denotes the standard normal distribution.

$$\begin{aligned} \int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t) \stackrel{\mathrm{d}}{=} N\left(0, \int_{(n-1)\Delta}^{n\Delta} \sigma^2(s)ds\right) \\ = N\left(0, \sigma_{n\Delta}^{2*} - \sigma_{(n-1)\Delta}^{2*}\right) = N\left(0, \sigma_n^2\right). \end{aligned}$$

We have a normally distributed variable with mean 0 and variance σ_n^2

$$\int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t) = \sigma_n X(1). \tag{3.3}$$

By integrating the instantaneous return rate (3.1) per component, we have:

$$\int_{(n-1)\Delta}^{n\Delta} dY_s^* = \beta\Delta + \delta \int_{(n-1)\Delta}^{n\Delta} d\sigma^{2*}(s) + \sigma \int_{(n-1)\Delta}^{n\Delta} \sigma(t)dW(t).$$

Based on (3.1), (3.2) and (3.3), we have the following equation over the interval $[(n-1)\Delta;\, n\Delta]$

$$Y_n = \mu + \delta\sigma_n^2 + \sigma\sigma_n X(1) \tag{3.4}$$

where $\mu = \beta\Delta$, $X(1) \stackrel{\mathrm{d}}{=} N(0,1)$ and $\sigma_n^2 \stackrel{\mathrm{d}}{=} \Gamma(\alpha,\theta)$.

In case Δ is daily length, Y_n becomes the daily return rate. The equation (3.4) was analyzed in [9, 10] as a daily return rate and the parameters were estimated.
The density function of the Variance-Gamma variable Y_n in (3.4) is proven[9, 10] to be (3.5).

$$f(y) = \frac{1}{\sigma\Gamma(\alpha)\theta^\alpha} \int_0^{+\infty} \frac{1}{\sqrt{2\pi v}} e^{-\frac{(y-\mu-\delta v)^2}{2v\sigma^2}} v^{\alpha-1} e^{-\frac{v}{\theta}}\, dv. \tag{3.5}$$

The integral (3.5) makes it challenging to utilize the density function and its derivatives and to perform the Maximum likelihood method.
The characteristic function of the Variance-Gamma variable Y_n in (3.4) and the inverse

Fourier transform is proved in [8] to be:

$$F[f](x) = \frac{e^{-i\mu x}}{\left(1+\frac{1}{2}\theta\sigma^2x^2+i\delta\theta x\right)^{\alpha}} \qquad f(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F[f](x)e^{iyx}\,\mathrm{d}y. \tag{3.6}$$

The Fractional Fourier Transform (FRFT) technique was used to compute the probability density function (3.6) and its derivatives: $\left\{\frac{df(y,V)}{dV_j}\right\}_{1\leq j\leq 5}$ and $\left\{\frac{d^2f(y,V)}{dV_kdV_j}\right\}_{\substack{1\leq k\leq 5\\ 1\leq j\leq 5}}$. See [9, 10, 14, 15] for more details on the methodologies.
The data came from the daily SPY ETF historical data (adjustment for splits and dividends); the period spans from January 4, 2010, to December 30, 2020.
Table 1 presents the estimation results of the five parameters $(\mu,\delta,\alpha,\theta,\sigma)$ of the Variance-Gamma variable Y_n in (3.4).

Table 1. FRFT Maximum Likelihood VG Parameter Estimations

Model	μ	δ	σ	α	θ
VG	0.08476896	−0.0577418	1.02948292	0.88450029	0.93779517

As shown in Table 2, with initial parameter values: $\sigma=\alpha=\theta=1, \delta=\mu=0$, the maximization procedure convergences after 21 iterations. $\log(ML)$ is the function to maximize and $||\frac{d\log(ML)}{dV}||$ is the norm of the partial derivative function $(\log(ML))$. During the maximization process, both quantities converge to -3549.692 and 0 respectively, and the parameter vector becomes stable. It appears that the location parameter μ is positive, the symmetric parameter δ is negative, and other parameters have the expected sign.

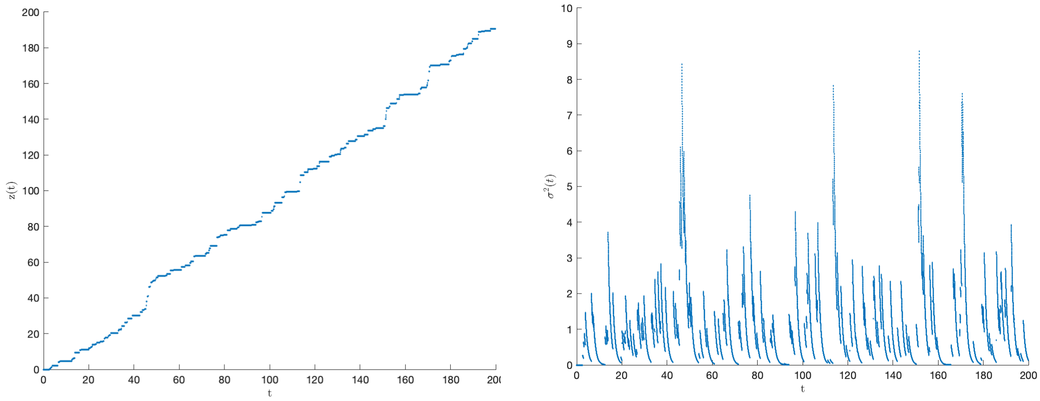
Table 2. Results of VG Model Parameter Estimations

Iterations	μ	δ	σ	α	θ	$Log(ML)$	$\vert\vert\frac{dLog(ML)}{dV}\vert\vert$
1	0	0	1	1	1	-3582.8388	598.743231
2	0.05905599	-0.0009445	1.03195903	0.9130208	1.03208412	-3561.5099	833.530396
3	0.06949925	0.00400035	1.04101444	0.88478895	1.05131996	-3559.5656	447.807305
4	0.07514039	0.00055771	1.17577397	0.67326429	1.17778666	-3569.6221	211.365781
5	0.08928373	-0.0263716	1.03756321	0.83842661	0.94304967	-3554.4434	498.289445
6	0.08676498	-0.0521887	1.03337015	0.85591875	0.95066351	-3550.6419	204.467192
7	0.086995	-0.0608517	1.02788937	0.87382621	0.95054954	-3549.8465	66.8039738
8	0.08542912	-0.058547	1.02705241	0.88258411	0.94321299	-3549.7023	15.3209117
9	0.08478622	-0.0576654	1.02995166	0.88447791	0.93670036	-3549.6921	1.14764198
10	0.08477798	-0.0577736	1.02922308	0.88449072	0.93831041	-3549.692	0.17287708
11	0.08476475	-0.0577271	1.02960343	0.88450434	0.93755549	-3549.692	0.07850459
12	0.08477094	-0.0577488	1.02942608	0.8844984	0.93790784	-3549.692	0.03723941
13	0.08476804	-0.0577386	1.02950937	0.88450117	0.93774266	-3549.692	0.01732146
14	0.0847694	-0.0577434	1.02947043	0.88449987	0.93781995	-3549.692	0.00813465
15	0.08476876	-0.0577411	1.02948868	0.88450048	0.93778375	-3549.692	0.00380345
16	0.08476906	-0.0577422	1.02948014	0.88450019	0.9378007	-3549.692	0.00178206
17	0.08476892	-0.0577417	1.02948414	0.88450033	0.93779276	-3549.692	0.00083415
18	0.08476898	-0.0577419	1.02948226	0.88450026	0.93779648	-3549.692	0.00039063
19	0.08476895	-0.0577418	1.02948314	0.88450029	0.93779474	-3549.692	0.00018289
20	0.08476897	-0.0577419	1.02948273	0.88450028	0.93779555	-3549.692	8.56E-05
21	0.08476896	-0.0577418	1.02948292	0.88450029	0.93779517	-3549.692	4.01E-05

3.2. Variance - Gamma Process Simulations

$$z(t) = \sum_{k=1}^{N(t)} \xi_k \qquad \sigma^2(t) = \sigma^2(0)e^{\lambda t} + \sum_{k=1}^{N(t)} exp(-\lambda(t - a_k))\xi_k. \tag{3.7}$$

For $\lambda = 1$ and $\sigma^2(0) = 0$, the compound Poisson process and the $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck process (3.7) were simulated. The simulation results are shown in Fig 1. As



(a) Compound Poison process: $\hat{z}(t)$ (b) Gamma process: $\hat{\sigma}^2(t)$

Figure 1. Simulations: $\hat{\alpha} = 0.8845$, $\hat{\theta} = 0.9378$

shown in Fig 1a, we have a no drift process with nonnegative, independent and stationary increments. The compound Poisson process $(Z(t))$ is a subordinator. This is not the case for the volatility $\sigma^2(t)$ in Fig 1b. $\sigma^2(t)$ moves up entirely by jumps and then tails off exponentially. The process has been used in storage theory [12].

$$\sigma^{2*}(t) = \int_0^t \sigma^2(s)ds. \tag{3.8}$$

The integrated volatility $\sigma^{2*}(t)$ (3.8) is a continuous process and shaped by the compound Poisson process $(Z(t))$. As shown in Fig 2, $\sigma^{2*}(t)$ is the continuous version of $Z(t)$, also called subordinator.

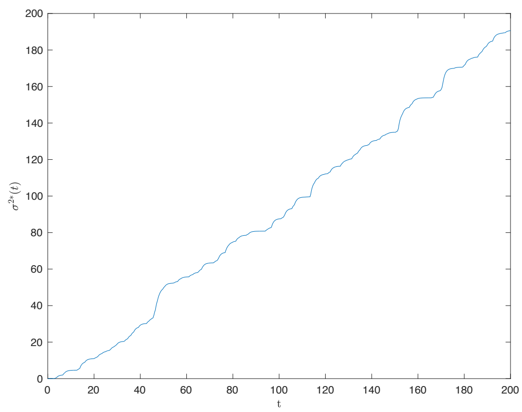
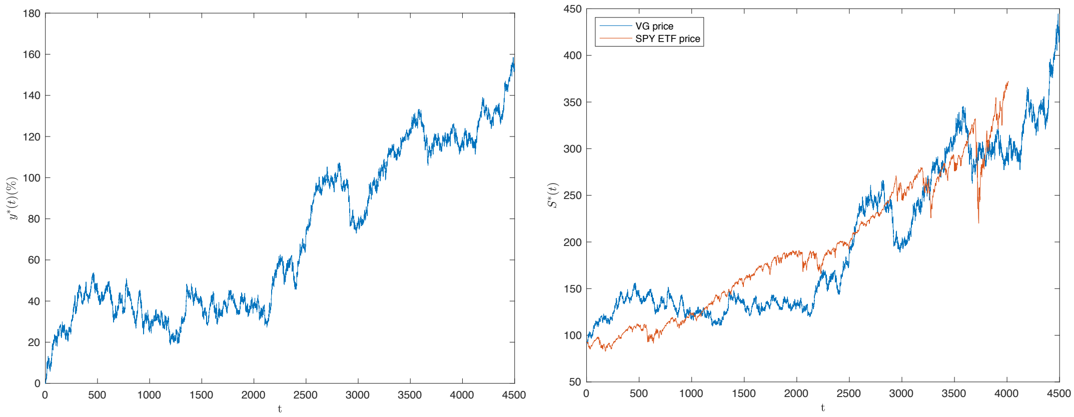


Figure 2. Subordinator: $\hat{\sigma}^{2*}(t)$, $\hat{\alpha} = 0.8845$, $\hat{\theta} = 0.9378$

The Variance-Gamma process, $Y^* = \{Y_t^*\}$, is used to model the daily SPY ETF return.

$$Y_t^* = \beta t + \delta\sigma^{2*}(t) + \sigma \int_0^t \sigma(s) dW(s) \tag{3.9}$$

$$S_t^* = S_0 e^{Y_t^*}. \tag{3.10}$$



(a) Simulations: $Y^* = \{Y_t^*\}$ (b) VG Price versus SPY ETF Price

FIGURE 3. VG Model: $\hat{\mu} = 0.0848$, $\hat{\delta} = -0.0577$, $\hat{\sigma} = 1.0295$, $\hat{\alpha} = 0.8845$, $\hat{\theta} = 0.9378$

The simulation of the Variance-Gamma process (Y^*) in (3.9) shows an increasing trend in the long run with important fluctuations in the short run. The sample path is displayed in blue colour in Fig 3a.
The price generated by the Variance-Gamma process (Y^*) was compared with the actual data, the daily SPY ETF (SPY) price. The SPY ETF price was collected from January 4, 2010, to December 30, 2020, and used for parameter estimation. The dynamics of the VG price (3.10) in blue colour and the daily SPY ETF (SPY) price in red colour are shown in Fig 3b.

4. Conclusion

In the paper, the stationary process $\sigma^2(t)$ was built from the Ornstein-Uhlenbeck (OU) process type with the compound Poisson process as a background driving Lévy process (BDLP). The $\Gamma(\alpha, \theta)$ Ornstein-Uhlenbeck type process was used to build a continuous sample path Variance Gamma (VG) process with parameters $(\mu, \delta, \sigma, \alpha, \theta)$. The parameter estimation results were produced based on the data from the daily SPY ETF historical prices. The estimated parameters were used to simulate the Gamma process ($\sigma^2(t)$) and the continuous sample path process ($\sigma^{2*}(t)$). Both simulations were used as inputs to simulate the continuous sample path Variance Gamma (VG) process.
It will be interesting to compare the performance of the European option price under the five-parameter Variance Gamma (VG) process with the Black-Scholes model.

References

[1] Mohammed Alhagyan, Masnita Misiran, and Zurni Omar. Discussions on continuous stochastic volatility models. Global and Stochastic Analysis, 7(1):55-64, 2020.

[2] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. The Review of Financial Studies, 6(2):327-343, 1993.

[3] John Hull and Alan White. The pricing of options on assets with stochastic volatilities. The Journal of Finance, 42(2):281-300, 1987.

[4] Ole E. Barndorff-Nielsen and Neil Shephard. Integrated ou processes and non-Gaussian ou-based stochastic volatility models. Scandinavian Journal of Statistics, 30(2):277-295, 2003.

[5] Ole E. Barndorff-Nielsen, Jens Ledet Jensen, and Michael Sørensen. Some stationary processes in discrete and continuous time. Advances in Applied Probability, 30(4):989-1007,1998.

[6] Ole E. Barndorff-Nielsen and Neil Shephard. Modelling by Lévy processes for financial econometrics. Lévy Processes, pages 283-318, 2001.

[7] Ole E. Barndorff-Nielsen and Neil Shephard. Non-Gaussian Ornstein–Uhlenbeck based models and some of their uses in financial economics. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 63(2):167-241, 2001.

[8] A. H. Nzokem. Pricing European options under stochastic volatility models: Case of five parameter gamma-variance processes. arXiv 2201.03378 [q-fin.PR], 2022.

[9] A. H. Nzokem. Fitting infinitely divisible distribution: Case of gamma-variance model. arXiv 2104.07580 [stat.ME], 2021.

[10] A. H. Nzokem. Gamma variance model: Fractional Fourier transform (FRFT). Journal of Physics: Conference Series, 2090(1):012094, 2021.

[11] G. E. Uhlenback and L. S. Ornstein. On the theory of the Brownian motion. Phys. Rev, 36:823-841, 1930.

[12] Ole E. Barndorff-Nielsen and Neil Shephard. Non-Gaussian OU-based models and some of their uses in financial economics. Nuffield College Oxford, 1999.

[13] Philip Protter. Stochastic Integration and Differential Equations. A New Approach, Volume 21, 2005.

[14] A. H. Nzokem. Numerical solution of a gamma–integral equation using a higher-order composite Newton–Cotes formula. Journal of Physics: Conference Series, 2084(1):012019, 2021.

[15] A. H. Nzokem and V. T. Montshiwa. Fitting generalized tempered stable distribution: Fractional Fourier Transform (FRFT) approach. arxiv.2205.00586 Statistical Finance (q-fin.ST), 2022.