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Remarks on the Boundedness of Poles of Padé-orthogonal and Padé-Faber approximants

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Abstract Given a function F holomorphic on a neighborhood of some compact subset of the complex plane, we prove that if the zeros of the denominators of (n, m_n) orthogonal Padé and Padé-Faber approximants remain uniformly bounded on a sequence of indices $\{(n, m_n)\}$ satisfying $\sup\{m_n : n \in \mathbb{N}\} < \infty$, then either F is a polynomial or F has a singularity in the complex plane. In this paper, we relax a condition on the indices m_n in results from [N. Bosuwan, On the boundedness of poles of generalized Padé approximants, Adv. Differ. Equ. 2019 (137) (2019) https://doi.org/10.1186/s13662-019-2081-9].

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1. Introduction

Let E be a compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected and E contains infinitely many points. From now on, the set E will satisfy this condition. Let μ be a finite positive Borel measure with infinite support $\operatorname{supp}(\mu)$ contained in E. We can write $\mu \in \mathcal{M}(E)$ and define the corresponding inner product by

$$\langle g, h \rangle_\mu := \int g(\zeta)\overline{h(\zeta)} d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, \ldots,$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle p_n, p_m \rangle_\mu = \delta_{n,m}$. By $\mathcal{H}(E)$ and $\mathbb{P}_m$, we denote the space of all functions holomorphic in some neighborhood of E and the space of all polynomials of

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degree at most m, respectively. In order to simplify our notation, for $F \in \mathcal{H}(E)$, we define

$$\langle F \rangle_n := \langle F, p_n \rangle_\mu.$$

Definition 1.1. Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. For any pair of integers $n \geq 0$ and $m \geq 1$, there exists $Q^\mu_{n,m} \in \mathbb{P}_m$ such that $Q^\mu_{n,m} \not\equiv 0$ and $\langle Q^\mu_{n,m} F \rangle_{n+k} = 0$ for all $k = 1, 2, \ldots, m$. The associated rational function

$$R^\mu_{n,m} := \frac{\Sigma^n_{j=0} \langle Q^\mu_{n,m} F \rangle_j p_j}{Q^\mu_{n,m}}$$

is called an (n, m) *standard orthogonal Padé (SOP) approximant of* F *with respect to* μ.

Next, let us define the standard Padé-Faber approximation. Let Φ be the unique Riemann mapping function from $\overline{\mathbb{C}} \backslash E$ to $\overline{\mathbb{C}} \backslash \overline{\mathbb{B}(0,1)}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For any $\rho > 1$, we define

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index* ρ and the *canonical domain of index* ρ, respectively. We denote by $\rho_0(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function. The *Faber polynomial of* E *of degree* n, where $z \in D_\rho$, $n = 0, 1, 2, \ldots$, is defined by the formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt.$$

It is easy to check that

$$\Phi_n(z) = \frac{z^n}{\operatorname{cap}^n(E)} + \text{lower degree terms},$$

where $\operatorname{cap}(E)$ is the logarithmic capacity of the set E. The *n-th Faber coefficient of* $F \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $1 < \rho < \rho_0(F)$. For any integers $n \geq 0$ and $m \geq 0$, it is known that

$$[\Phi_n]_m = \begin{cases} 1 & ; \quad n = m \\ 0 & ; \quad n \neq m. \end{cases} \tag{1.1}$$

Moreover,

$$\lim_{n \to \infty} |\Phi_n(z)|^{1/n} = |\Phi(z)|, \tag{1.2}$$

uniformly on each compact subset of $\mathbb{C} \setminus E$.

Definition 1.2. Let $F \in \mathcal{H}(E)$. For any pair of integers $n \geq 0$ and $m \geq 1$, there exists $Q^E_{n,m} \in \mathbb{P}_m$ such that $Q^E_{n,m} \not\equiv 0$ and $[Q^E_{n,m} F]_{n+k} = 0$ for all $k = 1, 2, \ldots, m$. The associated rational function

$$R^E_{n,m} := \frac{\Sigma^n_{j=0} [Q^E_{n,m} F]_j \Phi_j}{Q^E_{n,m}}$$

is called an (n, m) *standard Padé-Faber (SPF) approximant of* F *with respect to* E.

In order to find $Q^{\mu}_{n,m}$ or $Q^{E}_{n,m}$ in Definitions 1.1 and 1.2, one has to solve a homogeneous system of m linear equations on $m+1$ unknowns. Therefore, for any pair of integers $(n,m) \in \mathbb{N}_0 \times \mathbb{N}$, polynomials $Q^{\mu}_{n,m}$ and $Q^{E}_{n,m}$ always exist but they may not be unique. Since $Q^{\mu}_{n,m}$ and $Q^{E}_{n,m}$ are not the zero function, we normalize them to be "monic" polynomials. Unlike the classical Padé approximants, for any $(n,m) \in \mathbb{N}_0 \times \mathbb{N}$, $R^{\mu}_{n,m}$ and $R^{E}_{n,m}$ may not be unique. The rational functions $R^{\mu}_{n,m}$ and $R^{E}_{n,m}$ are natural extensions of $R_{n,m}$ and were introduced by Maehly [1] in 1960. Almost all of the studies of SOP approximants considered the case where the support of the measure μ is a subset of a finite interval or the unit circle (see [2–11]). Many recent results were dedicated to the convergence of SOP approximants on row sequences. The pioneering one in this direction was the work of Suetin [10] where he proved an analogue of Montessus de Ballore's theorem for SOP approximants when $E = [-1,1]$. Bosuwan, López Lagomasino, and Saff [12] and Bosuwan and López Lagomasino [13] investigated the convergence of row sequences of SOP approximants corresponding to a measure supported on a general compact set E. Studies of the convergence of SPF approximants on row sequences can be found for example in [4, 5, 10, 14]. The main advantage of the study of row sequences of SOP and SPF approximants is that these rational functions allow us to locate singularities of $F \in \mathcal{H}(E)$ near a compact set E. We would like to point out that there are related constructions called modified orthogonal Padé and modified Padé-Faber approximants (see [15, 16]). For interested readers, we also refer to recent developments of vector orthogonal Padé and Padé-Faber approximants (see [17–22]).

The main goal of this paper is to extend the following result [15, Theorem 2.1]:

Theorem 1.3. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}^*_1(E)$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}^{\mu}_n$ and $\mathcal{P}^{E}_n$ the sets of all zeros of $Q^{\mu}_{n,m}$ and $Q^{E}_{n,m}$, respectively. Assume that one of the following conditions holds.*

(a) The cardinality of $\mathcal{P}^{\mu}_n$ is at least 1 for all n sufficiently large and

$$\inf_{N\geq m} \sup_{n\geq N} \{|\zeta| : \zeta \in \mathcal{P}^{\mu}_n\} < \infty.$$

(b) The cardinality of $\mathcal{P}^{E}_n$ is at least 1 for all n sufficiently large and

$$\inf_{N\geq m} \sup_{n\geq N} \{|\zeta| : \zeta \in \mathcal{P}^{E}_n\} < \infty.$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Theorem 1.3 provides us analytic properties of the approximated function when the sets of all zeros of $Q^{\mu}_{n,m}$ or $Q^{E}_{n,m}$ are uniformly bounded for m fixed. The first result of this kind appeared earlier in [23] where the authors considered incomplete Padé approximants. In this paper, we prove that Theorem 1.3 can be extended from the case that m is fixed to the case that the indices m are uniformly bounded.

The outline of this paper is as follows. We keep all needed notations and auxiliary lemmas in Section 2. The main results and their proofs are in Section 3.

2. Notations and Auxiliary lemmas

For all analytical studies of SOP approximation, we need to limit our interest to some classes of measures in $\mathcal{M}(E)$.

Definition 2.1. Let $\mu \in \mathcal{M}(E)$. Then $\mu \in \mathbf{Reg}_1(E)$ if and only if

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|,$$

uniformly on each compact subset of $\mathbb{C} \setminus E$.

Definition 2.2. Let $\mu \in \mathcal{M}(E)$. Then $\mu \in \mathbf{Reg}_1^*(E)$ if and only if $\mu \in \mathbf{Reg}_1(E)$ and there exist $n_0 \in \mathbb{N}$ and $c > 0$ such that $\kappa_{n-1}/\kappa_n \geq c$, for all $n \geq n_0$.

Domains of convergence of orthogonal and Faber series are shown in the following lemma.

Lemma 2.3. *Let $\mu \in \mathbf{Reg}_1(E)$. Suppose that*

$$L := \limsup_{n\to\infty} |c_n|^{1/n}.$$

(a) *If $L = 0$, then $\Sigma_{n=0}^{\infty} c_n p_n(z)$ and $\Sigma_{n=0}^{\infty} c_n \Phi_n(z)$ converge uniformly on each compact subset of $\mathbb{C}$.*
(b) *If $1 \leq L \leq \infty$, then $\Sigma_{n=0}^{\infty} c_n p_n(z)$ and $\Sigma_{n=0}^{\infty} c_n \Phi_n(z)$ diverge for all $z \in \mathbb{C} \setminus E$.*
(c) *If $0 < L < 1$, then $\Sigma_{n=0}^{\infty} c_n p_n(z)$ and $\Sigma_{n=0}^{\infty} c_n \Phi_n(z)$ converge uniformly on each compact subset of $D_{1/L}$ and diverge for all $z \in \mathbb{C} \setminus \overline{D_{1/L}}$.*

Proof of Lemma 2.3. We will consider only the orthogonal polynomial case. The proof of the Faber polynomial case is similar and relies on equation (1.2).

(a) Assume that $L = 0$. Let $\rho > 1$ be fixed. By $\mu \in \mathbf{Reg}_1(E)$,

$$\limsup_{n\to\infty} |c_n p_n(z)|^{1/n} = 0 \cdot \rho = 0,$$

for all $z \in \Gamma_\rho$. Then, there exists $N \in \mathbb{N}$ such that

$$|c_n p_n(z)| < \frac{1}{2^n} \tag{2.1}$$

for all $n \geq N$ and $z \in \Gamma_\rho$. By the maximum-modulus principle, (2.1) holds for all $n \geq N$ and $z \in \overline{D_\rho}$. By the Weierstrass M-test, $\Sigma_{n=0}^{\infty} c_n p_n(z)$ converges uniformly on $\overline{D_\rho}$. Since ρ is arbitrary, $\Sigma_{n=0}^{\infty} c_n p_n(z)$ converges uniformly on each compact subset of $\mathbb{C}$.

(b) Assume that $1 \leq L \leq \infty$. Fix $z \in \mathbb{C} \setminus E$. By $\mu \in \mathbf{Reg}_1(E)$, and since

$$\limsup_{n\to\infty} |c_n p_n(z)|^{1/n} = L|\Phi(z)| > 1,$$

there are infinitely many n such that $|c_n p_n(z)| \geq 1$. Hence, $\Sigma_{n=0}^{\infty} c_n p_n(z)$ diverges for all $z \in \mathbb{C} \setminus E$.

(c) Assume that $0 < L < 1$. Set

$$\rho := \frac{1}{L}.$$

Let us show that $\sum_{n=0}^{\infty} c_n p_n(z)$ converges uniformly on each compact subset of D_ρ. For any compact subset $K \subset D_\rho$, choose $1 < \sigma < \rho$ such that $K \subset D_\sigma \subset D_\rho$. Since $\mu \in \mathbf{Reg}_1(E)$, for any $z \in \Gamma_\sigma$, there is $\delta > 0$ such that

$$\limsup_{n\to\infty} |c_n p_n(z)|^{1/n} = \frac{\sigma}{\rho} = 1 - 2\delta.$$

This implies that for sufficiently large n,

$$|c_n|^{1/n} |p_n(z)|^{1/n} \leq 1 - \delta, \quad z \in \Gamma_\sigma.$$

Using the maximum modulus principle, we have

$$|c_n p_n(z)| \leq (1-\delta)^n,$$

for all $z \in \overline{D_\sigma}$ and sufficiently large n. Then, $\sum_{n=0}^{\infty} \langle F \rangle_n p_n(z)$ converges uniformly on K.

Next, let us show that $\sum_{n=0}^{\infty} c_n p_n(z)$ diverges for all $z \in \mathbb{C} \setminus \overline{D_\rho}$. Let $z \in \mathbb{C} \setminus \overline{D_\rho}$. By $\mu \in \mathbf{Reg}_1(E)$, and since

$$\limsup_{n\to\infty} |c_n p_n(z)|^{1/n} > \frac{\rho}{\rho} = 1,$$

$\sum_{n=0}^{\infty} c_n p_n(z)$ diverges for all $z \in \mathbb{C} \setminus \overline{D_\rho}$.

■

Next, we will show that if F is entire, then orthogonal polynomial and Faber polynomial expansions converge to F uniformly on each compact subset of $\mathbb{C}$.

Lemma 2.4. *Assume that F is entire.*

(a) *If $\mu \in \mathbf{Reg}_1(E)$, then $\lim_{n\to\infty} |\langle F \rangle_n|^{1/n} = 0$ and $\Sigma_{n=0}^{\infty} \langle F \rangle_n p_n(z)$ converges to $F(z)$ uniformly on each compact subset of $\mathbb{C}$.*

(b) *Then, $\lim_{n\to\infty} |[F]_n|^{1/n} = 0$ and $\Sigma_{n=0}^{\infty} [F]_n \Phi_n(z)$ converges to $F(z)$ uniformly on each compact subset of $\mathbb{C}$.*

Proof of Lemma 2.4. (a) Since F is entire, $\rho_0(F) = \infty$. It follows from [24, Theorem 6.6.1] that

$$\lim_{n\to\infty} |\langle F \rangle_n|^{1/n} = \frac{1}{\rho_0(F)} = 0.$$

By Lemma 2.3, $\Sigma_{n=0}^{\infty} \langle F \rangle_n p_n(z)$ converges uniformly on each compact subset of $\mathbb{C}$.

Next, we show that the series converges to F. Let F_1 be the uniform limit. Obviously, $F_1 \in \mathcal{H}(\mathbb{C})$. Since $\mathbb{C} \setminus E$ is connected and $F \in \mathcal{H}(E)$, there is a sequence of polynomials $\{f_n\}_{n\in\mathbb{N}}$ with $\deg(f_n) = n$ such that $\lim_{n\to\infty} \|F - f_n\|_E = 0$, where $\|\cdot\|_E$ is the sup-norm on E. Then,

$$0 \leq \lim_{n\to\infty} \left(\int |(F - f_n)(z)|^2 d\mu(z) \right)^{1/2} \leq \mu(E)^{1/2} \lim_{n\to\infty} \|F - f_n\|_E = 0.$$

So, f_n converges to F in $L_2(\mu)$. Since

$$0 \leq \lim_{n\to\infty} \left(\int |(F - \sum_{j=0}^{n} \langle F \rangle_n p_n)(z)|^2 d\mu(z) \right)^{1/2}$$
$$\leq \lim_{n\to\infty} \left(\int |(F - f_n)(z)|^2 d\mu(z) \right)^{1/2} \leq 0,$$

the partial sums of orthogonal polynomial expansion also converge to F in $L_2(\mu)$. Therefore, $F = F_1$ μ-a.e. in E. Since F is entire, $F(z) = F_1(z)$ for all $z \in \mathbb{C}$.

(b) The proof for the Faber expansion case is similar and is well-known. We refer to [26] for the proof.

■

The next lemma (see [25, p. 583] or [26, p. 43] for its proof) gives an estimate of Faber polynomials Φ_n on a level curve. For completeness, we provide the proof here.

Lemma 2.5. *Let $\rho > 1$. Then, there is $c > 0$ such that for every $n \in \mathbb{N}_0$,*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n,$$

where $\|\cdot\|_{\Gamma_\rho}$ is the sup-norm on Γ_ρ.

Proof of Lemma 2.5. Let $\rho > 1$ and choose ρ_1 and ρ_2 such that $1 < \rho_2 < \rho < \rho_1$. Let $z \in \Gamma_\rho$. Recall that

$$\Phi_n(z) := \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} \frac{\Phi^n(t)}{t-z}dt, \quad z \in D_{\rho_1},\ n = 0, 1, 2, \ldots.$$

Then $\frac{\Phi^n(t)}{t-z}$ as a function of t is analytic in a neighborhood of the closure of $D_{\rho_1}\setminus D_{\rho_2}$, except for a simple pole at the point $t = z$. The residue at the pole is

$$\lim_{t\to z}(t-z)\frac{\Phi^n(t)}{t-z} = \Phi^n(z).$$

By the Cauchy residue theorem, for $\frac{\Phi^n(t)}{t-z}$ at $t = z$ we get

$$\int_{\Gamma_{\rho_1}-\Gamma_{\rho_2}} \frac{\Phi^n(t)}{t-z}dt = \int_{\Gamma_{\rho_1}} \frac{\Phi^n(t)}{t-z}dt - \int_{\Gamma_{\rho_2}} \frac{\Phi^n(t)}{t-z}dt = 2\pi i \mathrm{Res}\left(\frac{\Phi^n(t)}{t-z}, z\right) = 2\pi i\Phi^n(z).$$

Then, for all $z \in \Gamma_\rho$,

$$\Phi_n(z) = \Phi^n(z) + \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} \frac{\Phi^n(t)}{t-z}dt.$$

Since $\rho_2 < \rho < \rho_1$, then $\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n$, for all $n \geq 0$, where c does not depends on n. ■

The following technical lemma, whose proof may be found in [13, Lemma 3], is mainly used in the proofs of our results. Again, for completeness, we provide the proof here.

Lemma 2.6. *If a sequence of complex numbers $(A_N)_{N\in\mathbb{N}}$ has the following properties:*

(a) $\lim_{N\to\infty} |A_N|^{1/N} = 0$

(b) *there are $N_0 \in \mathbb{N}$ and $c > 0$ such that $|A_N| \leq c\Sigma_{k=N+1}^{\infty}|A_k|$ for all $N \geq N_0$,*

then there exists $N_1 \in \mathbb{N}$ such that $A_N = 0$ for all $N \geq N_1$.

Proof of Lemma 2.6. Given the assumptions, there is $M \in \mathbb{N}$ such that for all $N \geq M$,

$$|A_N|^{1/n} < \frac{1}{c+2} \quad \text{and} \quad |A_N| \leq c\sum_{k=N+1}^{\infty}|A_k|. \tag{2.2}$$

Claim that for any non-negative integer n,

$$|A_N| \leq \left(\frac{c}{c+1}\right)^n \left(\frac{1}{c+2}\right)^N. \tag{2.3}$$

Then, letting $n \to \infty$, we see that $|A_N| = 0$. To prove the claim, we use the principle of mathematical induction on n. When $n = 0$, the formula follows immediately from the first inequality in (2.2). Now, assume that (2.3) holds for n; we will show that (2.3) holds for $n+1$ as well. In fact,

$$\begin{aligned}|A_N| \le c\sum_{k=N+1}^{\infty}|A_k| &\le c\sum_{k=N+1}^{\infty}\left(\frac{c}{c+1}\right)^n\left(\frac{1}{c+2}\right)^k\\ &= c\left(\frac{c}{c+1}\right)^n\frac{\left(\frac{1}{c+2}\right)^{N+1}}{1-\frac{1}{c+2}}\\ &= \left(\frac{c}{c+1}\right)^{n+1}\left(\frac{1}{c+2}\right)^N.\end{aligned}$$

This completes the proof. ■

A simple observation concerning an inner product of orthonormal polynomials is shown in the following lemma.

Lemma 2.7. *Let p_n be the orthonormal polynomial of degree n. Then,*

$$\langle z^m p_{n-m}, p_n\rangle_\mu = \frac{\kappa_{n-m}}{\kappa_n},$$

where $m,n\in\mathbb{N}_0$ and $n\ge m$.

Proof of Lemma 2.7. We know that

$$z^m p_{n-m} = \left(\frac{\kappa_{n-m}}{\kappa_n}\right)p_n + \sum_{j=1}^{n-1}\langle z^m p_{n-m}, p_j\rangle_\mu p_j.$$

By the orthonormality of p_n, we get

$$\langle z^m p_{n-m}, p_n\rangle_\mu = \left\langle\left(\frac{\kappa_{n-m}}{\kappa_n}\right)p_n + \sum_{j=1}^{n-1}\langle z^m p_{n-m}, p_j\rangle_\mu p_j, p_n\right\rangle_\mu = \frac{\kappa_{n-m}}{\kappa_n}.$$

■

Lemma 2.8. *Let $(Q_n)_{n\in\mathbb{N}} = \left(\sum_{j=1}^{l_n} q_{n,j}z^j\right)$ be a sequence of monic polynomials such that $\deg(Q_n) = l_n \le M$ for some $M>0$ and $\mathcal{P}_n$ be the set of all zeros of Q_n. Suppose that $\mathcal{P}_n$ is nonempty for every sufficiently large n and*

$$\inf_{N\ge M}\sup_{n\ge N}\{|\zeta| : \zeta\in\mathcal{P}_n\} < \infty.$$

Then, there exists $n_0\in\mathbb{N}$ such that

$$\sup\{|q_{n,j}| : 0\le j\le l_n, n\ge n_0\} < \infty.$$

Proof of Lemma 2.8. From Vieta's fomulas, one obtains that

$$\sum_{1\le i_1<i_2<\cdots<i_{l_n-j}\le l_n}\left(\prod_{\gamma=1}^{l_n-j}\zeta_{i_\gamma}\right) = (-1)^{l_n-j}q_{n,j}, \qquad j=1,2,\dots,l_n,$$

where $\zeta_j\in\mathcal{P}_n$ for all $j=1,\dots,l_n$. Since

$$\inf_{N\ge M}\sup_{n\ge N}\{|\zeta| : \zeta\in\mathcal{P}_n\} < \infty \quad\text{and}\quad l_n\le M,$$

there exists $n_0 \in \mathbb{N}$ such that for each $j \in \{1, 2, \ldots, l_n\}$ and for all $n \geq n_0$,

$$|q_{n,j}| = \left| \sum_{1 \leq i_1 < i_2 < \cdots < i_{l_n - j} \leq l_n} \left(\prod_{\gamma=1}^{l_n - j} \zeta_{i_\gamma} \right) \right| \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_{l_n - j} \leq l_n} \left(\prod_{\gamma=1}^{l_n - j} |\zeta_{i_\gamma}| \right) \leq L,$$

for some $0 < L < \infty$. This completes the proof. ∎

Lemma 2.9. *Let p_n be the orthonormal polynomial of degree n. Then, for all $\nu, n, m, k \in \mathbb{N}_0$ and $j = 0, \ldots, m$,*

$$|\langle z^j p_\nu, p_{n+k} \rangle_\mu| < \infty.$$

Proof of Lemma 2.9. By the Cauchy-Schwarz inequality and the orthonormality of (p_ν), for all $n, m, k, \nu \in \mathbb{N}_0$ and $j = 1, \ldots, m$, we obtain

$$|\langle z^j p_\nu, p_{n+k} \rangle_\mu| \leq \|z^j p_\nu\|_2 \|p_{n+k}\|_2 = \left(\int_E |z^j p_\nu|^2 d\mu(z) \right)^{1/2}$$

$$\leq \left(\|z^{2j}\|_E \right)^{1/2} \left(\int_E |p_\nu|^2 d\mu(z) \right)^{1/2} \leq \left(\|z^{2j}\|_E \right)^{1/2} < \infty.$$

∎

Lemma 2.10. *Let $m, n \in \mathbb{N}_0$ such that $m \leq n$. Then, $[z^m \Phi_{n-m}]_n = \mathrm{cap}^m(E)$, where we recall that $\mathrm{cap}(E)$ is the logarithmic capacity of the set E.*

Proof of Lemma 2.10. By making use of the equality (1.1) and applying Lemma 2.4 for the Faber expansion of $z^m \Phi_{n-m}$, the proof of this lemma is very similar to the proof of Lemma 2.7. We leave this for the reader. ∎

3. Main Results and Their Proofs

An extension of Theorem 1.3 is stated in the following theorems.

Theorem 3.1. *Let $F \in \mathcal{H}(E)$, $\mu \in \mathbf{Reg}_1^*(E)$, and $(n, m_n) \in \mathbb{N}_0 \times \mathbb{N}$. Suppose that there exists $M > 0$ such that $m_n \leq M$ for all $n \in \mathbb{N}_0$, $\mathcal{P}_{n,m_n}^\mu$ is nonempty for all sufficiently large n, and*

$$\inf_{N \geq M} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_{n,m_n}^\mu\} < \infty.$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Proof of Theorem 3.1. Suppose that the assumption is true. We want to show that if F is entire, then F is a polynomial. Let $Q_{n,m_n}^\mu(z) := \sum_{j=0}^{l_n} q_{n,j} z^j$, with $q_{n,l_n} = 1$ (Q_{n,m_n}^μ is a monic polynomial). By our assumption, there exists $M > 0$ such that $1 \leq l_n \leq m_n \leq M$ for all sufficiently large n. By the definition of standard orthogonal Padé approximants and Lemma 2.4, for every $k = 1, 2, \ldots, m_n$,

$$\begin{aligned}0 =&\langle Q^{\mu}_{n,m_n}F\rangle_{n+k} = \sum_{j=0}^{l_n}\sum_{\nu=0}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k} = \sum_{j=0}^{l_n}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}\\ =&\sum_{\nu=n+k-l_n}^{\infty}\langle F\rangle_{\nu}\langle z^{l_n} p_{\nu}\rangle_{n+k} + \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}\\ =&\langle F\rangle_{n+k-l_n}\langle z^{l_n} p_{n+k-l_n}\rangle_{n+k} + \sum_{\nu=n+k-l_n+1}^{\infty}\langle F\rangle_{\nu}\langle z^{l_n} p_{\nu}\rangle_{n+k}\\ &+ \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}.\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}0 =& \frac{\kappa_{n+k-l_n}}{\kappa_{n+k}}\langle F\rangle_{n+k-l_n} + \sum_{\nu=n+k-l_n+1}^{\infty}\langle F\rangle_{\nu}\langle z^{l_n} p_{\nu}\rangle_{n+k}\\ &+ \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}.\end{aligned} \tag{3.1}$$

Using Lemma 2.8, since $\inf_{N\geq M}\sup_{n\geq N}\{|\zeta| : \zeta \in \mathcal{P}^{\mu}_{n,m_n}\} < \infty$, there are $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sup\{|q_{n,j}| : 0 \leq j \leq l_n, n \geq n_0\} \leq c_1. \tag{3.2}$$

Applying Lemma 2.9, for all $n, k, \nu \in \mathbb{N}_0$ and $j = 1, \ldots, l_n$, then there exists $c_2 > 0$ such that

$$|\langle z^j p_{\nu}\rangle_{n+k}| \leq c_2. \tag{3.3}$$

Since $\mu \in \mathbf{Reg}^*_1(E)$, there are $c_3 > 0$ and $n_1 \in \{n \in \mathbb{N} : n > n_0\}$ such that for all $n \geq n_1$,

$$\frac{\kappa_{n+k-l_n}}{\kappa_{n+k}} \geq c_3. \tag{3.4}$$

The equation (3.1) implies that for all $k = 1, 2, \ldots, m_n$ and for all $n \geq n_1$,

$$\begin{aligned}-\frac{\kappa_{n+k-l_n}}{\kappa_{n+k}}\langle F\rangle_{n+k-l_n} =& \sum_{\nu=n+k-l_n+1}^{\infty}\langle F\rangle_{\nu}\langle z^{l_n} p_{\nu}\rangle_{n+k}\\ &+ \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}.\end{aligned}$$

Then,

$$\begin{aligned}\left|\frac{\kappa_{n+k-l_n}}{\kappa_{n+k}}\langle F\rangle_{n+k-l_n}\right| \leq& \sum_{\nu=n+k-l_n+1}^{\infty}|\langle F\rangle_{\nu}\langle z^{l_n} p_{\nu}\rangle_{n+k}|\\ &+ \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}|\langle F\rangle_{\nu} q_{n,j}\langle z^j p_{\nu}\rangle_{n+k}|.\end{aligned}$$

Hence, by (3.2), (3.3), and (3.4),

$$c_3|\langle F\rangle_{n+k-l_n}| \le c_2 \sum_{\nu=n+k-l_n+1}^{\infty} |\langle F\rangle_\nu| + c_1c_2 \sum_{j=0}^{l_n-1} \sum_{\nu=n+k-j}^{\infty} |\langle F\rangle_\nu|.$$

So,

$$|\langle F\rangle_{n+k-l_n}| \le c_4 \sum_{\nu=n+k-l_n+1}^{\infty} |\langle F\rangle_\nu|,$$

where c_4 is a positive constant which does not depend on n and k. For every $n \ge n_1$, we choose $k = l_n$ in the previous inequality and we obtain, for all $n \ge n_1$,

$$|\langle F\rangle_n| \le c_4 \sum_{\nu=n+1}^{\infty} |\langle F\rangle_\nu|.$$

By Lemma 2.4, since F is entire, $\lim_{n\to\infty} |\langle F\rangle_n|^{1/n} = 0$. Using Lemma 2.6 when $A_n = \langle F\rangle_n$, because the above inequality holds and

$$\lim_{n\to\infty} |A_n|^{1/n} = \lim_{n\to\infty} |\langle F\rangle_n|^{1/n} = 0,$$

then $\langle F\rangle_n = 0$ for all sufficiently large n and F is a polynomial. ∎

Theorem 3.2. *Let $F \in \mathcal{H}(E)$. Suppose that there exists $M > 0$ such that $m_n \le M$ for all $n \in \mathbb{N}_0$, $\mathcal{P}^E_{n,m_n}$ is nonempty for all sufficiently large n, and*

$$\inf_{N\ge M} \sup_{n\ge N} \{|\zeta| : \zeta \in \mathcal{P}^E_{n,m_n}\} < \infty.$$

Then, either F is a polynomial or $\rho_0(F) < \infty$.

Proof of Theorem 3.2. Suppose that the assumption is true. We shall follow the same plan by proving that if F is an entire function, then F is a polynomial. Assume that F is entire. By Lemma 2.4, $\sum_{\nu=0}^{\infty}[F]_\nu \Phi_\nu(z)$ converges uniformly to $F(z)$ on each compact subset of $\mathbb{C}$. Let $Q^E_{n,m_n}(z) := \sum_{j=0}^{l_n} q_{n,j}z^j$, with $q_{n,l_n} = 1$. By our assumption, there exists $M > 0$ such that $1 \le l_n \le m_n \le M$ for all sufficiently large n. By the definition of standard Padé-Faber approximants, Lemma 2.10, and (1.1), for every $k = 1, 2, \ldots, m_n$,

$$
\begin{aligned}
0 =&[Q^E_{n,m}F]_{n+k} = \sum_{j=0}^{l_n}\sum_{\nu=0}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k} = \sum_{j=0}^{l_n}\sum_{\nu=n+k-j}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}\\
=&\sum_{\nu=n+k-l_n}^{\infty}[F]_\nu[z^{l_n}\Phi_\nu]_{n+k} + \sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}\\
=&[z^{l_n}\Phi_{n+k-l_n}]_{n+k}[F]_{n+k-l_n} + \sum_{\nu=n+k-l_n+1}^{\infty}[F]_\nu[z^{l_n}\Phi_\nu]_{n+k}\\
&+\sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}\\
=&(\mathrm{cap}(E))^{l_n}[F]_{n+k-l_n} + \sum_{\nu=n+k-l_n+1}^{\infty}[F]_\nu[z^{l_n}\Phi_\nu]_{n+k}\\
&+\sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}.
\end{aligned}
\tag{3.5}
$$

Using Lemma 2.8, since $\inf_{N\geq M}\sup_{n\geq N}\{|\zeta| : \zeta \in \mathcal{P}^E_{n,m_n}\} < \infty$, there are $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$
\sup\{|q_{n,j}| : 0 \leq j \leq l_n, n \geq n_0\} \leq c_1. \tag{3.6}
$$

Let $\rho > 1$. Using Lemma 2.5 and the M-L inequality, for $j = 0, \ldots, l_n$, $k = 1, \ldots, m_n$, and $n, \nu \in \mathbb{N}_0$,

$$
|[z^j\Phi_\nu]_{n+k}| = \left|\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{z^j\Phi_\nu(z)\Phi'(z)}{\Phi^{n+k+1}(z)}\right| \leq c_2\frac{\rho^\nu}{\rho^n}. \tag{3.7}
$$

Clearly, there exists $c_3 > 0$ such that for all $n \in \mathbb{N}$,

$$
(\mathrm{cap}(E))^{l_n} > c_3. \tag{3.8}
$$

The equation (3.5) implies that for all $k = 1, 2, \ldots, m_n$ and for all $n \in \mathbb{N}$,

$$
\begin{aligned}
-(\mathrm{cap}(E))^{l_n}[F]_{n+k-l_n} =& \sum_{\nu=n+k-l_n+1}^{\infty}[F]_\nu[z^{l_n}\Phi_\nu]_{n+k}\\
&+\sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
|(\mathrm{cap}(E))^{l_n}[F]_{n+k-l_n}| \leq& \sum_{\nu=n+k-l_n+1}^{\infty}|[F]_\nu[z^{l_n}\Phi_\nu]_{n+k}|\\
&+\sum_{j=0}^{l_n-1}\sum_{\nu=n+k-j}^{\infty}|[F]_\nu q_{n,j}[z^j\Phi_\nu]_{n+k}|.
\end{aligned}
$$

Hence, by (3.6), (3.7), and (3.8),

$$\begin{aligned} c_3|[F]_{n+k-l_n}| \leq & \frac{c_2}{\rho^n} \sum_{\nu=n+k-l_n+1}^{\infty} |[F]_\nu \rho^\nu| \\ & + \frac{c_1 c_2}{\rho^n} \sum_{j=0}^{l_n-1} \sum_{\nu=n+k-j}^{\infty} |[F]_\nu \rho^\nu|. \end{aligned}$$

So,

$$|[F]_{n+k-l_n} \rho^n| \leq c_4 \sum_{\nu=n+k-l_n+1}^{\infty} |[F]_\nu \rho^\nu|,$$

where c_4 is a positive constant which does not depend on n and k. For every $n \geq n_0$, we choose $k = l_n$ in the previous inequality and we obtain for all $n \geq n_0$,

$$|[F]_n \rho^n| \leq c_4 \sum_{\nu=n+1}^{\infty} |[F]_\nu \rho^\nu|.$$

By Lemma 2.4, since F is entire, $\lim_{n\to\infty} |[F]_n|^{1/n} = 0$. Using Lemma 2.6 when $A_n = [F]_n \rho^n$, because the above inequality holds and

$$\lim_{n\to\infty} |A_n|^{1/n} = \rho \lim_{n\to\infty} |[F]_n|^{1/n} = 0,$$

$[F]_n = 0$ for all sufficiently large n and F is a polynomial.

■

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