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An Invariant-Preserving Three-Level Linearized Finite Difference Method for the Korteweg-de Vries-Kawahara Equation with Added Viscosity Term*

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Abstract In this work, we aim to study numerical solutions of a nonlinear partial differential equation using a family of three-level linearized finite difference θ-methods for a shallow water waves having surface tension in the form of Viscous Korteweg-de Vries-Kawahara equation with initial and boundary conditions. The model admits two invariants: momentum and energy. The scheme is proved to preserve both momentum and energy in the discrete sense when $\theta = 1/3$. In addition, we proved that the method converges uniformly. The method gives second-order of accuracy in space. Several numerical examples are presented to demonstrate the accuracy and efficiency of the proposed method.

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1. Introduction

Nonlinear partial differential equations have an important role in mathematics, physics, and engineering. They are used as models to describe phenomena in solid state physics, fluid mechanics, chemical physics, plasma physics, geochemistry, and chemical kinematics fields, etc. See [1–5], for example.

One of the well-known nonlinear equations is the Korteweg-de Vries (KdV) equation which appears in the theories of shallow water waves having surface tension [6],[7].

The KdV equation [8] started with experiments by Russell in 1834, followed by theoretical investigations by Rayleigh and Boussinesq around 1870 and, finally, Korteweg and

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De Vries in 1895. The **Korteweg de Vries** equation (1.1). Its dimensionless from is given by

$$u_t + u_{xxx} - 6uu_x = 0, \tag{1.1}$$

and the **Kawahara** equation is given by

$$u_t + auu_x + bu_{xxx} - cu_{xxxxx} = 0, \tag{1.2}$$

The **Korteweg de Vries - Kawahara** [9] equation is given by

$$u_t - \eta u_{xxxxx} + u_{xxx} + uu_x + u_x = 0,\; x \in [a, b], t \in [0, T], \tag{1.3}$$

$\eta \in \mathbb{R}$, where $u = u(x, t)$ is fluid velocity and t, x are temporal and spatial variables, respectively.

Ceballos et al. [10] studied the KdV–Kawahara equation in a bounded domain and obtained some numerical results. They showed that the KdV–Kawahara equation has smoothing effects that are uniform with respect to the size of the interval and they also proposed a simple finite different scheme for the problem and proved its stability. Natali [11] established the nonlinear stability of solitary traveling-wave solutions for the KdV–Kawahara equation. The main approach used to determine the stability of solitary traveling-waves come from the theory developed by Albert in [12]. Badali et al.[13] introduced a new solution for the KdV–Kawahara equation. Lie group analysis was used to carry out the integration of this equations. Coclite et al.[14] proved the well-posedness of the classical solutions for the Cauchy problem associated with the KdV–Kawahara equation.

In this article, we consider the KdV-Kawahara Equation with added viscosity term as follows:

$$u_t - \eta u_{xxxxx} + u_{xxx} + uu_x + u_x = \gamma u_{xx}, \quad x \in [a, b], \quad t \in [0, T], \tag{1.4}$$

with initial condition

$$u(x, 0) = f(x), \quad x \in [a, b], \tag{1.5}$$

and boundary conditions

$$\begin{aligned} &u(a, t) = u(b, t) = 0, \\ &u_x(a, t) = u_x(b, t) = 0, \\ &u_{xx}(a, t) = u_{xx}(b, t) = 0, \quad t \in [0, T], \end{aligned} \tag{1.6}$$

where $\gamma \geq 0$.

The nonlinear term uu_x in equation (1.4) is difficult to approximate accurately using a simple numerical method. In 1981, Guo and Sanz-Serna [15] used two finite different sums to approximate the term in the KdV equation. Since then, there have been several attempts to estimate the term uu_x by splitting it into a sum of derivatives of u. Each term in the sum is estimated by using the central difference with or without some average values. In the early years, two layers of numerical solution from consecutive time steps are involved in the schemes. See [16–20] for example. In [18] and [21], Pan and Zhang developed linearized difference schemes which are three-level and conservative implicit for both Rosenau–RLW and general Rosenau–RLW equations. The second-order accuracy and unconditional stability were also proved. In [22], Wongsaijai and Poochinapan used the term Rosenua-KdV-RLW to refer to all three equations and applied an average value of two explicit finite differences to approximate the nonlinear term. This gives a collection of second-order θ-schemes proven to preserve invariants when $\theta = 1/3$. We adapt their idea to the present problem. In their more recent works, Sun, Wang and Zhang [23] prove the pointwise estimates of a conservative difference scheme for Burgers' equation. Adapting the ideas from these works, Darayon et al. [24] applied second-order operators to the viscous Burgers-Poisson system and obtained point-wise convergence analysis. We adopt all these techniques in designing our finite difference method and in our analysis.

The content of this paper is organized as follows. In Section 2, we discuss some analytic properties of the solution that are essential to the stability and error analysis. In Section 3, we outline basic settings for finite difference method framework and propose a collection of θ-schemes to approximate the solution of the KdV-Kawahara Equation with added viscosity term. We also prove the invariant properties of the proposed schemes and the existence and uniqueness of the resulting numerical solution in this section. In Section 4, we present the error analysis to show that the method has second-order accuracy. In Section 5, we discuss an extension to the non-homogeneous and test the proposed scheme on various examples to verify the theoretical results. Finally, discussion and conclusions are made in Sections 6-7.

2. Analytic Property

We will show that the solution of (1.4) with conditions (1.5) and (1.6) preserves the momentum and energy.

Theorem 2.1 (Momentum Preserving). *Let u be a solution of (1.4)–(1.6). Define*

$$Q(t) = \int_a^b u(x,t)\,dx.$$

If $u_0 \in L^1(a,b)$, then

$$Q(t) = Q(0), \qquad \textit{for all} \quad t > 0.$$

Proof. From (1.4), we have $u_t = \eta u_{xxxxx} - u_{xxx} - uu_x - u_x + \gamma u_{xx}$. Therefore,

$$\begin{aligned}\frac{d}{dt}Q(t) &= \int_a^b \eta u_{xxxxx}(x,t)\,dx - \int_a^b u_{xxx}(x,t)\,dx - \int_a^b u(x,t)u_x(x,t)\,dx \\ &\quad - \int_a^b u_x(x,t)\,dx + \int_a^b \gamma u_{xx}(x,t)\,dx \\ &= \Big[\eta u_{xxxx}(x,t) - u_{xx}(x,t) - \frac{1}{2}u^2(x,t) - u(x,t) + \gamma u_x(x,t)\Big]_a^b = 0\end{aligned}$$

because of the boundary conditions (1.6). Therefore, the function $Q(t)$ does not change over time. That is,

$$Q(t) = Q(0), \qquad \text{for all} \quad t > 0.$$

■

Theorem 2.2 (Energy Preserving). *Let u be a solution of (1.4)–(1.6). Define*

$$E(t) = \int_a^b u^2(x,t)\,dx.$$

If $u_0 \in L^2(a,b)$*, then*

$$E(t) \leq E(0), \quad \text{for all} \quad t > 0.$$

The inequality becomes equality when $\gamma = 0$*.*

Proof. Upon differentiating E, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E(t) &= \eta \int_a^b u(x,t)u_{xxxxx}(x,t)\,dx - \int_a^b u(x,t)u_{xxx}(x,t)\,dx \\
&\quad - \int_a^b u^2(x,t)u_x(x,t)\,dx - \int_a^b u(x,t)u_x(x,t)\,dx + \gamma \int_a^b u(x,t)u_{xx}(x,t)\,dx \\
&= \Big[\eta u_x(x,t)u_{xxx}(x,t) + \frac{\eta}{2}u_{xx}^2(x,t) - u(x,t)u_{xx}(x,t) + \frac{1}{2}u_x^2(x,t) \\
&\quad - \frac{1}{3}u^3(x,t) - \frac{1}{2}u^2(x,t)\Big]_a^b + \gamma \int_a^b u(x,t)u_{xx}(x,t)\,dx \\
&= -\gamma \int_a^b u_x^2(x,t)\,dx \leq 0.
\end{aligned}$$

Therefore, the function $E(t)$ does not increase over time; that is,

$$E(t) \leq E(0), \quad \text{for all} \quad t > 0.$$

If $\gamma = 0$, then the proof above shows that $E'(0) = 0$. This gives $E(t) = E(0)$. This completes the proof. ∎

3. Finite Difference Method

3.1. Discretization

In this section, we present a complete description of our finite difference scheme and an algorithm for the formulation of the problem (1.4)-(1.6).We discretize the spatial domain $[a,b]$ into the partition $x_i = a + ih,\ i = 0, ..., M-1$ for the step size $h = (b-a)/M$. For a time step $\tau = T/N$, we define $t^n = n\tau$ for $n = 0, ..., N$. Let u_i^n be an approximation of $u(x_i, t^n)$. The numerical solution $\mathbf{u}^n$ at the time t^n is taken from the solution space Z_h defined as

$$\begin{aligned}
Z_h = \{u = [u_i]_{i=-2}^{M+2} \mid u_0 = u_M = 0,\ D_h^- u_0 = 0, D_h^+ u_M = 0, \\
D_h^- D_h^- u_0 = 0,\ D_h^+ D_h^+ u_M = 0\}.
\end{aligned}$$

For any $\mathbf{u}, \mathbf{v} \in Z_h$, we define an inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle_h = h \sum_{i=1}^{M-1} u_i v_i$$

which allows us to define the discrete L_2 norm

$$\|\mathbf{u}\|_h^2 = \langle \mathbf{u}, \mathbf{u} \rangle_h. \tag{3.1}$$

We also define the discrete uniform norm

$$\|\mathbf{u}\|_{h,\infty} = \max_{i=1,...,M-1} |u_i|, \tag{3.2}$$

for later use in the analysis.

With the set up above, we define finite difference methods for approximating some partial derivatives and some averages of u at (x_i, t^n) as follows:

$$\mathcal{D}_\tau^+ u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau}, \quad \mathcal{D}_\tau^0 u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$
$$u_i^{n+1/2} = \frac{u_i^{n+1} + u_i^n}{2}, \quad \overline{u}_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2},$$
$$D_h^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \quad D_h^0 u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}, \quad D_h^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h}.$$

Note that we can apply these operators to a vector in Z_h by acting on each element of the vector.

Using the boundary conditions in the definition of Z_h and summation by parts, one can easily prove the following results.

Lemma 3.1. *For any two mesh functions $\mathbf{u}, \mathbf{v} \in Z_h$, we have*

$$\begin{aligned}
\langle D_h^0 \mathbf{u}, \mathbf{v} \rangle_h &= -\langle \mathbf{u}, D_h^0 \mathbf{v} \rangle_h \\
\langle D_h^+ \mathbf{u}, \mathbf{v} \rangle_h &= -\langle \mathbf{u}, D_h^- \mathbf{v} \rangle_h \\
\langle D_h^+ D_h^- \mathbf{u}, \mathbf{v} \rangle_h &= -\langle D_h^+ \mathbf{u}, D_h^+ \mathbf{v} \rangle_h \\
\langle \mathbf{u}, D_h^+ D_h^- \mathbf{u} \rangle_h &= -\langle D_h^+ \mathbf{u}, D_h^+ \mathbf{u} \rangle_h = -\|D_h^+ \mathbf{u}\|_h^2.
\end{aligned}$$

Lemma 3.2. *Let $\mathbf{u} \in Z_h$, The following relations hold.*

$$\begin{aligned}
\langle D_h^0 \mathbf{u}, \mathbf{u} \rangle_h &= 0 \\
\langle D_h^+ D_h^- D_h^0 \mathbf{u}, \mathbf{u} \rangle_h &= 0 \\
\langle D_h^+ D_h^+ D_h^- D_h^- D_h^0 \mathbf{u}, \mathbf{u} \rangle_h &= 0.
\end{aligned}$$

Lemma 3.3. *Let $\mathbf{u} \in Z_h$, The following relations hold.*

$$\begin{aligned}
\langle D_h^0 \mathbf{u}, D_h^0 \mathbf{u} \rangle_h &\le \langle D_h^+ \mathbf{u}, D_h^+ \mathbf{u} \rangle_h \\
\|D_h^0 \mathbf{u}\|_{h,\infty} &\le \|D_h^+ \mathbf{u}\|_{h,\infty}.
\end{aligned}$$

3.2. Formulation of the Scheme

We propose a family of explicit three-level finite difference schemes for solving the Viscous Korteweg-de Vries-Kawahara equation (1.4) with conditions (1.5) and (1.6) as follows: for $\theta \in [0, 1]$, find $\mathbf{u}^{n+1} \in Z_h$ satisfying

$$\begin{aligned}
&\mathcal{D}_\tau^+ u_i^0 - \eta D_h^+ D_h^+ D_h^- D_h^- D_h^0 u_i^{1/2} + D_h^+ D_h^- D_h^0 u_i^{1/2} \\
&\quad + \frac{1}{2} \Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i + D_h^0 u_i^{1/2} = \gamma D_h^+ D_h^- u_i^{1/2} \qquad (3.3)
\end{aligned}$$

$$u_i^0 = u_0(x_i) \qquad (3.4)$$

for $i = 1 \ldots, M-1$, and

$$\begin{aligned}
&\mathcal{D}_\tau^0 u_i^n - \eta D_h^+ D_h^+ D_h^- D_h^- D_h^0 \overline{u}_i^n + D_h^+ D_h^- D_h^0 \overline{u}_i^n \\
&\quad + \frac{1}{2} \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i + D_h^0 \overline{u}_i^n = \gamma D_h^+ D_h^- \overline{u}_i^n \qquad (3.5)
\end{aligned}$$

for $i = 1\ldots, M-1$ and $n = 1\ldots, N-1$, where

$$\Psi_\theta(\mathbf{u},\mathbf{v})_i := 2\theta u_i D_h^0 v_i + (1-\theta)D_h^0\,(uv)_i\,. \tag{3.6}$$

For the implementation of the three-level scheme, the first two initial steps of the solutions are required. Thus, we use (3.3)–(3.4) to compute $\mathbf{u}^1$. Then, in (3.5) we use $\mathbf{u}^{n-1}$ and $\mathbf{u}^n$ to compute $\mathbf{u}^{n+1}$.

In the initial step, the equation (3.3), with $i = 1,\ldots,M-1$, can be written in a matrix-vector form:

$$\begin{aligned}
&\mathbf{u}^1 + \left(\frac{\eta}{2}\right)\frac{\tau}{2h^5}A_1^0\mathbf{u}^1 + \frac{1}{2}\left(\frac{\tau}{2h^3}\right)A_2^0\mathbf{u}^1 + \frac{1}{2}\left(\frac{\tau\theta}{2h}\right)A_3^0\mathbf{u}^1\\
&+\frac{1}{2}\left(\frac{\tau(1-\theta)}{4h}\right)A_4^0\mathbf{u}^1 + \frac{\tau}{4h}A_5^0\mathbf{u}^1 - \frac{1}{2}\left(\frac{\gamma\tau}{h^2}\right)A_6^0\mathbf{u}^1\\
&\quad= \mathbf{u}^0 - \left(\frac{\eta}{2}\right)\frac{\tau}{2h^5}A_1^0\mathbf{u}^0 - \frac{1}{2}\left(\frac{\tau}{2h^3}\right)A_2^0\mathbf{u}^0 - \frac{1}{2}\left(\frac{\tau\theta}{2h}\right)A_3^0\mathbf{u}^0\\
&\quad-\frac{1}{2}\left(\frac{\tau(1-\theta)}{4h}\right)A_4^0\mathbf{u}^0 - \frac{\tau}{4h}A_5^0\mathbf{u}^0 + \frac{1}{2}\left(\frac{\gamma\tau}{h^2}\right)A_6^0\mathbf{u}^0.
\end{aligned} \tag{3.7}$$

For $n = 1,2,\ldots,N-1$, the equation (3.5), $i = 1,\ldots,M-1$, can be written as

$$\begin{aligned}
&\frac{1}{2}\mathbf{u}^{n+1} + \left(\frac{\eta}{2}\right)\frac{\tau}{2h^5}A_1^n\mathbf{u}^{n+1} + \frac{1}{2}\left(\frac{\tau}{2h^3}\right)A_2^n\mathbf{u}^{n+1} + \frac{1}{2}\left(\frac{\tau\theta}{2h}\right)A_3^n\mathbf{u}^{n+1}\\
&+\frac{1}{2}\left(\frac{\tau(1-\theta)}{4h}\right)A_4^n\mathbf{u}^{n+1} + \frac{\tau}{4h}A_5^n\mathbf{u}^{n+1} - \frac{1}{2}\left(\frac{\gamma\tau}{h^2}\right)A_6^n\mathbf{u}^{n+1}\\
&\quad= \frac{1}{2}\mathbf{u}^{n-1} - \left(\frac{\eta}{2}\right)\frac{\tau}{2h^5}A_1^n\mathbf{u}^{n-1} - \frac{1}{2}\left(\frac{\tau}{2h^3}\right)A_2^n\mathbf{u}^{n-1} - \frac{1}{2}\left(\frac{\tau\theta}{2h}\right)A_3^n\mathbf{u}^{n-1}\\
&\quad-\frac{1}{2}\left(\frac{\tau(1-\theta)}{4h}\right)A_4^n\mathbf{u}^{n-1} - \frac{\tau}{4h}A_5^n\mathbf{u}^{n-1} + \frac{1}{2}\left(\frac{\gamma\tau}{h^2}\right)A_6^n\mathbf{u}^{n-1}.
\end{aligned} \tag{3.8}$$

Here, the coefficients A_1^n, A_2^n, A_3^n, A_4^n, A_5^n and A_6^n are circulant pentadiagonal matrices or circulant tridiagonal matrices whose nonzero entries on the i^{th} row are given by $(-5,4,-1,0,5,-4,1)$, $(2,-1,0,-2,1)$, $(-u_i^n,0,u_i^n)$, $(-u_{i-1}^n,0,u_{i+1}^n)$, $(-1,0,1)$ and $(1,-2,1)$, respectively.

Next, we will show the numerical solution obtained from the proposed scheme is bounded. The following result is necessary for the proof of boundedness.

Lemma 3.4. *(Discrete Sobolev's inequality) [25–27]. Define $L = b-a$. If $\mathbf{u}^n \in Z_h$, then*

$$\|\mathbf{u}^n\|_{h,\infty} \le \frac{\sqrt{L}}{2}\|D_h^+\mathbf{u}^n\|_h. \tag{3.9}$$

Theorem 3.5. *Let $\mathbf{u}^1$ and $\mathbf{u}^{n+1}$ be solutions of the schemes (3.3) and (3.5). Then, $\|D_h^+\mathbf{u}^1\|_h$ and $\|D_h^+\mathbf{u}^{n+1}\|_h$ are bounded. It follows that $\|\mathbf{u}^1\|_{h,\infty}$ and $\|\mathbf{u}^{n+1}\|_{h,\infty}$ are also bounded.*

Proof. We take inner product of (3.3), $i = 1,\ldots,M-1$ with $-D_h^+D_h^-\mathbf{u}^{1/2}$ to get

$$\begin{aligned}
&\langle \mathcal{D}_\tau^+\mathbf{u}^0, -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h - \langle \eta D_h^+D_h^+D_h^-D_h^-D_h^0\mathbf{u}^{1/2}, -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h\\
&+\langle D_h^+D_h^-D_h^0\mathbf{u}^{1/2}, -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h + \langle D_h^0\mathbf{u}^{1/2}, -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h\\
&-\langle \gamma D_h^+D_h^-\mathbf{u}^{1/2}, -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h = -\langle \frac{1}{2}\Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2}), -D_h^+D_h^-\mathbf{u}^{1/2}\rangle_h.
\end{aligned} \tag{3.10}$$

Now we estimate each item in (3.10) as follows. For the first term on the left-hand side, using (3.1) and Lemma 3.1, we have

$$\begin{aligned}\mathcal{A}_1 &= \langle \mathcal{D}_\tau^+ \mathbf{u}^0, -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h \\ &= \Big\langle \frac{1}{2} D_h^+ \mathbf{u}^1 + \frac{1}{2} D_h^+ \mathbf{u}^0, \frac{1}{\tau} D_h^+ \mathbf{u}^1 - \frac{1}{\tau} D_h^+ \mathbf{u}^0 \Big\rangle_h \\ &= \frac{1}{2\tau} \|D_h^+ \mathbf{u}^1\|_h^2 - \frac{1}{2\tau} \|D_h^+ \mathbf{u}^0\|_h^2. \end{aligned} \tag{3.11}$$

For the second term to the fourth term on the left-hand side, using Lemma 3.2, we have

$$\mathcal{A}_2 = -\langle \eta D_h^+ D_h^+ D_h^- D_h^- D_h^0 \mathbf{u}^{1/2}, -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h = 0, \tag{3.12}$$

$$\mathcal{A}_3 = \langle D_h^+ D_h^- D_h^0 \mathbf{u}^{1/2}, -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h = 0, \tag{3.13}$$

$$\mathcal{A}_4 = \langle D_h^0 \mathbf{u}^{1/2}, -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h = 0. \tag{3.14}$$

For the fifth term on the left-hand side, we have

$$\begin{aligned}\mathcal{A}_5 &= -\langle \gamma D_h^+ D_h^- \mathbf{u}^{1/2}, -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h \\ &= \gamma \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h^2. \end{aligned} \tag{3.15}$$

Note that

$$D_h^0 (uv)_i = \frac{1}{2} u_{i+1} D_h^+ v_i + (D_h^0 u_i) v_i + \frac{1}{2} u_{i-1} D_h^+ v_{i-1}. \tag{3.16}$$

For the first term on the right-hand side of (3.10), using the Cauchy–Schwarz inequality, Young's inequality, Lemma 3.3 and (3.6), (3.16), we have

$$\begin{aligned}\mathcal{A}_6 &= -\langle \frac{1}{2} \Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2}), -D_h^+ D_h^- \mathbf{u}^{1/2} \rangle_h \\ &= \frac{1}{2} h \sum_{i=1}^{M-1} (\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i)(D_h^+ D_h^- u_i^{1/2}) \\ &= \frac{1}{2} h \sum_{i=1}^{M-1} (2\theta u_i^0 D_h^0 u_i^{1/2} + (1-\theta) D_h^0 (u^0 u^{1/2})_i)(D_h^+ D_h^- u_i^{1/2}) \\ &= \theta h \sum_{i=1}^{M-1} (u_i^0 D_h^0 u_i^{1/2})(D_h^+ D_h^- u_i^{1/2}) \\ &\quad + \frac{(1-\theta)}{2} h \sum_{i=1}^{M-1} \Big(\frac{1}{2} u_{i+1}^0 D_h^+ u_i^{1/2} + (D_h^0 u_i^0) u_i^{1/2} + \frac{1}{2} u_{i-1}^0 D_h^+ u_{i-1}^{1/2} \Big)(D_h^+ D_h^- u_i^{1/2}) \\ &\le \theta C \|D_h^0 \mathbf{u}^{1/2}\|_h \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h + \frac{1-\theta}{4} C \|D_h^+ \mathbf{u}^{1/2}\|_h \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h \\ &\quad + \frac{(1-\theta)}{2} \|\mathbf{u}^{1/2}\|_{h,\infty} \Big(\|D_h^0 \mathbf{u}^0\|_h \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h \Big) + \frac{1-\theta}{4} C \|D_h^+ \mathbf{u}^{1/2}\|_h \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h \\ &\le C \|D_h^+ \mathbf{u}^{1/2}\|_h \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h \\ &\le \frac{C^2}{4\gamma} \|D_h^+ \mathbf{u}^{1/2}\|_h^2 + \gamma \|D_h^+ D_h^- \mathbf{u}^{1/2}\|_h^2. \end{aligned} \tag{3.17}$$

We have that (3.10) can be written as

$$\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5 = \mathcal{A}_6,$$

which simplifies to

$$\frac{1}{2\tau}\Big(\|D_h^+\mathbf{u}^1\|_h^2 - \|D_h^+\mathbf{u}^0\|_h^2\Big) + \gamma\|D_h^+D_h^-\mathbf{u}^{1/2}\|_h^2 \le \frac{C^2}{4\gamma}\|D_h^+\mathbf{u}^{1/2}\|_h^2 + \gamma\|D_h^+D_h^-\mathbf{u}^{1/2}\|_h^2$$

$$\|D_h^+\mathbf{u}^1\|_h^2 - \|D_h^+\mathbf{u}^0\|_h^2 \le \frac{\tau C^2}{4\gamma}\Big(\|D_h^+\mathbf{u}^1\|_h^2 + \|D_h^+\mathbf{u}^0\|_h^2\Big).$$

Hence,

$$\|D_h^+\mathbf{u}^1\|_h^2 - \|D_h^+\mathbf{u}^0\|_h^2 \le C\tau\Big(\|D_h^+\mathbf{u}^1\|_h^2 + \|D_h^+\mathbf{u}^0\|_h^2\Big). \tag{3.18}$$

Define

$$B^1 = \|D_h^+\mathbf{u}^1\|_h^2, \quad B^0 = \|D_h^+\mathbf{u}^0\|_h^2, \tag{3.19}$$

then (3.18) can be rewritten as follows:

$$\begin{aligned} B^1 - B^0 &\le C\tau(B^1 + B^0)\\ (1-C\tau)B^1 &\le (1+C\tau)B^0. \end{aligned}$$

If τ is sufficiently small, i.e. $\tau \le \frac{1}{2C}$, then

$$B^1 \le \frac{(1+C\tau)}{(1-C\tau)}B^0 \le (1+4C\tau)B^0. \tag{3.20}$$

Hence, $\|D_h^+\mathbf{u}^1\|_h$ is bounded. It follows that $\|\mathbf{u}^1\|_{h,\infty}$ is also bounded. Next, assume that $\|D_h^+\mathbf{u}^k\|_h$ is bounded for $k = 2,\dots,n$. Then, there exists a constant C that satisfies

$$\|D_h^+u^k\|_h \le \frac{2}{\sqrt{L}}C, \quad \|u^k\|_{h,\infty} \le C, \quad k = 1,2,\dots,n. \tag{3.21}$$

We will show that $\|D_h^+\mathbf{u}^{n+1}\|_h, n \ge 1$, is bounded. We first take inner product of (3.5), $i = 1,\dots,M-1$ with $-2D_h^+D_h^-\overline{\mathbf{u}}^n$ to get

$$\begin{aligned} &\langle \mathcal{D}_\tau^0\mathbf{u}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h - \langle \eta D_h^+D_h^+D_h^-D_h^-D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h\\ &+ \langle D_h^+D_h^-D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h + \langle D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h\\ &- \langle \gamma D_h^+D_h^-\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h = -\langle \frac{1}{2}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h. \end{aligned} \tag{3.22}$$

Now estimate each item in (3.22) as follows. For the first term on the left-hand side, using (3.1), Lemma 3.1, we have

$$\begin{aligned} \mathcal{B}_1 &= \langle \mathcal{D}_\tau^0\mathbf{u}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h\\ &= 2\Big\langle \frac{1}{2}D_h^+\mathbf{u}^{n+1} + \frac{1}{2}D_h^+\mathbf{u}^{n-1}, \frac{1}{2\tau}D_h^+\mathbf{u}^{n+1} - \frac{1}{2\tau}D_h^+\mathbf{u}^{n-1}\Big\rangle_h\\ &= \frac{1}{2\tau}\|D_h^+\mathbf{u}^{n+1}\|_h^2 - \frac{1}{2\tau}\|D_h^+\mathbf{u}^{n-1}\|_h^2. \end{aligned} \tag{3.23}$$

For the second term to the fourth term on the left-hand side, using Lemma 3.2, we have

$$\mathcal{B}_2 = -\langle \eta D_h^+D_h^+D_h^-D_h^-D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h = 0, \tag{3.24}$$

$$\mathcal{B}_3 = \langle D_h^+D_h^-D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h = 0, \tag{3.25}$$

$$\mathcal{B}_4 = \langle D_h^0\overline{\mathbf{u}}^n, -2D_h^+D_h^-\overline{\mathbf{u}}^n\rangle_h = 0. \tag{3.26}$$

For the fifth term on the left-hand side, we have

$$\begin{aligned}\mathcal{B}_5 &= -\langle \gamma D_h^+ D_h^- \overline{\mathbf{u}}^n, -2D_h^+ D_h^- \overline{\mathbf{u}}^n \rangle_h \\ &= 2\gamma \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h^2. \end{aligned} \tag{3.27}$$

For the first term on the right-hand side of (3.22), using the Cauchy–Schwarz inequality, Young's inequality, Lemma 3.3 and (3.6), (3.16), (3.21), we have

$$\begin{aligned}\mathcal{B}_6 &= -\langle \frac{1}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n), -2D_h^+ D_h^- \overline{\mathbf{u}}^n \rangle_h \\ &= h \sum_{i=1}^{M-1} (\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i)(D_h^+ D_h^- \overline{u}_i^n) \\ &= h \sum_{i=1}^{M-1} (2\theta u_i^n D_h^0 \overline{u}_i^n + (1-\theta) D_h^0 (u^n \overline{u}^n)_i)(D_h^+ D_h^- \overline{u}_i^n) \\ &= 2\theta h \sum_{i=1}^{M-1} (u_i^n D_h^0 \overline{u}_i^n)(D_h^+ D_h^- \overline{u}_i^n) \\ &\quad + (1-\theta) h \sum_{i=1}^{M-1} \Big(\frac{1}{2} u_{i+1}^n D_h^+ \overline{u}_i^n + (D_h^0 u_i^n)\overline{u}_i^n + \frac{1}{2} u_{i-1}^n D_h^+ \overline{u}_{i-1}^n\Big)(D_h^+ D_h^- \overline{u}_i^n) \\ &\leq 2C \|D_h^+ \overline{\mathbf{u}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h \\ &\leq \frac{C^2}{2\gamma} \|D_h^+ \overline{\mathbf{u}}^n\|_h^2 + 2\gamma \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h^2. \end{aligned} \tag{3.28}$$

We have that (3.22) can be written as

$$\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5 = \mathcal{B}_6,$$

which simplifies to

$$\begin{aligned}\frac{1}{2\tau}\Big(\|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2\Big) + 2\gamma \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h^2 &\leq \frac{C^2}{2\gamma} \|D_h^+ \overline{\mathbf{u}}^n\|_h^2 \\ &\quad + 2\gamma \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h^2 \\ \|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2 &\leq \frac{\tau C^2}{2\gamma} \|D_h^+ \mathbf{u}^{n+1}\|_h^2 \\ &\quad + \frac{\tau C^2}{2\gamma} \|D_h^+ \mathbf{u}^{n-1}\|_h^2. \end{aligned}$$

Hence,

$$\|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2 \leq C\tau \Big(\|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \|D_h^+ \mathbf{u}^{n-1}\|_h^2\Big). \tag{3.29}$$

Define

$$B^n = \|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \|D_h^+ \mathbf{u}^n\|_h^2, \tag{3.30}$$

then (3.29) can be rewritten as follows:

$$\begin{aligned} &B^n - B^{n-1} \leq C\tau (B^n + B^{n-1}) \\ &(1 - C\tau) B^n \leq (1 + C\tau) B^{n-1}. \end{aligned}$$

If τ is sufficiently small, i.e. $\tau \leq \frac{1}{2C}$, then

$$B^n \leq \frac{(1+C\tau)}{(1-C\tau)} B^{n-1} \leq (1+4C\tau)B^{n-1} \leq e^{(4C\tau n)} B^1 \leq e^{(4CT)} B^1. \tag{3.31}$$

Hence, $\|D_h^+ \mathbf{u}^{n+1}\|_h$ is bounded. It follows that $\|\mathbf{u}^{n+1}\|_{h,\infty}$ is also bounded. ■

3.3. Stability Analysis

We will show that numerical solution of (3.3)–(3.5) preserves the invariants in the discrete sense.

Theorem 3.6. *Define*

$$Q_h^n = h \sum_{i=1}^{M-1} u_i^{n+1/2} + \frac{\theta\tau}{2} h \sum_{i=1}^{M-1} u_i^n D_h^0 u_i^{n+1}. \tag{3.32}$$

If $\mathbf{u}^n$ *is a solution of (3.3)–(3.5), then*

$$Q_h^n = Q_h^{n-1} = \ldots = Q_h^0 = h \sum_{i=1}^{M-1} u_i^0 + \frac{\theta\tau}{2} h \sum_{i=1}^{M-1} u_i^0 D_h^0 u_i^1, \tag{3.33}$$

for any $\theta \in [0,1]$.

Proof. Multiply (3.5) by τh and take summation on $i = 1, \ldots, M-1$ to arrive at

$$\tau h \sum_{i=1}^{M-1} (D_\tau^0 u_i^n) - \eta\tau h \sum_{i=1}^{M-1} (D_h^+ D_h^+ D_h^- D_h^- D_h^0 \overline{u}_i^n) + \tau h \sum_{i=1}^{M-1} (D_h^+ D_h^- D_h^0 \overline{u}_i^n)$$
$$+ \frac{1}{2}\tau h \sum_{i=1}^{M-1} \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i + \tau h \sum_{i=1}^{M-1} (D_h^0 \overline{u}_i^n) = \gamma\tau h \sum_{i=1}^{M-1} (D_h^+ D_h^- u_i^n).$$

We get

$$\frac{1}{2} h \sum_{i=1}^{M-1} (u_i^{n+1} - u_i^{n-1}) - \eta\tau h \sum_{i=1}^{M-1} (D_h^+ D_h^+ D_h^- D_h^- D_h^0 \overline{u}_i^n) + \tau h \sum_{i=1}^{M-1} (D_h^+ D_h^- D_h^0 \overline{u}_i^n)$$
$$+ \theta\tau h \sum_{i=1}^{M-1} (u_i^n D_h^0 \overline{u}_i^n) + \frac{1-\theta}{2} \tau h \sum_{i=1}^{M-1} D_h^0 (u_i^n \overline{u}_i^n) + \tau h \sum_{i=1}^{M-1} (D_h^0 \overline{u}_i^n) = \gamma\tau h \sum_{i=1}^{M-1} (D_h^+ D_h^- u_i^n).$$

Since all terms other than the first and fourth terms vanish, we obtain

$$\frac{1}{2} h \sum_{i=1}^{M-1} (u_i^{n+1} - u_i^{n-1}) + \theta\tau h \sum_{i=1}^{M-1} (u_i^n D_h^0 \overline{u}_i^n) = 0$$

and hence,

$$\frac{1}{2} h \sum_{i=1}^{M-1} (u_i^{n+1} - u_i^{n-1}) + \frac{\theta\tau}{4} h \sum_{i=1}^{M-1} (u_i^n u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1} + u_i^n u_{i+1}^{n-1} - u_i^n u_{i-1}^{n-1}) = 0.$$

After rewriting, we get

$$\frac{1}{2} h \sum_{i=1}^{M-1} (u_i^{n+1} - u_i^{n-1}) + \frac{\theta\tau}{4} h \sum_{i=1}^{M-1} (u_i^n u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1} + u_{i-1}^n u_i^{n-1} - u_{i+1}^n u_i^{n-1}) = 0$$

and thus,

$$\frac{1}{2}h\sum_{i=1}^{M-1}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{\theta\tau}{2}h\sum_{i=1}^{M-1}\left(u_i^nD_h^0u_i^{n+1}-u_i^{n-1}D_h^0u_i^n\right)=0.$$

This shows that $Q_h^n - Q_h^{n-1} = 0$ for $n = 1, \ldots, N-1$. Similarly, multiply (3.3) by τh and take summation over i to obtain

$$h\sum_{i=1}^{M-1}u_i^1 = h\sum_{i=1}^{M-1}u_i^0-\frac{\theta\tau}{2}h\sum_{i=1}^{M-1}u_i^0D_h^0u_i^1=0.$$

Substitute this into Q_h^0 to find that

$$Q_h^0 = h\sum_{i=1}^{M-1}u_i^0+\frac{\theta\tau}{4}h\sum_{i=1}^{M-1}u_i^0D_h^0u_i^1$$

as needed. ■

Theorem 3.7. *Define*

$$E_h^n=\frac{1}{2}\|\mathbf{u}^{n+1}\|_h^2+\frac{1}{2}\|\mathbf{u}^n\|_h^2. \tag{3.34}$$

Let $\mathbf{u}^n$ *be a solution of (3.3) and (3.5). If* $\theta = \frac{1}{3}$, *then*

$$E_h^n \le E_h^{n-1}\le \ldots \le E_h^0 \le \|\mathbf{u}^0\|_h^2 \tag{3.35}$$

for any $n > 0$.

Proof. First, consider the summation

$$\begin{aligned}
&h\sum_{i=1}^{M-1}\left[u_i^n\left(D_h^0\overline{u}_i^n\right)\overline{u}_i^n+D_h^0\left(u_i^n\overline{u}_i^n\right)\overline{u}_i^n\right]\\
&\quad=h\sum_{i=1}^{M-1}\left[u_i^n\overline{u}_{i+1}^n\overline{u}_i^n-u_i^n\overline{u}_{i-1}^n\overline{u}_i^n+u_{i+1}^n\overline{u}_{i+1}^n\overline{u}_i^n-u_{i-1}^n\overline{u}_{i-1}^n\overline{u}_i^n\right]=0.
\end{aligned} \tag{3.36}$$

We multiply (3.5) by $2\tau h\overline{u}_i^n$ and take summation over i to obtain

$$\begin{aligned}
&2\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_\tau^0u_i^n)-2\eta\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^+D_h^+D_h^-D_h^-D_h^0\overline{u}_i^n)+2\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^+D_h^-D_h^0\overline{u}_i^n)\\
&+\frac{1}{2}(2\tau h)\sum_{i=1}^{M-1}\overline{u}_i^n\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i+2\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^0\overline{u}_i^n)=2\gamma\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^+D_h^-u_i^n).
\end{aligned}$$

Using (3.36) and Lemma 3.1, we get

$$\frac{1}{2}h\sum_{i=1}^{M-1}\left[\left(u_i^{n+1}\right)^2-\left(u_i^{n-1}\right)^2\right]-2\eta\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^+D_h^+D_h^-D_h^-D_h^0\overline{u}_i^n)$$
$$+2\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^+D_h^-D_h^0\overline{u}_i^n)+(3\theta-1)\tau h\sum_{i=1}^{M-1}D_h^0\left(u_i^n\overline{u}_i^n\right)\overline{u}_i^n$$
$$+2\tau h\sum_{i=1}^{M-1}\overline{u}_i^n(D_h^0\overline{u}_i^n)=-2\gamma\tau h\sum_{i=1}^{M-1}(D_h^+\overline{u}_i^n)^2. \tag{3.37}$$

Take $\theta=\frac{1}{3}$. Using Lemma 3.2 and (3.37), we arrive at

$$\frac{1}{2}h\sum_{i=1}^{M-1}\left[\left(u_i^{n+1}\right)^2-\left(u_i^{n-1}\right)^2\right]+2\gamma\tau h\sum_{i=1}^{M-1}(D_h^+\overline{u}_i^n)^2=0. \tag{3.38}$$

This shows that

$$E_h^n-E_h^{n-1}+2\gamma\tau h\sum_{i=1}^{M-1}(D_h^+\overline{u}_i^n)^2=0, \tag{3.39}$$

for $n=1,\dots,N-1$.
Using similar idea, one can show from (3.3)–(3.4) that

$$\|\mathbf{u}^1\|_h^2-\|\mathbf{u}^0\|_h^2+2\gamma\tau h\sum_{i=1}^{M-1}(D_h^+u_i^{1/2})^2=0. \tag{3.40}$$

Substitute this into the definition of E_h^0 to derive the desired result. ∎

3.4. Existence and uniqueness

In this subsection, we prove the solvability of solutions for the scheme (3.3)-(3.5). This guarantees the existence and uniqueness of our numerical solution.

Theorem 3.8. *The finite difference schemes (3.3)-(3.5) is uniquely solvable.*

Proof. We first consider from (3.3)

$$\left(\frac{u_i^1-u_i^0}{\tau}\right)-\eta D_h^+D_h^+D_h^-D_h^-D_h^0\left(\frac{u_i^1+u_i^0}{2}\right)+D_h^+D_h^-D_h^0\left(\frac{u_i^1+u_i^0}{2}\right)$$
$$+\frac{1}{2}\Psi_\theta\left(\mathbf{u}^0,\left(\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)\right)_i+D_h^0\left(\frac{u_i^1+u_i^0}{2}\right)=\gamma D_h^+D_h^-\left(\frac{u_i^1+u_i^0}{2}\right).$$

We get

$$\frac{1}{\tau}u_i^1-\frac{\eta}{2}D_h^+D_h^+D_h^-D_h^-D_h^0u_i^1+\frac{1}{2}D_h^+D_h^-D_h^0u_i^1$$
$$+\frac{1}{4}\Psi_\theta(\mathbf{u}^0,\mathbf{u}^1)_i+\frac{1}{2}D_h^0u_i^1-\frac{\gamma}{2}D_h^+D_h^-u_i^1=0, \tag{3.41}$$

We take the inner product of (3.41), $i=1,\dots,M-1$ with $\mathbf{u}^1$ to obtain

$$\frac{1}{\tau}\langle\mathbf{u}^1,\mathbf{u}^1\rangle_h-\frac{\eta}{2}\langle D_h^+D_h^+D_h^-D_h^-D_h^0\mathbf{u}^1,\mathbf{u}^1\rangle_h+\frac{1}{2}\langle D_h^+D_h^-D_h^0\mathbf{u}^1,\mathbf{u}^1\rangle_h$$
$$+\frac{1}{4}\langle\Psi_\theta(\mathbf{u}^0,\mathbf{u}^1),\mathbf{u}^1\rangle_h+\frac{1}{2}\langle D_h^0\mathbf{u}^1,\mathbf{u}^1\rangle_h-\frac{\gamma}{2}\langle D_h^+D_h^-\mathbf{u}^1,\mathbf{u}^1\rangle_h=0.$$

Using Lemma 3.2, we have

$$\frac{1}{\tau}\langle \mathbf{u}^1, \mathbf{u}^1\rangle_h + \frac{1}{4}\langle \Psi_\theta(\mathbf{u}^0, \mathbf{u}^1), \mathbf{u}^1\rangle_h - \frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h = 0,$$

or

$$\frac{1}{\tau}\langle \mathbf{u}^1, \mathbf{u}^1\rangle_h = -\frac{1}{4}\langle \Psi_\theta(\mathbf{u}^0, \mathbf{u}^1), \mathbf{u}^1\rangle_h + \frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h. \tag{3.42}$$

Consider the summation

$$\begin{aligned} &h\sum_{i=1}^{M-1}\left[u_i^0\left(D_h^0 u_i^1\right) + D_h^0\left(u_i^0 u_i^1\right)\right]u_i^1 \\ &\quad = h\sum_{i=1}^{M-1}\left[u_i^0 u_{i+1}^1 u_i^1 - u_i^0 u_{i-1}^1 u_i^1 + u_{i+1}^0 u_{i+1}^1 u_i^1 - u_{i-1}^0 u_{i-1}^1 u_i^1\right] = 0. \end{aligned} \tag{3.43}$$

Now estimate each item in (3.42) as follows. For the first term, we have

$$\mathcal{D}_1 \equiv \frac{1}{\tau}\langle \mathbf{u}^1, \mathbf{u}^1\rangle_h = \frac{1}{\tau}\|\mathbf{u}^1\|_h^2. \tag{3.44}$$

For the second term, using Cauchy-Schwarz inequality, Young's inequality, Theorem 3.5 and (3.43), we obtain

$$\begin{aligned} \mathcal{D}_2 &\equiv -\frac{1}{4}\langle \Psi_\theta(\mathbf{u}^0, \mathbf{u}^1), \mathbf{u}^1\rangle_h \\ &= -\frac{1}{4}h\sum_{i=1}^{M-1}\left[2\theta u_i^0 D_h^0 u_i^1 + (1-\theta)D_h^0(u_i^0 u_i^1)\right]u_i^1 \\ &\le Ch\sum_{i=1}^{M-1}|D_h^0 u_i^1||u_i^1| \\ &\le C\left(\|D_h^+\mathbf{u}^1\|_h^2 + \|\mathbf{u}^1\|_h^2\right). \end{aligned} \tag{3.45}$$

For the third term, using Lemma 3.1, we have

$$\mathcal{D}_3 \equiv \frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h = -\frac{\gamma}{2}\|D_h^+\mathbf{u}^1\|_h^2. \tag{3.46}$$

We have that (3.42) can be written as

$$\mathcal{D}_1 = \mathcal{D}_2 + \mathcal{D}_3,$$

which simplifies to

$$\begin{aligned} \frac{1}{\tau}\|\mathbf{u}^1\|_h^2 &\le C\left(\|D_h^+\mathbf{u}^1\|_h^2 + \|\mathbf{u}^1\|_h^2\right) - \frac{\gamma}{2}\|D_h^+\mathbf{u}^1\|_h^2 \\ \|\mathbf{u}^1\|_h^2 &\le C\tau\left(\|D_h^+\mathbf{u}^1\|_h^2 + \|\mathbf{u}^1\|_h^2\right) - C\tau\|D_h^+\mathbf{u}^1\|_h^2. \end{aligned} \tag{3.47}$$

If τ is sufficiently small, i.e. $\tau \le \dfrac{1}{2C}$, then

$$(1 - C\tau)\|\mathbf{u}^1\|_h^2 = 0.$$

Consequently,

$$\|\mathbf{u}^1\|_h^2 = 0.$$

This implies that there uniquely exists the trivial solution satisfying (3.3).
Similarly, from (3.5), we have

$$\left(\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}\right)-\eta D_h^+D_h^+D_h^-D_h^-D_h^0\left(\frac{u_i^{n+1}+u_i^{n-1}}{2}\right)+D_h^+D_h^-D_h^0\left(\frac{u_i^{n+1}+u_i^{n-1}}{2}\right)$$
$$+\frac{1}{2}\Psi_\theta\left(\mathbf{u}^n,\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\right)_i+D_h^0\left(\frac{u_i^{n+1}+u_i^{n-1}}{2}\right)=\gamma D_h^+D_h^-\left(\frac{u_i^{n+1}+u_i^{n-1}}{2}\right).$$

We get

$$\frac{1}{2\tau}u_i^{n+1}-\frac{\eta}{2}D_h^+D_h^+D_h^-D_h^-D_h^0u_i^{n+1}+\frac{1}{2}D_h^+D_h^-D_h^0u_i^{n+1}$$
$$+\frac{1}{4}\Psi_\theta(\mathbf{u}^n,\mathbf{u}^{n+1})_i+\frac{1}{2}D_h^0u_i^{n+1}-\frac{\gamma}{2}D_h^+D_h^-u_i^{n+1}=0, \tag{3.48}$$

We take the inner product of (3.48), $i=1,\ldots,M-1$ with $\mathbf{u}^{n+1}$ to obtain

$$\frac{1}{2\tau}\langle\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h-\frac{\eta}{2}\langle D_h^+D_h^+D_h^-D_h^-D_h^0\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h+\frac{1}{2}\langle D_h^+D_h^-D_h^0\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h$$
$$+\frac{1}{4}\langle\Psi_\theta(\mathbf{u}^n,\mathbf{u}^{n+1}),\mathbf{u}^{n+1}\rangle_h+\frac{1}{2}\langle D_h^0\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h-\frac{\gamma}{2}\langle D_h^+D_h^-\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h=0.$$

Using Lemma 3.2, we have

$$\frac{1}{2\tau}\langle\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h+\frac{1}{4}\langle\Psi_\theta(\mathbf{u}^n,\mathbf{u}^{n+1}),\mathbf{u}^{n+1}\rangle_h-\frac{\gamma}{2}\langle D_h^+D_h^-\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h=0,$$

or

$$\frac{1}{2\tau}\langle\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h=-\frac{1}{4}\langle\Psi_\theta(\mathbf{u}^n,\mathbf{u}^{n+1}),\mathbf{u}^{n+1}\rangle_h+\frac{\gamma}{2}\langle D_h^+D_h^-\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h. \tag{3.49}$$

Considering the summation, we obtain

$$\begin{aligned}&h\sum_{i=1}^{M-1}\left[u_i^n\left(D_h^0u_i^{n+1}\right)+D_h^0\left(u_i^nu_i^{n+1}\right)\right]u_i^{n+1}\\&=h\sum_{i=1}^{M-1}\left[u_i^nu_{i+1}^{n+1}u_i^{n+1}-u_i^nu_{i-1}^{n+1}u_i^{n+1}+u_{i+1}^nu_{i+1}^{n+1}u_i^{n+1}-u_{i-1}^nu_{i-1}^{n+1}u_i^{n+1}\right]\\&=0.\end{aligned} \tag{3.50}$$

Now estimate each item in (3.49) as follows. For the first term, we have

$$\mathcal{F}_1\equiv\frac{1}{2\tau}\langle\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h=\frac{1}{2\tau}\|\mathbf{u}^{n+1}\|_h^2. \tag{3.51}$$

For the second term, using Cauchy-Schwarz inequality, Young's inequality, Theorem 3.5 and (3.50), we obtain

$$\begin{aligned}\mathcal{F}_2&\equiv-\frac{1}{4}\langle\Psi_\theta(\mathbf{u}^n,\mathbf{u}^{n+1}),\mathbf{u}^{n+1}\rangle_h\\&\leq C\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2+\|\mathbf{u}^{n+1}\|_h^2\right).\end{aligned} \tag{3.52}$$

For the third term, using Lemma 3.1, we have

$$\mathcal{F}_3\equiv\frac{\gamma}{2}\langle D_h^+D_h^-\mathbf{u}^{n+1},\mathbf{u}^{n+1}\rangle_h=-\frac{\gamma}{2}\|D_h^+\mathbf{u}^{n+1}\|_h^2. \tag{3.53}$$

We have that (3.49) can be written as

$$\mathcal{F}_1=\mathcal{F}_2+\mathcal{F}_3,$$

which simplifies to

$$\frac{1}{2\tau}\|\mathbf{u}^{n+1}\|_h^2 \le C\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2 + \|\mathbf{u}^{n+1}\|_h^2\right) - \frac{\gamma}{2}\|D_h^+\mathbf{u}^{n+1}\|_h^2$$
$$\|\mathbf{u}^{n+1}\|_h^2 \le 2C\tau\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2 + \|\mathbf{u}^{n+1}\|_h^2\right) - 2C\tau\|D_h^+\mathbf{u}^{n+1}\|_h^2.$$

If τ is sufficiently small, i.e. $\tau \le \dfrac{1}{2C}$, then

$$(1-2C\tau)\|\mathbf{u}^{n+1}\|_h^2 = 0.$$

Consequently,

$$\|\mathbf{u}^{n+1}\|_h^2 = 0.$$

This implies that there uniquely exists the trivial solution satisfying (3.5). Hence, $\mathbf{u}^1$ and $\mathbf{u}^{n+1}$ are uniquely solvable, and this completes the proof of the theorem.

■

4. Convergence Analysis

In this section, we study the error analysis of the numerical solution obtained from the proposed scheme.

Define $e_i^n = v_i^n - u_i^n$, where v_i^n is the exact solution of (1.4)-(1.6) at the point (x_i, t^n), and u_i^n is the numerical solution obtained from (3.3)-(3.5). From (3.3)–(3.4), we have that the truncation errors at the time step t^0, denoted $\mathcal{T}^0$, satisfies

$$\mathcal{D}_\tau^+ e_i^0 - \eta D_h^+D_h^+D_h^-D_h^-D_h^0 e_i^{1/2} + D_h^+D_h^-D_h^0 e_i^{1/2} + \frac{1}{2}[\Psi_\theta(\mathbf{v}^0,\mathbf{v}^{1/2})_i - \Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2})_i]$$
$$+ D_h^0 e_i^{1/2} = \gamma D_h^+D_h^- e_i^{1/2} + \mathcal{T}_i^0. \tag{4.1}$$

Using Taylor series expansion, one can show there exists c_1 such that

$$|\mathcal{T}_i^0| \le c_1(\tau + h^2).$$

On the other hand, we have from (3.5) that the truncation errors at the time step t^n, $n \ge 1$, denoted $\mathcal{T}^n$, satisfies

$$\mathcal{D}_\tau^0 e_i^n - \eta D_h^+D_h^+D_h^-D_h^-D_h^0 \bar{e}_i^n + D_h^+D_h^-D_h^0 \bar{e}_i^n + \frac{1}{2}[\Psi_\theta(\mathbf{v}^n,\bar{\mathbf{v}}^n)_i - \Psi_\theta(\mathbf{u}^n,\bar{\mathbf{u}}^n)_i]$$
$$+ D_h^0 \bar{e}_i^n = \gamma D_h^+D_h^- \bar{e}_i^n + \mathcal{T}_i^n. \tag{4.2}$$

Using Taylor series expansion, one can show there exists c_2 such that

$$|\mathcal{T}_i^n| \le c_2(\tau^2 + h^2).$$

Theorem 4.1. *The solution $\mathbf{u}^n$ of the scheme (3.3)-(3.5) converges in the sense*

$$\|\mathbf{e}^n\|_{h,\infty} \le c_6(\tau^{3/2} + h^2)$$

if $(\tau^{3/2}+h^2) \le \min\left\{\sqrt{2c_4}, \frac{1}{c_6}\right\}$ *and* $\tau \le \min\left\{1, \frac{1}{2c_4}, \frac{1}{4c_5}\right\}$,
where

$$
\begin{aligned}
c_3 &= \max_{x\in[a,b],\ t\in[0,T]} \{|v(x,t)|, |v_x(x,t)|\},\\
c_4 &= \max\left\{\Big(\frac{c_3^2(1+3\theta)^2+c_3^2(1-\theta)^2}{32\gamma}+\frac{c_3^2(1-\theta)^2}{8\gamma}\Big), \frac{2c_1}{\gamma}\right\},\\
c_5 &= \max\left\{\frac{1}{4}\Big(\frac{5c_3^2((1+3\theta)^2+(1-\theta)^2)}{16\gamma}+\frac{5((3+\theta)^2+5(1-\theta)^2)}{16\gamma}+\frac{5c_3^2(1-\theta)^2}{4\gamma}\Big),\right.\\
&\quad \left.\Big(\frac{5c_3^2\theta^2}{\gamma}+\frac{5c_3^2(1-\theta)^2}{8\gamma}+\frac{5c_3^2(1-\theta)^2}{4\gamma}\Big), \frac{5c_2}{\gamma}\right\},\\
c_6 &= \frac{\sqrt{L}}{2}\sqrt{4c_4 e^{c_5 T}}.
\end{aligned}
$$

Proof. Since $\mathbf{e}^0=\mathbf{0}$, we have

$$
\begin{aligned}
\Psi_\theta(\mathbf{v}^0,\mathbf{v}^{1/2})-\Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2}) &= \Psi_\theta(\mathbf{v}^0,\mathbf{e}^{1/2})+\Psi_\theta(\mathbf{e}^0,\mathbf{v}^{1/2})-\Psi_\theta(\mathbf{e}^0,\mathbf{e}^{1/2})\\
&= \Psi_\theta(\mathbf{v}^0,\mathbf{e}^{1/2}).
\end{aligned}
$$

Thus, the error equation (4.1) is reduced to

$$
\begin{aligned}
&\frac{e_i^1}{\tau}-\frac{1}{2}\eta D_h^+D_h^+D_h^-D_h^-D_h^0 e_i^1+\frac{1}{2}D_h^+D_h^-D_h^0e_i^1\\
&\quad+\frac{1}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1)_i+\frac{1}{2}D_h^0e_i^1=\frac{\gamma}{2}D_h^+D_h^-e_i^1+\mathcal{T}_i^0.
\end{aligned}\tag{4.3}
$$

Take the inner product of the system (4.3), where $i=1,\dots,M-1$, with $-D_h^+D_h^-\mathbf{e}^1$ to get

$$
\begin{aligned}
&\langle\frac{\mathbf{e}^1}{\tau},-D_h^+D_h^-\mathbf{e}^1\rangle_h-\langle\frac{1}{2}\eta D_h^+D_h^+D_h^-D_h^-D_h^0\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\frac{1}{2}D_h^+D_h^-D_h^0\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h\\
&+\langle\frac{1}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\frac{1}{2}D_h^0\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h\\
&=\langle\frac{\gamma}{2}D_h^+D_h^-\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\mathcal{T}^0,-D_h^+D_h^-\mathbf{e}^1\rangle_h.
\end{aligned}\tag{4.4}
$$

Using Lemma 3.1 and Lemma 3.2, (4.4) is reduced to

$$
\begin{aligned}
&\langle\frac{\mathbf{e}^1}{\tau},-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\frac{1}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),-D_h^+D_h^-\mathbf{e}^1\rangle_h\\
&=\langle\frac{\gamma}{2}D_h^+D_h^-\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\mathcal{T}^0,-D_h^+D_h^-\mathbf{e}^1\rangle_h.
\end{aligned}
$$

Rewrite

$$
\begin{aligned}
&\langle\frac{\mathbf{e}^1}{\tau},-D_h^+D_h^-\mathbf{e}^1\rangle_h-\langle\frac{\gamma}{2}D_h^+D_h^-\mathbf{e}^1,-D_h^+D_h^-\mathbf{e}^1\rangle_h\\
&=-\langle\frac{1}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),-D_h^+D_h^-\mathbf{e}^1\rangle_h+\langle\mathcal{T}^0,-D_h^+D_h^-\mathbf{e}^1\rangle_h.
\end{aligned}\tag{4.5}
$$

The left-hand side of (4.5) can be simplified into

$$\frac{1}{\tau}\|D_h^+\mathbf{e}^1\|_h^2 + \frac{\gamma}{2}\|D_h^+D_h^-\mathbf{e}^1\|_h^2. \tag{4.6}$$

For the first term on the right-hand side of (4.5), using Lemma 3.1 and Lemma 3.3, we arrive at

$$\begin{aligned}
\langle\frac{1}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1), D_h^+D_h^-\mathbf{e}^1\rangle_h &= \frac{\theta}{2}h\sum_{i=1}^{M-1} v_i^0 D_h^0 e_i^1 (D_h^+D_h^- e_i^1)\\
&\quad + \frac{1-\theta}{4}h\sum_{i=1}^{M-1}\left(\frac{1}{2}v_{i+1}^0 D_h^+ e_i^1 + (D_h^0 v_i^0)e_i^1 + \frac{1}{2}v_{i-1}^0 D_h^+ e_{i-1}^1\right)(D_h^+D_h^- e_i^1)\\
&\le \frac{(1+3\theta)}{8}c_3\|D_h^+\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h + \frac{(1-\theta)}{4}c_3\|\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h\\
&\quad + \frac{(1-\theta)}{8}c_3\|D_h^+\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h\\
&\le \frac{c_3^2(3\theta+1)^2 + c_3^2(1-\theta)^2}{32\gamma}\|D_h^+\mathbf{e}^1\|_h^2 + \frac{3\gamma}{8}\|D_h^+D_h^-\mathbf{e}^1\|_h^2\\
&\quad + \frac{c_3^2(1-\theta)^2}{8\gamma}\|\mathbf{e}^1\|_h^2.
\end{aligned} \tag{4.7}$$

As for the last term in (4.5), using Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned}
\langle\mathcal{T}^0, -D_h^+D_h^-\mathbf{e}^1\rangle_h &\le \|\mathcal{T}^0\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h\\
&\le \frac{2}{\gamma}\|\mathcal{T}^0\|_h^2 + \frac{\gamma}{8}\|D_h^+D_h^-\mathbf{e}^1\|_h^2.
\end{aligned} \tag{4.8}$$

Substituting (4.6), (4.7), and (4.8) into (4.5), we obtain

$$\begin{aligned}
\frac{1}{\tau}\|D_h^+\mathbf{e}^1\|_h^2 &\le c_4\|D_h^+\mathbf{e}^1\|_h^2 + c_4(\tau+h^2)^2\\
(1-c_4\tau)\|D_h^+\mathbf{e}^1\|_h^2 &\le c_4\tau(\tau+h^2)^2.
\end{aligned} \tag{4.9}$$

For $c_4\tau \le \frac{1}{2}$, we have

$$\|D_h^+\mathbf{e}^1\|_h^2 \le 2c_4\tau(\tau+h^2)^2. \tag{4.10}$$

From this, we get

$$\|D_h^+\mathbf{e}^1\|_h \le \sqrt{2c_4}(\tau^{3/2}+h^2) \tag{4.11}$$

if $\tau < 1$. From the hypothesis of the theorem, we have

$$\|D_h^+\mathbf{e}^1\|_h \le \frac{2}{\sqrt{L}}.$$

This shows that

$$\|\mathbf{e}^1\|_{h,\infty} \le 1.$$

For the time step $t^n, n > 0$, we use induction on n and proceed in a similar manner. Assume

$$\|D_h^+\mathbf{e}^k\|_h \le \frac{2}{\sqrt{L}}. \tag{4.12}$$

That is, $\|\mathbf{e}^k\|_{h,\infty} \le 1$, for $k = 0, 1, \dots, n$. We shall show that (4.12) holds for $k = n+1$. First, we take inner product of (4.2), where $i = 1, \dots, M-1$, with $-2D_h^+D_h^-\overline{\mathbf{e}}^n$ to get

$$\begin{aligned}&\langle \mathcal{D}_\tau^0 \mathbf{e}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h - \langle \eta D_h^+D_h^+D_h^-D_h^-D_h^0\overline{\mathbf{e}}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h\\&+\langle D_h^+D_h^-D_h^0\overline{\mathbf{e}}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h + \langle \frac{1}{2}[\Psi_\theta(\mathbf{v}^n,\overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)], -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h\\&+\langle D_h^0\overline{\mathbf{e}}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h = \langle \gamma D_h^+D_h^-\overline{\mathbf{e}}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h + \langle \mathcal{T}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h.\end{aligned} \tag{4.13}$$

Using Lemma 3.2, we get

$$\begin{aligned}&\langle \mathcal{D}_\tau^0 \mathbf{e}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h - \langle \gamma D_h^+D_h^-\overline{\mathbf{e}}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h\\&= -\langle \frac{1}{2}[\Psi_\theta(\mathbf{v}^n,\overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)], -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h + \langle \mathcal{T}^n, -2D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h.\end{aligned} \tag{4.14}$$

The left-hand side of (4.14) can be simplified into

$$\frac{1}{2\tau}\left(\|D_h^+\mathbf{e}^{n+1}\|_h^2 - \|D_h^+\mathbf{e}^{n-1}\|_h^2\right) + 2\gamma\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2. \tag{4.15}$$

As for the first term on the right-hand side of (4.14), note that

$$\Psi_\theta(\mathbf{v}^n,\overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n) = \Psi_\theta(\mathbf{v}^n,\overline{\mathbf{e}}^n) + \Psi_\theta(\mathbf{e}^n,\overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{e}^n,\overline{\mathbf{e}}^n).$$

Using (3.16), Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}&\langle \Psi_\theta(\mathbf{v}^n,\overline{\mathbf{e}}^n), D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h = 2\theta h\sum_{i=1}^{M-1} v_i^n D_h^0\overline{e}_i^n(D_h^+D_h^-\overline{e}_i^n)\\&\quad + (1-\theta)h\sum_{i=1}^{M-1}\left(\frac{1}{2}v_{i+1}^n D_h^+\overline{e}_i^n + (D_h^0 v_i^n)\overline{e}_i^n + \frac{1}{2}v_{i-1}^n D_h^+\overline{e}_{i-1}^n\right)(D_h^+D_h^-\overline{e}_i^n)\\&\quad\le \frac{(3\theta+1)}{2}c_3\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h + (1-\theta)c_3\|\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\&\quad+\frac{(1-\theta)}{2}c_3\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\&\quad\le \frac{5c_3^2((3\theta+1)^2+(1-\theta)^2)}{16\gamma}\|D_h^+\overline{\mathbf{e}}^n\|_h^2 + \frac{5c_3^2(1-\theta)^2}{4\gamma}\|\overline{\mathbf{e}}^n\|_h^2 + \frac{3\gamma}{5}\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2.\end{aligned} \tag{4.16}$$

Similarly, we have

$$\begin{aligned}\langle \Psi_\theta(\mathbf{e}^n,\overline{\mathbf{v}}^n), D_h^+D_h^-\overline{\mathbf{e}}^n\rangle_h &\le 2\theta c_3\|\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h + \frac{(1-\theta)}{2}c_3\|\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\&\quad+(1-\theta)c_3\|D_h^+\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\&\quad+\frac{(1-\theta)}{2}c_3\|\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\&\le\left(\frac{5c_3^2\theta^2}{\gamma}+\frac{5c_3^2(1-\theta)^2}{8\gamma}\right)\|\mathbf{e}^n\|_h^2 + \frac{5c_3^2(1-\theta)^2}{4\gamma}\|D_h^+\mathbf{e}^n\|_h^2\\&\quad+\frac{4\gamma}{5}\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2.\end{aligned} \tag{4.17}$$

We find that

$$
\begin{aligned}
-\langle \Psi_\theta(\mathbf{e}^n, \overline{\mathbf{e}}^n), D_h^+ D_h^- \overline{\mathbf{e}}^n \rangle_h &= -2\theta h \sum_{i=1}^{M-1} e_i^n D_h^0 \overline{e}_i^n (D_h^+ D_h^- \overline{e}_i^n) \\
&\quad - (1-\theta) h \sum_{i=1}^{M-1} \left(\frac{1}{2} e_{i+1}^n D_h^+ \overline{e}_i^n + (D_h^0 e_i^n) \overline{e}_i^n + \frac{1}{2} e_{i-1}^n D_h^+ \overline{e}_{i-1}^n \right) (D_h^+ D_h^- \overline{e}_i^n) \\
&\le \frac{(3+\theta)}{2} \|D_h^+ \overline{\mathbf{e}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{e}}^n\|_h + \frac{(1-\theta)}{2} \|D_h^+ \overline{\mathbf{e}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{e}}^n\|_h \\
&\le \frac{5(\theta+3)^2 + 5(1-\theta)^2}{16\gamma} \|D_h^+ \overline{\mathbf{e}}^n\|_h^2 + \frac{2\gamma}{5} \|D_h^+ D_h^- \overline{\mathbf{e}}^n\|_h^2. \qquad (4.18)
\end{aligned}
$$

As for the last term in (4.14), we have

$$
\begin{aligned}
-2\langle \mathcal{T}^n, D_h^+ D_h^- \overline{\mathbf{e}}^n \rangle_h &\le 2\|\mathcal{T}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{e}}^n\|_h \\
&\le \frac{5}{\gamma} \|\mathcal{T}^n\|_h^2 + \frac{\gamma}{5} \|D_h^+ D_h^- \overline{\mathbf{e}}^n\|_h^2. \qquad (4.19)
\end{aligned}
$$

Substituting (4.15)–(4.19) into (4.14), we obtain

$$
\begin{aligned}
\|D_h^+ \mathbf{e}^{n+1}\|_h^2 - \|D_h^+ \mathbf{e}^{n-1}\|_h^2 &\le 2c_5\tau \Big(\|D_h^+ \mathbf{e}^{n+1}\|_h^2 + \|D_h^+ \mathbf{e}^{n-1}\|_h^2 + \|D_h^+ \mathbf{e}^n\|_h^2 \Big) \\
&\quad + 2c_5\tau(\tau^2 + h^2)^2. \qquad (4.20)
\end{aligned}
$$

Define

$$
S^n = \|D_h^+ \mathbf{e}^n\|_h^2 + \|D_h^+ \mathbf{e}^{n+1}\|_h^2.
$$

One can show that (4.20) leads to

$$
\begin{aligned}
S^n - S^{n-1} &\le 2c_5\tau(S^n + S^{n-1}) + 2c_5\tau(\tau^2 + h^2)^2 \\
(1 - 2c_5\tau)\, S^n &\le (1 + c_5\tau)\, S^{n-1} + 2c_5\tau(\tau^2 + h^2)^2.
\end{aligned}
$$

For $2c_5\tau \le \dfrac{1}{2}$, we arrive at

$$
S^n \le 2\,(1 + c_5\tau)\, S^{n-1} + 4c_5\tau(\tau^2 + h^2)^2. \qquad (4.21)
$$

A use of discrete Grönwall inequality shows

$$
\begin{aligned}
S^n &\le 2e^{c_5 n\tau} \Big(S^0 + 4c_5 n\tau(\tau^2 + h^2)^2 \Big) \\
&\le 2e^{c_5 T} \Big(S^0 + 4c_5 T(\tau^2 + h^2)^2 \Big). \qquad (4.22)
\end{aligned}
$$

Take the value of S^0 from (4.11), we get

$$
\begin{aligned}
S^n &\le 2e^{c_5 T} \Big(2c_4(\tau^{3/2} + h^2)^2 + 4c_5 T(\tau^2 + h^2)^2 \Big) \\
&\le \frac{c_6^2}{\left(\frac{\sqrt{L}}{2}\right)^2} (\tau^{3/2} + h^2)^2. \qquad (4.23)
\end{aligned}
$$

We have from (4.23) that

$$
\|D_h^+ \mathbf{e}^{n+1}\|_h \le \frac{2c_6}{\sqrt{L}} (\tau^{3/2} + h^2). \qquad (4.24)
$$

Using Lemma 3.4, we get $\|\mathbf{e}^{n+1}\|_{h,\infty} \le c_6(\tau^{3/2} + h^2)$ as needed. ■

5. Numerical Results

In this section, we test the proposed scheme on various examples. We begin by extending the scheme to a more general setting.

5.1. An Extension To The Non-homogeneous Cases

For the non-homogeneous KdV-Kawahara equation

$$u_t - \eta u_{xxxxx} + u_{xxx} + uu_x + u_x - \gamma u_{xx} = f(x,t), \quad x \in [a,b], \quad t \in [0,T], \tag{5.1}$$

with conditions (1.5) and (1.6), we use the following difference scheme

$$\begin{aligned}&\mathcal{D}_\tau^+ u_i^0 - \eta D_h^+ D_h^+ D_h^- D_h^- D_h^0 u_i^{1/2} + D_h^+ D_h^- D_h^0 u_i^{1/2}\\ &+ \frac{1}{2}\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i + D_h^0 u_i^{1/2} - \gamma D_h^+ D_h^- u_i^{1/2} = f(x_i, t^{1/2}),\end{aligned} \tag{5.2}$$

for $i = 1 \ldots, M-1$ and

$$\begin{aligned}&\mathcal{D}_\tau^0 u_i^n - \eta D_h^+ D_h^+ D_h^- D_h^- D_h^0 \overline{u}_i^n + D_h^+ D_h^- D_h^0 \overline{u}_i^n\\ &+ \frac{1}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i + D_h^0 \overline{u}_i^n - \gamma D_h^+ D_h^- \overline{u}_i^n = f(x_i, t^n),\end{aligned} \tag{5.3}$$

for $i = 1 \ldots, M-1$, for $n = 1 \ldots, N-1$. We can prove that difference scheme (5.2)-(5.3) has second-order of accuracy in space with respect to the uniform norm using the idea similar to Theorem 4.1.

5.2. Numerical Tests

Some numerical experiments were undergone to verify the correctness of our theoretical analysis. We chose three test problems, both viscous ($\gamma \neq 0$) and inviscous ($\gamma = 0$), to measure the performance on the proposed scheme.

In Theorem 3.7, the optimal value of θ is $1/3$. Therefor, the simulations using $\theta = 1/3$ was conducted against other values of θ to compare their performances. The errors were measured by the discrete Euclidean norm $\|\cdot\|_h$ and the discrete uniform norm $\|\cdot\|_{h,\infty}$ defined in (3.1) and (3.2), respectively.

Let r be the order of accuracy with respect to the norm $\|\cdot\|$. If the exact value u **is known**, order of convergence is computed using the formula

$$r = \log_2\left(\frac{\|\mathbf{e}_h\|}{\|\mathbf{e}_{h/2}\|}\right),$$

where $\mathbf{e}_h$ and $\mathbf{e}_{h/2}$ are the errors resulting from the divisions of the domain into subranges of sizes h and $h/2$ respectively. When u **is not known**, order is computed using the formula

$$r = \log_2\left(\frac{\|\mathbf{u}_h - \mathbf{u}_{h/2}\|}{\|\mathbf{u}_{h/2} - \mathbf{u}_{h/4}\|}\right),$$

where $\mathbf{u}_h, \mathbf{u}_{h/2}$ and $\mathbf{u}_{h/4}$ are the estimated solutions resulting from the divisions of the domain into subranges of size $h, h/2$ and $h/4$ respectively.

Example 1. Accuracy Test for the Homogeneous Viscous case.
We used the homogeneous example with no exact solution. The initial data is given by

$$u(x,0) = \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right).$$

The simulation was conducted on the domain $[-40, 40]$ with parameters $T = 1$, $\eta = 1$, $\gamma = 1$ and $\tau = h$. Orders showing optimal convergence rates are given in Table 1.

M	$\theta = 0$		$\theta = 1/3$		$\theta = 2/3$		$\theta = 1$	
	$\|u_h - u_{h/2}\|_h$	order	$\|u_h - u_{h/2}\|_h$	order	$\|u_h - u_{h/2}\|_h$	order	$\|u_h - u_{h/2}\|_h$	order
80	1.7203e-03	1.98	1.6846e-03	1.97	1.7097e-03	1.99	1.7354e-03	1.99
160	4.3573e-04	2.00	4.3075e-04	1.99	4.3063e-04	1.99	4.3597e-04	2.00
320	1.0904e-04	2.00	1.0806e-04	2.00	1.0808e-04	2.00	1.0910e-04	2.00
640	2.7265e-05		2.7039e-05		2.7054e-05		2.7280e-05	
1280								

TABLE 1. Orders showing optimal convergence rates for the viscous homogeneous problem (Example 1) with $\theta = 0, 1/3, 2/3, 1$.

Example 2. Accuracy Test for the Non-Homogeneous Viscous case. We used the non-homogeneous example with exact solution

$$u(x, t) = e^{-(x-t)^2}.$$

The simulation was conducted on the domain $[-20, 40]$ with parameters $T = 1$, $\eta = 1$, $\gamma = 1$ and $\tau = h$. The errors and orders of accuracy are given in Tables 2 and 3. The comparison between exact and approximate solutions is represented in Figure 1 for $T = 0, 5, 15, 20$.

M	$\theta = 0$				$\theta = 1/3$			
	$\|\mathbf{u} - \mathbf{u}_h\|_h$		$\|\mathbf{u} - \mathbf{u}_h\|_{h,\infty}$		$\|\mathbf{u} - \mathbf{u}_h\|_h$		$\|\mathbf{u} - \mathbf{u}_h\|_{h,\infty}$	
	error	order	error	order	error	order	error	order
160	8.8058e-02		9.9948e-02		8.7100e-02		9.8985e-02	
320	1.9590e-02	2.17	2.1627e-02	2.21	1.9405e-02	2.17	2.1447e-02	2.21
640	4.7898e-03	2.03	5.2382e-03	2.05	4.7462e-03	2.03	5.1961e-03	2.05
1280	1.1911e-03	2.01	1.3028e-03	2.01	1.1803e-03	2.01	1.2916e-03	2.01
2560	2.9738e-04	2.00	3.2533e-04	2.00	2.9471e-04	2.00	3.2265e-04	2.00

TABLE 2. Errors and orders showing optimal convergence rates for the viscous non-homogeneous problem (Example 2) with $\theta = 0, 1/3$.

M	$\theta = 2/3$				$\theta = 1$			
	$\|\mathbf{u} - \mathbf{u}_h\|_h$		$\|\mathbf{u} - \mathbf{u}_h\|_{h,\infty}$		$\|\mathbf{u} - \mathbf{u}_h\|_h$		$\|\mathbf{u} - \mathbf{u}_h\|_{h,\infty}$	
	error	order	error	order	error	order	error	order
160	8.6311e-02		9.8029e-02		8.5692e-02		9.7082e-02	
320	1.9249e-02	2.16	2.1268e-02	2.20	1.9124e-02	2.16	2.1089e-02	2.20
640	4.7095e-03	2.03	5.1539e-03	2.04	4.6798e-03	2.03	5.1119e-03	2.04
1280	1.1713e-03	2.01	1.2804e-03	2.01	1.1639e-03	2.01	1.2695e-03	2.01
2560	2.9245e-04	2.00	3.1996e-04	2.00	2.9062e-04	2.00	3.1728e-04	2.00

TABLE 3. Errors and orders showing optimal convergence rates for the viscous non-homogeneous problem (Example 2) with $\theta = 2/3, 1$.

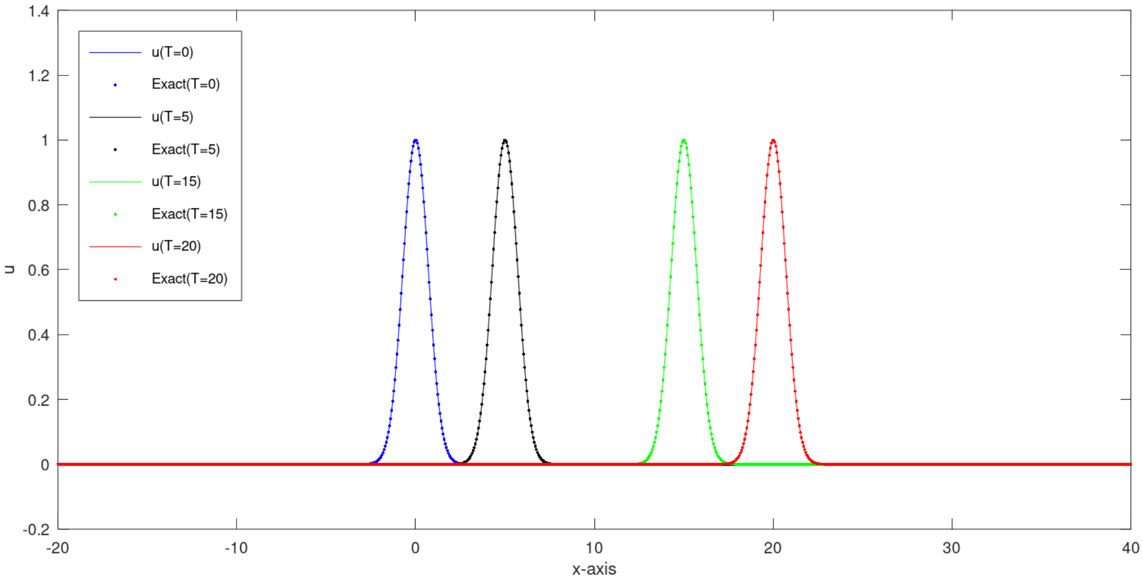

FIGURE 1. Comparison between the exact and approximate solutions for the viscous non-homogeneous problem (Example 2) at $T = 0, 5, 15, 20$.

Example 3. The Homogeneous Inviscous case.
For the case where $\gamma = 0$, we used the homogeneous example from [28]. The exact solution is given by

$$u(x,t) = \frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}(x - \frac{205t}{169} - x_0)\right)$$

where x_0 is an arbitrary constant.

Accuracy Test.

To test the convergence, the simulation was conducted on the domain $[-80, 80]$ with parameters $T = 1$, $x_0 = 2$, $\eta = 1$ and $\tau = h$. The errors and orders of accuracy are given in Tables 4 and 5. The comparison between exact and approximate solutions is represented in Figure 2 for $T = 0, 5, 15, 20$. The performances for each value of θ are shown in Figure 3 where we see that all the errors are vanishing for any value of θ. However, we will show later that the energy is stable only when $\theta = 1/3$.

M	$\theta = 0$				$\theta = 1/3$			
	$\|u-u_h\|_h$		$\|u-u_h\|_{h,\infty}$		$\|u-u_h\|_h$		$\|u-u_h\|_{h,\infty}$	
	error	order	error	order	error	order	error	order
80	1.7942e-02		1.1498e-02		1.7212e-02		1.0702e-02	
160	4.4815e-03	2.00	2.8070e-03	2.03	4.3222e-03	1.99	2.7338e-03	1.97
320	1.1199e-03	2.00	7.0555e-04	1.99	1.0821e-03	2.00	6.8581e-04	2.00
640	2.7993e-04	2.00	1.7657e-04	2.00	2.7062e-04	2.00	1.7247e-04	1.99

TABLE 4. Errors and orders showing optimal convergence rates for the inviscous homogeneous problem (Example 3) with $\theta = 0, 1/3$.

M	$\theta = 2/3$				$\theta = 1$			
	$\|u-u_h\|_h$		$\|u-u_h\|_{h,\infty}$		$\|u-u_h\|_h$		$\|u-u_h\|_{h,\infty}$	
	error	order	error	order	error	order	error	order
80	1.6648e-02		1.0303e-02		1.6266e-02		1.0054e-02	
160	4.2149e-03	1.98	2.6610e-03	1.95	4.1635e-03	1.97	2.6287e-03	1.94
320	1.0579e-03	1.99	6.8078e-04	1.97	1.0480e-03	1.99	6.8277e-04	1.94
640	2.6472e-04	2.00	1.7049e-04	2.00	2.6246e-04	2.00	1.7087e-04	2.00

TABLE 5. Errors and orders showing optimal convergence rates for the inviscous homogeneous problem (Example 3) with $\theta = 2/3, 1$.

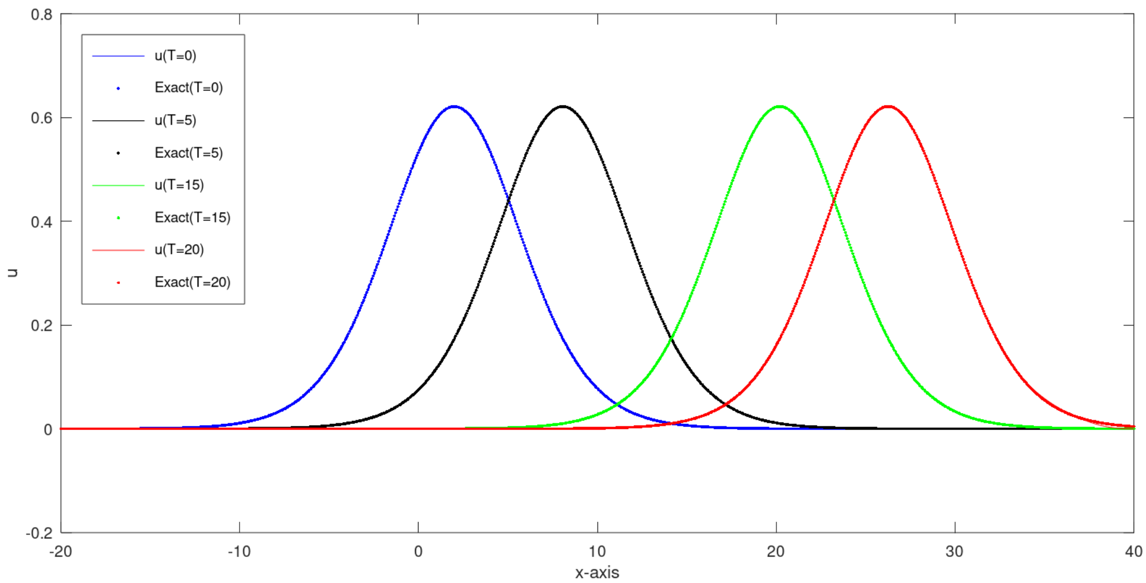

FIGURE 2. Comparison between the exact and approximate solutions for the inviscous homogeneous problem (Example 3) at $T = 0, 5, 15, 20$.

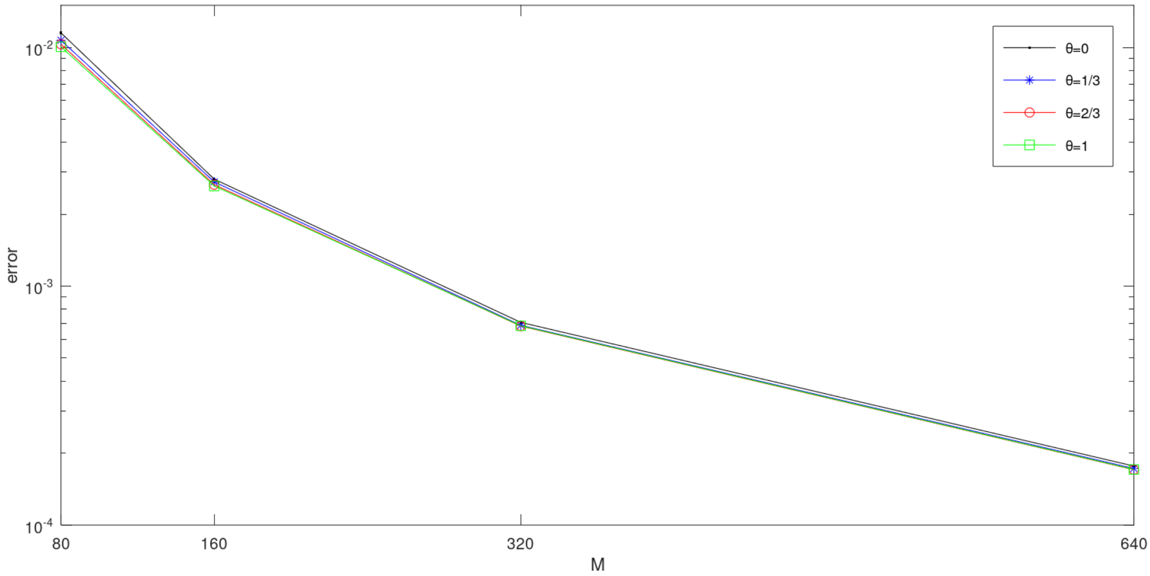

FIGURE 3. Comparison of the uniform errors at $t = 1$ when using different values of θ for the inviscous homogeneous problem (Example 3).

Invariant-Preserving Test

To see how well the scheme performs on a long-time simulation, we compare the performances for each value of θ in Figures 4 and 5. We see that the quantity $\|u^n\|_h$ is stable only when $\theta = 1/3$.

We also compare the percentage of the quantity $\|u^n\|_h$ with respect to its original value at the initial time. The simulation was done using $M = 1000$ on the domain $[-30, 150]$ with $\tau = 0.046$, as shown in Table 6. We see that the quantity $\|\mathbf{u}^n\|_h$ is stable only when $\theta = 1/3$, but when we used $\theta = 0, 2/3, 1$, the numerical solution becomes unstable as time increases. This shows that the invariant-preserving method performs better in the long run.

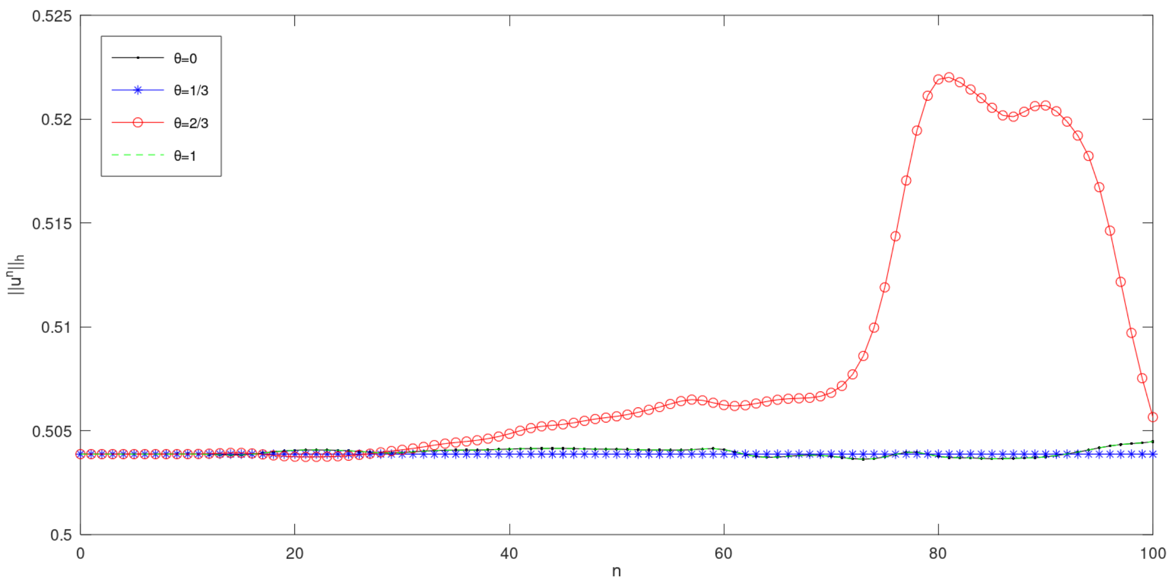

FIGURE 4. Values of $\|\mathbf{u}^n\|_h$ at $t = 0, 1, \ldots, 100$ when using the scheme with different choices of θ with $M = 160$ for the inviscous homogeneous problem Example 3.

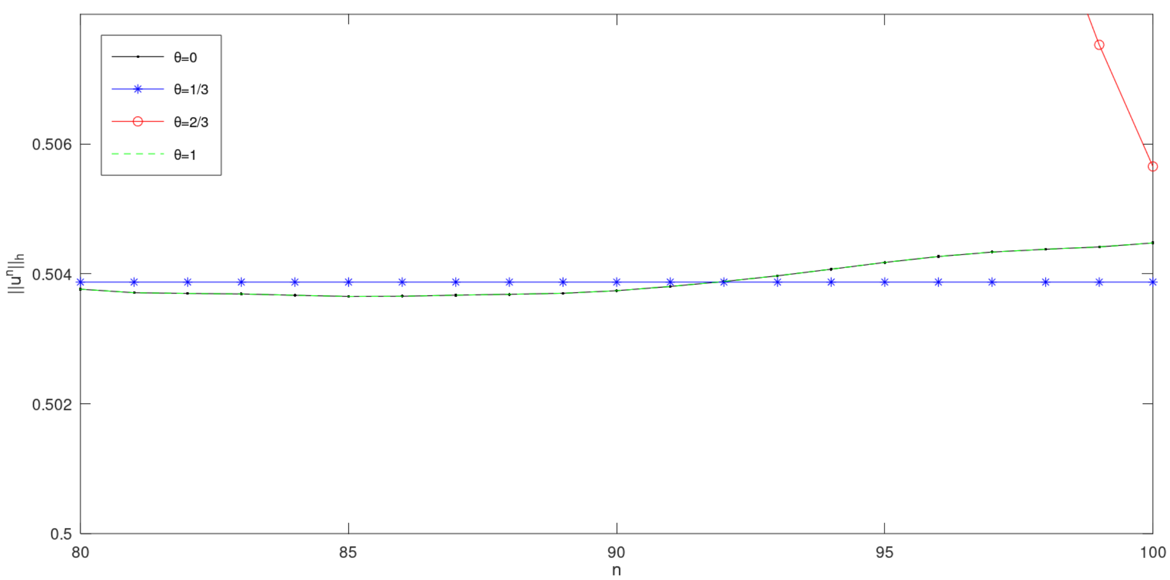

FIGURE 5. Values of $\|\mathbf{u}^n\|_h$ at $t = 80, \ldots, 100$ when using the scheme with different choices of θ with $M = 160$ for the inviscous homogeneous problem Example 3.

T	$\theta = 0$		$\theta = 1/3$		$\theta = 2/3$		$\theta = 1$	
	$\Vert\mathbf{u}^n\Vert_h$	$\%\frac{\Vert\mathbf{u}^n\Vert_h}{\Vert\mathbf{u}^0\Vert_h}$	$\Vert\mathbf{u}^n\Vert_h$	$\%\frac{\Vert\mathbf{u}^n\Vert_h}{\Vert\mathbf{u}^0\Vert_h}$	$\Vert\mathbf{u}^n\Vert_h$	$\%\frac{\Vert\mathbf{u}^n\Vert_h}{\Vert\mathbf{u}^0\Vert_h}$	$\Vert\mathbf{u}^n\Vert_h$	$\%\frac{\Vert\mathbf{u}^n\Vert_h}{\Vert\mathbf{u}^0\Vert_h}$
0	3.7602		3.7602		3.7602		3.7602	
15	3.7602	100.00	3.7602	100.00	3.7602	100.00	3.7602	100.00
30	3.7602	100.00	3.7602	100.00	3.7602	100.00	3.7602	100.00
45	3.7602	100.00	3.7602	100.00	3.7602	100.00	3.7603	100.00
60	3.7602	100.00	3.7602	100.00	3.7602	100.00	3.7612	100.02
75	3.7602	100.00	3.7602	100.00	3.7603	100.00	3.7671	100.16
90	3.7602	100.00	3.7602	100.00	3.7605	100.01	3.8022	100.93
105	3.7602	100.00	3.7602	100.00	3.7610	100.02	4.0015	105.24
120	3.7600	99.99	3.7602	100.00	3.7628	100.05	5.0621	126.51

TABLE 6. Comparison of the performance of the invariant-preserving scheme ($\theta = 1/3$) and other schemes ($\theta = 0, 2/3, 1$) for Example 3.

6. Discussion

Results from numerical examples verified that the approximated solutions are consistent with the result of the previously proven theorems; that is, the numerical solutions converge to the exact solutions with second order of accuracy with respect to the spatial variable. The convergence can be inferred from the error plot in Figure 3 while the rate of convergence can be inferred from the numerical evidence in Tables 1-5. Note that second-order convergence occurs at any value of θ. However, Figures 4 and 5 show that the invariant is preserved only when $\theta = 1/3$. This behavior agrees with the results we proved in Section 4. The comparisons in Example 3 (Invariant-Preserving Test) confirmed that the invariant-preserving scheme performs better in the long run. This hints that the invariant-preserving property, if any, should be taken into account when one wants to design a numerical method.

7. Conclusion

In this work, we proposed a family of three-level linearized finite difference θ-schemes to solve the Korteweg-de Vries – Kawahara equation with added viscosity term. When $\theta = 1/3$, we showed that the resulting scheme has a discrete invariant-preserving property. We also proved that the numerical solution converges uniformly and has second order of accuracy in space. Finally, the proposed numerical method was verified using various numerical examples. The numerical results are consistent with the theoretical results.

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