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Applications of the Laplace variational iteration and Laplace homotopy perturbation methods for solving fractional integro-differential equations

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Abstract The main purpose of this paper is to apply the Laplace variational iteration method (LVIM) and the Laplace homotopy perturbation method (LHPM) to certain fractional integro-differential equations (FIDEs) in the sense of the Caputo derivative such as the fractional-order beam problem for constructing approximate analytical solutions. The LVIM is based on constructing a correction functional, identifying a Laplace-Lagrange multiplier and using a recursive relation to generate an approximate analytical solution, while the LHPM utilizes the homotopy equation, the Laplace transform, He's polynomials and recursive formulas to produce an infinite series solution. Employing the proposed methods, we implement symbolic computer programs to solve some interesting FIDEs including a system of FIDEs for their approximate analytical and numerical solutions. In addition, graphical results and absolute errors of the obtained solutions are presented. As seen in the numerical examples, the LVIM and LHPM are straightforward, effective, and accurate compared with other existing methods. Consequently, the two approaches can be efficiently utilized to solve other fractional integro-differential equations arising in applied science problems.

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1. Introduction

Recently, fractional integro-differential equations (FIDEs) have been utilized for modeling some natural phenomena arising in physics [1], biology [2], engineering [3], and control theory [4], etc. The use of FIDEs have been found in some real applications such

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as visco-elasticity [5], electricity swaption [6], grain growth [7], and population dynamics [8]. Finding accurate solutions of FIDEs is very important for explaining behaviors of all variables in such equations. Consequently, this leads to correct descriptions of the natural occurrences modelled by FIDEs. About a decade ago, many researchers proposed and developed powerful and efficient methods to construct approximate analytical solutions of FIDEs consisting both one and more than one independent variables. Recent reviews for constructing approximate analytical solutions of FIDEs are shown as follows. In 2016, Elbeleze et al. [9] used the homotopy perturbation and the variational iteration methods to obtain approximate analytical solutions for the fractional Fredholm integro-differential equations with constant coefficients. In the same year, Komashynska et al. [10] applied the residual power series method to the system of Fredholm integral equations for its analytical solution. In 2018, the Adomian decomposition method and modified Laplace Adomian decomposition method were used to obtain approximate analytical solutions of the Caputo fractional Volterra-Fredholm integro-differential equations [11]. Moreover, Ahmen and Kirtiwant [12] applied the homotopy analysis method to obtain analytical solutions of the Caputo fractional Volterra-Fredholm integro-differential equations. In the following year, Ahmed et al. [13] provided the approximate solutions of the Caputo fractional Volterra-Fredholm integro-differential equations using the modified Adomian decomposition method. They also discussed some new existence, uniqueness, and convergence results. In 2019, the Fredholm-Hammerstein type of multi-higher order nonlinear integro-fractional differential equations with variable coefficients in the Caputo sense was solved using He's modified homotopy perturbation method (MHPM) for analytical solution when given mixed conditions [14]. In 2020, the system of fractional-order Volterra integro-differential equations was investigated and analyzed using the optimal homotopy asymptotic method by Akbar et al. [15]. In 2021, Ahmad et al. [16] worked on the fuzzy fractional Volterra-Fredholm integro-differential equation using the modified Adomian decomposition method.

However, the prominent and useful methods, which will be used in the article, are the Laplace variational iteration method (LVIM) and the Laplace homotopy perturbation method (LHPM). Therefore, some significant literature review of these methods and their modifications is briefly described as follows. Alawad et al. [17] investigated the space-time fractional telegraph equations via the LVIM to obtain approximate solutions written as a series form rapidly converging into its closed exact solution. Using the method, the numerical calculations are simple and straightforward. In 2014, Biala et al. [18] applied the LVIM to the FIDEs. Utilizing this scheme, a recursive formula with a Lagrange multiplier for each of the proposed problems can be obtained easily. The implementation of the method on some FIDEs exhibits its simplicity, efficiency and accuracy. In 2018, Ziane and Cherif [19] investigated the LVIM to obtain approximate analytical solutions for certain linear and nonlinear time-fractional equations of order $\alpha \in (1, 2]$ in the Caputo sense. As a result, the series solutions converge very rapidly to their exact solutions. In 2021, Mohamed et al. [20] obtained approximate analytical solutions expressed in terms of infinite approximate series for some nonlinear Caputo fractional partial differential equations using the modified Laplace variational iteration method (MLVIM) and the Laplace Adomian decomposition method (LADM). Their numerical examples demonstrated fast convergence and accuracy of these two methods.

Next, a review of applications of the LHPM is given as follows. In 2012, the fractional nonlinear equations were first investigated for approximate solutions using the Laplace homotopy perturbation method (LHPM) by Lui [21]. Using the method, which is based on the homotopy perturbation method and Laplace transform, nonlinear terms in the equations can be manipulated by the use of He's polynomials. In 2015, Gupta et al. [22] applied the LHPM to various linear and nonlinear convection-diffusion problems occurring in physical phenomena for their approximate analytical solutions. In the following year, the approximate solutions, expressed as a convergent series using the LHPM, of the space-fractional telegraph equation was studied by Prakash [23]. The fractional partial derivative used in the proposed equation is the left sided Caputo fractional derivative. The approach is simple to implement and computationally very accurate. The LHPM was employed to find approximate analytical solutions of the initial valued autonomous systems of time-fractional partial differential equations (TFPDEs) with proportional delay [24]. The obtained solutions are described in series forms converging very fast. The proposed scheme is very reliable, effective and powerful. In 2019, the approximate solution of Black-Scholes partial differential equation for a European put option with two assets was obtained using the Laplace homotopy perturbation method (LHPM) [25]. The equation is used to explain the behavior of the financial market and its explicit solution generated by the method is written in the form of a Mellin-Ross function.

The objective of this paper is to apply the Laplace variational iteration method (LVIM) and the Laplace homotopy perturbation method (LHPM) to obtain approximate analytical solutions for certain Caputo fractional integro-differential equations (FIDEs). In general, we consider the nonlinear Volterra-Fredholm fractional integro-differential equation of order α, $m-1<\alpha\le m$, $m\in\mathbb{N}$ as follows [11, 26]:

$$ {}_C D_t^{\alpha} u(t) = g(t) + \lambda_1 \int_a^t K_1(t,\tau)F_1(u(\tau))\, d\tau + \lambda_2 \int_a^b K_2(t,\tau)F_2(u(\tau))\, d\tau, \tag{1.1} $$

subject to the initial conditions

$$ u^{(k)}(a) = c_k,\ \ k = 0,1,...,m-1, \tag{1.2} $$

where the unknown $u(t)$ is a real-valued function of $t\in[a,b]$. The symbol ${}_C D_t^{\alpha}$ represents the Caputo fractional derivative of order α with respect to t, whose definition will be provided later. The real-valued functions g, F_1, F_2 and the nonlinear kernels K_1, K_2 are known functions. The notation $u^{(k)}(a)$ represents for the k-th order derivative of u evaluated at $t=a$. The constants c_k, λ_1, and λ_2 are real parameters.

The organization of this article is managed as follows. In section 2, we introduce some preliminary definitions and important properties of the relevant fractional derivative and the Laplace transform. Section 3 compactly delineates the concept of the Laplace variational iteration and Laplace homotopy perturbation methods, while Section 4 illustrates the applications of the methods for some fractional integro-differential equations. Numerical results of the selected problems, which are reported in terms of numerical comparisons and graphs, are included in this section as well. Finally, discussions and conclusions for the results obtained using the approaches are given in Section 5.

2. Caputo derivative and its properties

In this section, we give some basic definitions of fractional calculus and Laplace transform.

Definition 2.1. [27–29] A function $f(t)$ $(t>0)$ is said to be in the space C_α $(\alpha\in\mathbb{R})$ if it can be expressed as $f(t)=t^p f_1(t)$ for some $p>\alpha$, where $f_1(t)$ is continuous in $[0,\infty)$, and is said to be in the space C_α^n if $f^{(n)} \in C_\alpha,\ n \in \mathbb{N}$. Clearly, $C_\alpha \subseteq C_\beta$ if $\beta,\ \alpha \in \mathbb{R}$ and $\beta \leq \alpha$.

Definition 2.2. The Caputo fractional derivative of fractional order $\alpha > 0$ of a function $f(t)$ is defined as [17, 30]

$$ {}_C D_a^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n,\ n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n,\ n \in \mathbb{N}, \end{cases} \tag{2.1} $$

where $a,\ t \in \mathbb{R},\ t > a$ and $f \in C_{-1}^m$. The Euler's gamma function $\Gamma(\cdot)$ is defined by

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,\ \ \Re(z) > 0, \tag{2.2} $$

where $\Re(z)$ is the real part of a complex number z.

Let $n-1 < \alpha < n,\ n \in \mathbb{N}$ and $f(t)$ and $g(t)$ be two functions such that their Caputo and Riemann-Liouville fractional derivatives exist. Some important properties of the Caputo fractional derivative are described as follows [30–32].

(1) Representation:

$$ {}_C D_a^\alpha f(t) = {}_{RL} J_a^{n-\alpha} \left(f^{(n)}(t) \right), \tag{2.3} $$

where

$$ {}_{RL} J_a^\alpha (g(t)) := \frac{1}{\Gamma(\alpha)} \int_a^t g(\tau)(t-\tau)^{\alpha-1}\, d\tau \tag{2.4} $$

is the Riemann-Liouville fractional integral of order α.

(2) Interpolation:

$$ \lim_{\alpha \to n^-} {}_C D_a^\alpha f(t) = f^{(n)}(t), \tag{2.5} $$

$$ \lim_{\alpha \to (n-1)^+} {}_C D_a^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0). \tag{2.6} $$

(3) Derivative of a constant:

$$ {}_C D_a^\alpha c = 0,\ c \in \mathbb{R}. \tag{2.7} $$

(4) Linearity:

$$ {}_C D_a^\alpha (\lambda f(t) + g(t)) = \lambda\, {}_C D_a^\alpha f(t) + {}_C D_a^\alpha g(t),\ \lambda \in \mathbb{R}. \tag{2.8} $$

Theorem 2.3. *[30, 33] The Caputo fractional derivative of order α $(n-1 < \alpha < n,\ n \in \mathbb{N})$ of the power function satisfies*

$$ {}_C D_a^\alpha (t-a)^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} (t-a)^{p-\alpha}, & p \in \{0,\ 1,\ \ldots,\ n-1\}, \\ 0, & p > n-1, \\ \text{non existing}, & \text{otherwise}. \end{cases} \tag{2.9} $$

Using the definition in (2.1), we can define the partial Caputo fractional derivative of order α $(n-1 < \alpha < n,\ n \in \mathbb{N})$ of a function $u(x,t)$ with respect to t as [28, 34]

$$ {}_C \partial_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\frac{\partial^n}{\partial \tau^n} u(x,\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau,\ t > a. \tag{2.10} $$

The last part of this section is devoted for the Laplace transform of the Caputo fractional derivative and definitions of the Mittag-Leffler functions [30].

Theorem 2.4. *[24, 35, 36] The Laplace transform of the Caputo fractional derivative of order α $(n-1<\alpha<n,\ n\in\mathbb{N})$ can be expressed as*

$$\mathscr{L}\left\{{}_C D_a^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(a^+), \tag{2.11}$$

where $F(s)=\mathscr{L}\left\{f(t)\right\}$.

Definition 2.5. [30] The single parameter Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0. \tag{2.12}$$

Moreover, the two-parameter Mittag-Leffler function can be defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\ \beta>0. \tag{2.13}$$

It is quite obvious that $E_{\alpha,1}(z)=E_\alpha(z)$.

From [37–39], the nth-order derivative of the Mittag-Leffler function in two parameters can be evaluated by

$$\frac{d^n}{dz^n}E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(k+n)!z^k}{k!\Gamma(\alpha k+\alpha n+\beta)}, \quad n\in\mathbb{N},\ \alpha>0,\ \beta\in\mathbb{R},\ z\in\mathbb{R}. \tag{2.14}$$

Lemma 2.6. *[28, 40] Let $\alpha,\ \lambda\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$, and $a\in\mathbb{R}$. Then*

$$\left({}_C D_a^\alpha E_\alpha\left(\lambda(z-a)^\alpha\right)\right)(t) = \sum_{k=1}^{\infty} \frac{\lambda^k(t-a)^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)} = \lambda E_\alpha\left(\lambda(t-a)^\alpha\right). \tag{2.15}$$

3. Description of the Methods

The fractional initial value problem (1.1) and (1.2) can be considered in terms of the linear operator L, the nonlinear operator N and the source term $g(t)$ as follows

$$L\left(u(t)\right) = N\left(u(t)\right) + g(t), \tag{3.1}$$

where $L\left(u(t)\right) = {}_C D_t^\alpha u(t)$, $m-1<\alpha\le m$, $m\in\mathbb{N}$ and

$$N(u(t)) = \lambda_1 \int_a^t K_1(t,\tau)F_1(u(\tau))\,d\tau + \lambda_2 \int_a^b K_2(t,\tau)F_2(u(\tau))\,d\tau \tag{3.2}$$

with the initial conditions

$$u^{(k)}(a) = c_k,\ k=0,1,...,m-1. \tag{3.3}$$

The compact description of the Laplace variational iteration method (LVIM) and the Laplace homotopy perturbation method (LHPM) for solving the initial value problem consisting of (3.1) and (3.3) is provided in the following parts.

3.1. Laplace variational iteration method (LVIM)

The important steps of the LVIM are described as follows [18, 36].
Step 1: Taking the Laplace transform on both sides of (3.1) and using (2.11), we then construct the correction functional from the resulting Laplace transform as

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left(s^\alpha U_n(s) - \sum_{k=0}^{m-1} u^{(k)}(a^+)s^{\alpha-k-1} - \mathscr{L}\left\{h_n(t)\right\}\right), \tag{3.4}$$

where $U_n(s)$ is the nth order approximate solution in the Laplace space, $\lambda(s)$ is the general Lagrange multiplier and

$$h_n(t) = N(u_n(t)) + g(t), \tag{3.5}$$

where the operator $N(\cdot)$ is defind in (3.2).
Step 2: Applying the variational operator δ on both sides of Eq. (3.4) by regarding the terms $\int_a^t K_1(t,\tau)F_1(u_n(\tau))\, d\tau$ and $\int_a^b K_2(t,\tau)F_2(u_n(\tau))\, d\tau$ as restricted variations, we then have

$$\delta U_{n+1}(s) = \delta U_n(s) + \lambda(s)[s^\alpha(\delta U_n(s))]. \tag{3.6}$$

Using the optimality condition $\frac{\delta U_{n+1}}{\delta U_n} = 0$, thus the Lagrange multiplier $\lambda(s)$ can be determined as

$$\lambda(s) = -\frac{1}{s^\alpha}. \tag{3.7}$$

Step 3: Taking the inverse Laplace transform of both sides of (3.4), we obtain the successive approximations for $n = 0, 1, 2, ...,$ as

$$\begin{aligned} u_{n+1}(t) &= u_n(t) - \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left(s^\alpha U_n(s) - \sum_{k=0}^{m-1} u^{(k)}(a^+)s^{\alpha-k-1} - \mathscr{L}\left\{h_n(t)\right\}\right)\right\} \\ &= u_0(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(t)\right\}\right\}, \end{aligned} \tag{3.8}$$

where the initial approximation $u_0(t)$ is defined as

$$u_0(t) = \mathscr{L}^{-1}\left\{\sum_{k=0}^{m-1} u^{(k)}(a^+)s^{-k-1}\right\}. \tag{3.9}$$

The exact solution $u(t)$ of the initial value problem consisting of (3.1) and (3.3) can be obtained by letting $n \to \infty$ in (3.8), in other words, $u(t) = \lim_{n\to\infty} u_n(t)$. The convergence analysis including the existence and uniqueness theorem, the convergence proof and the error analysis of the LVIM and its associated methods can be discovered in [41–44].

3.2. Laplace homotopy perturbation method (LHPM)

In this section, we briefly explain the basic idea and standard steps of the LHPM [21, 45, 46] for solving initial value problem including (3.1) and (3.3) as follows. Using the homotopy method [47, 48], we construct a homotopy $v(t,p) : [a,b] \times [0,1] \to \mathbb{R}$ satisfying the homotopy equation

$$H(v(t,p),p) = (1-p)\big[L(v(t,p)) - u_0\big] + p\big[L(v(t,p)) - N(v(t,p)) - g(t)\big] = 0, \tag{3.10}$$

or

$$H(v(t,p),p) = L(v(t,p)) - u_0 + p\big[u_0 - N(v(t,p)) - g(t)\big] = 0, \tag{3.11}$$

where $p \in [0,1]$ is a homotopy parameter and u_0 is an initial approximation of the solution of (3.1). Clearly, from Eq. (3.10) we obtain

$$H(v(t,0),0) = L(v(t,0)) - u_0 = 0, \tag{3.12}$$
$$H(v(t,1),1) = L\left(v(t,1)\right) - N\left(v(t,1)\right) - g(t) = 0. \tag{3.13}$$

Eqs. (3.12)-(3.13) imply that when p changes from zero to unity, $v(t,p)$ changes from $L^{-1}(u_0)$ to $u(t)$. This is called deformation and $L(v(t,0))-u_0$ and $L\left(v(t,1)\right)-N\left(v(t,1)\right)-g(t)$ are homotopic to each other. By the homotopy perturbation method, the solution $v(t,p)$ of (3.11) can be written in terms of a power series in p

$$v(t,p) = \sum_{i=0}^{\infty} p^i v_i(t) = v_0(t) + pv_1(t) + p^2 v_2(t) + p^3 v_3(t) + \dots\ . \tag{3.14}$$

The following steps are the process of the LHPM.
Step 1: Taking the Laplace transform $\mathscr{L}$ on both sides of homotopy Eq. (3.11), we obtain

$$\mathscr{L}\Big\{L(v(t,p)) - u_0 + p\big[u_0 - N(v(t,p)) - g(t)\big]\Big\} = 0. \tag{3.15}$$

Using the Laplace transform (2.11), we then get

$$s^\alpha \mathscr{L}\left\{v(t,p)\right\} - \sum_{k=0}^{m-1} v^{(k)}(a^+,p)s^{\alpha-k-1} = \mathscr{L}\Big\{u_0 + p\big[-u_0 + N(v(t,p)) + g(t)\big]\Big\}, \tag{3.16}$$

or

$$\mathscr{L}\left\{v(t,p)\right\} = \frac{1}{s^\alpha}\Bigg(\sum_{k=0}^{m-1} v^{(k)}(a^+,p)s^{\alpha-k-1} + \mathscr{L}\Big\{u_0 + p\big[-u_0 + N(v(t,p)) + g(t)\big]\Big\}\Bigg). \tag{3.17}$$

Step 2: Taking the inverse Laplace transform $\mathscr{L}^{-1}$ on both sides of Eq. (3.17), we have

$$v(t,p) = \mathscr{L}^{-1}\Bigg\{\frac{1}{s^\alpha}\Bigg(\sum_{k=0}^{m-1} v^{(k)}(a^+,p)s^{\alpha-k-1} + \mathscr{L}\Big\{u_0 + p\big[-u_0 + N(v(t,p)) + g(t)\big]\Big\}\Bigg)\Bigg\}. \tag{3.18}$$

Now, we assume that the nonlinear term $N(v(t,p))$ can be decomposed as

$$N(v(t,p)) = \sum_{n=0}^{\infty} p^n H_n(v(t,p)), \tag{3.19}$$

where $H_n(v(t,p))$ is He's polynomials [49] which can be defined by

$$H_n\left(v_0,\ v_1,\ \dots,\ v_n\right) = \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right)_{p=0},\quad n = 0,\ 1,\ 2,\ \dots\ . \tag{3.20}$$

Step 3: Substituting Eqs. (3.14) and (3.19) into Eq. (3.18), we have

$$\sum_{i=0}^{\infty} p^i v_i(t) = I(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{p\left[-u_0 + \sum_{n=0}^{\infty} p^n H_n(v(t,p)) + g(t)\right]\right\}\right\}, \tag{3.21}$$

where

$$I(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\sum_{k=0}^{m-1} v^{(k)}(a^+,p)s^{\alpha-k-1} + \mathscr{L}\{u_0\}\right]\right\}. \tag{3.22}$$

Equating the terms with identical powers in p appearing in (3.21), then it leads to the following recursive formulas

$$\begin{aligned} p^0 &: v_0(t) = I(t), \\ p^1 &: v_1(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-u_0 + H_0(v(t,p)) + g(t)\right\}\right\}, \\ &\vdots \\ p^n &: v_n(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_{n-1}(v(t,p))\right\}\right\},\ n \geq 2. \end{aligned} \tag{3.23}$$

Here, the above recurrence relations can be obtained by assuming that the initial approximation has the form $v(a^+,p) = u_0 = c_0,\ v'(a^+,p) = c_1,\ \ldots,\ v^{(n-1)}(a^+,p) = c_{n-1}$.
Step 4: By letting $p \to 1$ in (3.14), the exact analytical solution $u(t)$ of the problem (1.1)-(1.2) can be expressed as

$$u(t) = \lim_{p\to 1} v(t,p) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i v_i(t) = v_0(t) + v_1(t) + v_2(t) + \ldots\ . \tag{3.24}$$

The convergence analysis and error estimate of the LHPM and its relevant methods can be found in [24, 50, 51].

4. Applications and Comparisons of LVIM and LHPM

In this section, we demonstrate the applications of the proposed analytical techniques including the LVIM and the LHPM on some fractional integro-differential equations. Simulation results and comparisons are presented numerically and graphically.

Example 1. Consider the following nonhomogeneous Volterra FIDE:

$${}_C D_t^\alpha u(t) = \frac{\Gamma(2\alpha+1)t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(8\alpha+1)t^{7\alpha}}{\Gamma(7\alpha+1)} + \frac{t^{2\alpha+2}}{2\alpha+1} + \frac{t^{8\alpha+2}}{8\alpha+1} - \int_0^t t u(\tau)\, d\tau,\ 0 < \alpha \leq 1, \tag{4.1}$$

with the initial condition

$$u(0) = 0. \tag{4.2}$$

The exact solution of this problem is $u(t) = t^{2\alpha} + t^{8\alpha}$.
LVIM : Defining $h_n(t) = N(u_n(t)) + g(t)$, where

$$N(u_n(t)) = -\int_0^t t u_n(\tau)\, d\tau \ \text{ and }\ g(t) = \frac{\Gamma(2\alpha+1)t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(8\alpha+1)t^{7\alpha}}{\Gamma(7\alpha+1)} + \frac{t^{2\alpha+2}}{2\alpha+1} + \frac{t^{8\alpha+2}}{8\alpha+1}, \tag{4.3}$$

and using successive relation (3.8), we obtain

$$\begin{aligned} u_{n+1}(t) &= u_0(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(t)\right\}\right\}, \\ &= \mathscr{L}^{-1}\left\{\frac{u(0)}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(t)\right\}\right\}. \end{aligned} \tag{4.4}$$

From (3.9) and (4.4), the first few approximate analytical solutions for $0<\alpha\le 1$ are

$$\begin{aligned} &u_0(t) = 0, \\ &u_1(t) = t^{2\alpha} + t^{8\alpha} + \frac{\Gamma(2\alpha+3)\,t^{3\alpha+2}}{(2\alpha+1)\,\Gamma(3\alpha+3)} + \frac{\Gamma(8\alpha+3)\,t^{9\alpha+2}}{(8\alpha+1)\,\Gamma(9\alpha+3)}, \\ &u_2(t) = t^{2\alpha} + t^{8\alpha} - \frac{(3\alpha+4)\Gamma(2\alpha+3)\,t^{4\alpha+4}}{(2\alpha+1)\,\Gamma(4\alpha+5)} - \frac{\Gamma(8\alpha+3)\,\Gamma(9\alpha+5)\,t^{10\alpha+4}}{(8\alpha+1)\,\Gamma(9\alpha+4)\,\Gamma(10\alpha+5)}, \\ &u_3(t) = t^{2\alpha} + t^{8\alpha} + \frac{8\,(4+3\alpha)\,\Gamma(2\alpha+4)\,t^{5\alpha+6}}{(8\alpha+4)\,\Gamma(5\alpha+7)} + \frac{\left(90\alpha^2+94\alpha+24\right)\Gamma(8\alpha+3)\,t^{11\alpha+6}}{(8\alpha+1)\,\Gamma(11\alpha+7)}, \\ &u_4(t) = t^{2\alpha} + t^{8\alpha} - \frac{(5\alpha+8)\Gamma(2\alpha+4)t^{6\alpha+8}}{(2\alpha+1)\Gamma(6\alpha+8)} - \frac{(990\alpha^3+1754\alpha^2+1016\alpha+192)\Gamma(8\alpha+3)t^{12\alpha+8}}{(8\alpha+1)\Gamma(12\alpha+9)}, \\ &u_5(t) = t^{2\alpha} + t^{8\alpha} + \frac{(10\alpha+16)\,(4+3\alpha)\,\Gamma(2\alpha+4)\,\Gamma(6\alpha+11)\,t^{7\alpha+10}}{\Gamma(6\alpha+10)\,\Gamma(7\alpha+11)\,(2\alpha+1)} \\ &\qquad + \frac{(22\alpha+16)(4+9\alpha)(5\alpha+3)\Gamma(8\alpha+3)\Gamma(12\alpha+11)t^{13\alpha+10}}{\Gamma(12\alpha+10)\Gamma(13\alpha+11)(8\alpha+1)}. \end{aligned} \tag{4.5}$$

Theoretically, $u_n(t)$ as $n\to\infty$ is the best approximation solution of (4.1) tending to the exact solution $u(t)=t^{2\alpha}+t^{8\alpha}$. For the computational convenience, we let

$$u_5^V(t) = u_5(t) \tag{4.6}$$

be the approximate solution of IVP (4.1)-(4.2) which is utilized to calculate the following approximate analytical solutions for $\alpha = 1,\ 0.9,\ 0.7$.

$$\begin{aligned} \alpha = 1 \quad &: \quad u_5^V(t) = t^2 + t^8 + 3.27\times10^{-10}t^{17} + 1.35\times10^{-12}t^{23}, \\ \alpha = 0.9 \quad &: \quad u_5^V(t) = t^{1.8} + t^{7.2} + 1.62\times10^{-9}t^{16.3} + 1.06\times10^{-11}t^{21.7}, \\ \alpha = 0.7 \quad &: \quad u_5^V(t) = t^{1.4} + t^{5.6} + 3.72\times10^{-8}t^{14.9} + 6.18\times10^{-10}t^{19.1}. \end{aligned} \tag{4.7}$$

LHPM : From (3.20) and (4.3), it yields

$$H_n\left(v(t,p)\right) = \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(-\int_0^t t\left(\sum_{i=0}^{\infty}p^i v_i(\tau)\right)d\tau\right)\right)_{p=0} = -\int_0^t t v_n(\tau)\,d\tau. \tag{4.8}$$

Using relation (3.23) along with (3.22) and (4.8), we then have the first few recursive formulas for $0<\alpha\le1$ as follows.

$$\begin{aligned} p^0 &: v_0(t) = I(t) = 0, \\ p^1 &: v_1(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-u_0 + H_0(v(t,p)) + g(t)\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{g(t)\right\}\right\} \\ &\qquad = t^{2\alpha} + t^{8\alpha} + \frac{\Gamma(2\alpha+3)\,t^{3\alpha+2}}{(2\alpha+1)\,\Gamma(3\alpha+3)} + \frac{\Gamma(8\alpha+3)\,t^{9\alpha+2}}{(8\alpha+1)\,\Gamma(9\alpha+3)}, \end{aligned}$$

$$\begin{aligned}
p^2: v_2(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_1(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\int_0^t t v_1(\tau)\, d\tau\right\}\right\} \\
&= -\frac{\Gamma(2\alpha+3)\, t^{3\alpha+2}}{(2\alpha+1)\,\Gamma(3\alpha+3)} - \frac{(4+3\alpha)\,\Gamma(2\alpha+3)\, t^{4\alpha+4}}{(2\alpha+1)\,\Gamma(4\alpha+5)} - \frac{\Gamma(8\alpha+3)\, t^{9\alpha+2}}{(8\alpha+1)\,\Gamma(9\alpha+3)} \\
&\quad - \frac{\Gamma(8\alpha+3)\,\Gamma(9\alpha+5)\, t^{10\alpha+4}}{\Gamma(9\alpha+4)\,\Gamma(10\alpha+5)\,(8\alpha+1)}, \\
p^3: v_3(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_2(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\int_0^t t v_2(\tau)\, d\tau\right\}\right\} \\
&= \frac{(4+3\alpha)\Gamma(2\alpha+3)t^{4\alpha+4}}{(2\alpha+1)\Gamma(4\alpha+5)} + \frac{8(4+3\alpha)\Gamma(2\alpha+4)t^{5\alpha+6}}{(8\alpha+4)\Gamma(7+5\alpha)} + \frac{\Gamma(8\alpha+3)\Gamma(9\alpha+5)t^{10\alpha+4}}{(8\alpha+1)\Gamma(9\alpha+4)\Gamma(10\alpha+5)} \\
&\quad + \frac{(4+9\alpha)\,\Gamma(8\alpha+3)\,\Gamma(10\alpha+7)\, t^{11\alpha+6}}{(8\alpha+1)\,\Gamma(11\alpha+7)\,\Gamma(10\alpha+6)}, \\
p^4: v_4(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_3(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\int_0^t t v_3(\tau)\, d\tau\right\}\right\} \\
&= -\frac{8\,(4+3\alpha)\,\Gamma(2\alpha+4)\, t^{5\alpha+6}}{(8\alpha+4)\,\Gamma(7+5\alpha)} - \frac{2\Gamma(2\alpha+4)\,\Gamma(5\alpha+9)\,(3\alpha+4)\, t^{6\alpha+8}}{\Gamma(5\alpha+8)\,\Gamma(6\alpha+9)\,(2\alpha+1)} \\
&\quad - \frac{(4+9\alpha)\Gamma(8\alpha+3)\Gamma(10\alpha+7)t^{11\alpha+6}}{(8\alpha+1)\Gamma(11\alpha+7)\Gamma(10\alpha+6)} - \frac{2\Gamma(8\alpha+3)\Gamma(11\alpha+9)(5\alpha+3)(9\alpha+4)t^{12\alpha+8}}{\Gamma(11\alpha+8)\Gamma(12\alpha+9)(8\alpha+1)}, \\
p^5: v_5(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_4(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\int_0^t t v_4(\tau)\, d\tau\right\}\right\} \\
&= \frac{2\Gamma(2\alpha+4)\Gamma(5\alpha+9)(3\alpha+4)t^{6\alpha+8}}{\Gamma(5\alpha+8)\Gamma(6\alpha+9)(2\alpha+1)} + \frac{2\Gamma(2\alpha+4)\Gamma(6\alpha+11)(5\alpha+8)(3\alpha+4)t^{7\alpha+10}}{\Gamma(7\alpha+11)\Gamma(6\alpha+10)(2\alpha+1)} \\
&\quad + \frac{2\Gamma(8\alpha+3)\,\Gamma(11\alpha+9)\,(5\alpha+3)\,(9\alpha+4)\, t^{12\alpha+8}}{\Gamma(11\alpha+8)\,\Gamma(12\alpha+9)\,(8\alpha+1)} \\
&\quad + \frac{2\Gamma(8\alpha+3)\,\Gamma(12\alpha+11)\,(9\alpha+4)\,(5\alpha+3)\,(11\alpha+8)\, t^{13\alpha+10}}{\Gamma(12\alpha+10)\,\Gamma(13\alpha+11)\,(8\alpha+1)}.
\end{aligned}$$

By the process of the LHPM described above, the solution of IVP (4.1)-(4.2) is assumed in the following form

$$u(t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i v_i(t) = v_0(t) + v_1(t) + v_2(t) + \dots . \tag{4.9}$$

For the computational convenience, we denote $u_5^H(t) = \lim_{p\to 1}\sum_{i=0}^{5} p^i v_i(t)$ as the fifth-order approximate solution of (4.1), which can be expressed as

$$\begin{aligned}
u_5^H(t) = t^{2\alpha} + t^{8\alpha} &+ \frac{2\Gamma(2\alpha+4)\,\Gamma(6\alpha+11)\,(5\alpha+8)\,(3\alpha+4)\, t^{7\alpha+10}}{\Gamma(7\alpha+11)\,\Gamma(6\alpha+10)\,(2\alpha+1)} \\
&+ \frac{2\Gamma(8\alpha+3)\,\Gamma(12\alpha+11)\,(9\alpha+4)\,(5\alpha+3)\,(11\alpha+8)\, t^{13\alpha+10}}{\Gamma(12\alpha+10)\,\Gamma(13\alpha+11)\,(8\alpha+1)}.
\end{aligned} \tag{4.10}$$

The approximate analytical solution $u_5^H(t)$ can be expressed for $\alpha = 1,\ 0.9,\ 0.7$ as follows:

$$\begin{aligned}
\alpha = 1 &: \quad u_5^H(t) = t^2 + t^8 + 3.27\times 10^{-10} t^{17} + 1.35\times 10^{-12} t^{23}, \\
\alpha = 0.9 &: \quad u_5^H(t) = t^{1.8} + t^{7.2} + 1.62\times 10^{-9} t^{16.3} + 1.06\times 10^{-11} t^{21.7}, \\
\alpha = 0.7 &: \quad u_5^H(t) = t^{1.4} + t^{5.6} + 3.72\times 10^{-8} t^{14.9} + 6.18\times 10^{-10} t^{19.1}.
\end{aligned} \tag{4.11}$$

It can be observed from (4.7) and (4.11) that the last two terms of $u_5^V(t)$ and $u_5^H(t)$ for each value of α have the infinitesimal coefficients, which are close to zero. Consequently, the numerical solution graphs, obtained by these two methods, converge to the exact solution graphs for $\alpha = 1,\ 0.9,\ 0.7$ as shown in Figure 1. The arrow appearing in Figure 1 points in the direction of increasing value of α. In other words, the groups of the solution graphs for $\alpha = 0.7$, $\alpha = 0.9$ and $\alpha = 1$ are presented in the top, middle and bottom parts of the figure, respectively.

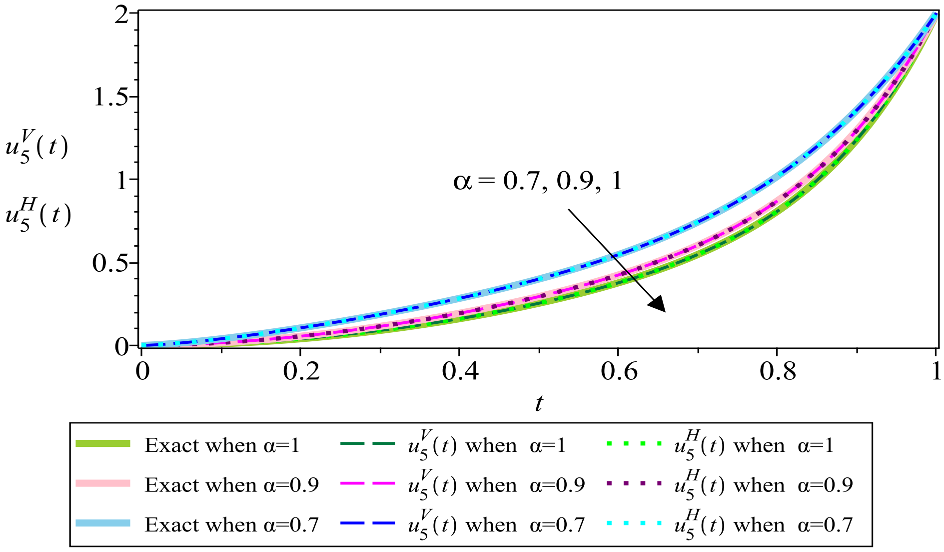


FIGURE 1. Comparison between the exact solution $u(t) = t^{2\alpha} + t^{8\alpha}$ and the approximate analytical solutions $u_5^V(t)$ and $u_5^H(t)$ for IVP (4.1)-(4.2) when $\alpha = 1,\ 0.9,\ 0.7$.

t	$u(t)$	$u_5^V(t)$	$u_5^H(t)$	Absolute error $\lvert u(t) - u_5^V(t)\rvert$	Absolute error $\lvert u(t) - u_5^H(t)\rvert$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.04000256	0.040002560000000000000429	0.040002560000000000000429	4.29232×10^{-22}	4.29232×10^{-22}
0.4	0.16065536	0.16065536000000005626133	0.16065536000000005626133	5.62613×10^{-17}	5.62613×10^{-17}
0.6	0.37679616	0.37679616000005544188628	0.37679616000005544188628	5.54418×10^{-14}	5.54418×10^{-14}
0.8	0.80777216	0.80777216000738216569797	0.80777216000738216569797	7.38216×10^{-12}	7.38216×10^{-12}
1.0	2.00000	2.0000000003288345597895	2.0000000003288345597895	3.28834×10^{-10}	3.28834×10^{-10}

TABLE 1. Exact solution $u(t) = t^{2\alpha} + t^{8\alpha}$, approximate solutions $u_5^V(t)$, $u_5^H(t)$ and their absolute errors for IVP (4.1)-(4.2) for $\alpha = 1$.

Example 2. Consider the following nonhomogeneous Volterra-Fredholm FIDE:

$$ {}_C D_t^{\alpha} u(t) = -\frac{1}{4\alpha+1} + \frac{\Gamma(4\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{u(t)t^2}{4\alpha+2} + \int_0^t \tau u(\tau)\, d\tau + \int_0^1 u(\tau)\, d\tau,\ 0 < \alpha \le 1, \tag{4.12} $$

t	$u(t)$	$u_5^V(t)$	$u_5^H(t)$	Absolute error $\lvert u(t)-u_5^V(t)\rvert$	Absolute error $\lvert u(t)-u_5^H(t)\rvert$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.1051829460282285057	0.1051829460282285071	0.1051829460282285071	1.43289×10^{-18}	1.43289×10^{-18}
0.4	0.1051829460282285057	0.2831672360766983274	0.2831672360766983274	4.38234×10^{-14}	4.38234×10^{-14}
0.6	0.5463489304818683652	0.5463489305003251760	0.5463489305003251760	1.84568×10^{-11}	1.84568×10^{-11}
0.8	1.018306390595508837	1.018306391943597838	1.018306391943597838	1.34808×10^{-9}	1.34808×10^{-9}
1.0	2.00000	2.000000037845882720	2.000000037845882720	3.78458×10^{-8}	3.78458×10^{-8}

TABLE 2. Exact solution $u(t) = t^{2\alpha} + t^{8\alpha}$, approximate solutions $u_5^V(t)$, $u_5^H(t)$ and their absolute errors for IVP (4.1)-(4.2) for $\alpha = 0.7$.

with the initial condition

$$u(0) = 0. \tag{4.13}$$

The exact solution of this problem is $u(t) = t^{4\alpha}$.

LVIM : Defining $h_n(t) = N(u_n(t)) + g(t)$, where

$$N(u_n(t)) = -\frac{u_n(t)t^2}{4\alpha+2} + \int_0^t \tau u_n(\tau)\, d\tau + \int_0^1 u_n(\tau)\, d\tau \text{ and } g(t) = -\frac{1}{4\alpha+1} + \frac{\Gamma(4\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{4.14}$$

and using successive relation (3.8), we obtain

$$\begin{aligned} u_{n+1}(t) &= u_0(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(t)\right\}\right\}, \\ &= \mathscr{L}^{-1}\left\{\frac{u(0)}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(t)\right\}\right\}. \end{aligned} \tag{4.15}$$

From (3.9) and (4.15), the first few approximate analytical solutions for $0 < \alpha \leq 1$ are expressed as

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= t^{4\alpha} - \frac{t^\alpha}{(4\alpha+1)\,\Gamma(1+\alpha)}, \\ u_2(t) &= t^{4\alpha} - \frac{t^\alpha}{(4\alpha+1)\,\Gamma(\alpha+2)\Gamma(1+\alpha)} - \frac{3\alpha(1+\alpha)\,t^{2+2\alpha}}{(8\alpha+2)(2\alpha+1)\,\Gamma(2\alpha+3)}, \\ u_3(t) &= t^{4\alpha} - \frac{\left((4\alpha+2)\,\Gamma(2\alpha+4) + 3\alpha\,(\Gamma(1+\alpha))^2(1+\alpha)^3\right)t^\alpha}{(16\alpha^2+12\alpha+2)\,(\Gamma(2+\alpha))^2\,\Gamma(1+\alpha)\,\Gamma(2\alpha+4)} \\ &\quad - \frac{3\alpha t^{2\alpha+2}}{(16\alpha^2+12\alpha+2)\,\Gamma(1+\alpha)\,\Gamma(2\alpha+3)} - \frac{3\alpha(2\alpha+3)\left(\alpha^2-1\right)t^{3\alpha+4}}{(8\alpha+2)(2\alpha+1)^2\,\Gamma(5+3\alpha)}. \end{aligned} \tag{4.16}$$

Theoretically, $u_n(t)$ as $n \to \infty$ is the best approximation solution of IVP (4.12)-(4.13) tending to the exact solution $u(t) = t^{4\alpha}$. With the symbolically computational limitation of Maple software when n is higher, it is quite difficult to show the analytical solutions in terms of $0 < \alpha \leq 1$. We instead compute the analytical solutions with the numerical values of α when $n = 20$. Now, we denote

$$u_{20}^V(t) = u_{20}(t) \tag{4.17}$$

as the approximate solution of (4.12) which is used to compute the following solutions when $\alpha = 1,\ 0.9,\ 0.7$.

$$\begin{aligned} \alpha = 1 \quad &: \quad u_{20}^V(t) = -6.76\times 10^{-7}t + t^4, \\ \alpha = 0.9 \quad &: \quad u_{20}^V(t) = -4.41\times 10^{-6}t^{0.9} + t^{3.6} - 3.85\times 10^{-7}t^{3.8} + 7.51\times 10^{-10}t^{6.7} + \ldots \\ &\qquad\qquad +8.14\times 10^{-41}t^{53.1} - 6.37\times 10^{-43}t^{56}, \\ \alpha = 0.7 \quad &: \quad u_{20}^V(t) = -1.46\times 10^{-4}t^{0.7} + t^{2.8} - 1.45\times 10^{-5}t^{3.4} + 1.39\times 10^{-7}t^{6.1} + \ldots \\ &\qquad\qquad +1.81\times 10^{-33}t^{49.3} - 3.32\times 10^{-35}t^{52}. \end{aligned} \tag{4.18}$$

LHPM : From (3.20) and (4.14), it yields

$$\begin{aligned} H_n\left(v(t,p)\right) &= \left[\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(-\frac{t^2}{4\alpha+2}\left(\sum_{i=0}^{\infty}p^i v_i(t)\right) + \int_0^t \tau\left(\sum_{i=0}^{\infty}p^i v_i(\tau)\right)d\tau + \int_0^1\left(\sum_{i=0}^{\infty}p^i v_i(\tau)\right)d\tau\right)\right]_{p=0} \\ &= -\frac{v_n(t)t^2}{4\alpha+2} + \int_0^t \tau v_n(\tau)\ d\tau + \int_0^1 v_n(\tau)\ d\tau. \end{aligned} \tag{4.19}$$

Using relation (3.23) along with (3.22) and (4.19), we then have the first few recursive relations for any $0 < \alpha \le 1$ as follows.

$$p^0 : v_0(t) = I(t) = 0,$$

$$\begin{aligned} p^1 : v_1(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\{u_0 + H_0(v(t,p)) + g(t)\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\{g(t)\}\right\} \\ &= t^{4\alpha} - \frac{t^\alpha}{(4\alpha+1)\,\Gamma\left(1+\alpha\right)}, \end{aligned}$$

$$\begin{aligned} p^2 : v_2(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_1(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\frac{v_1(t)t^2}{4\alpha+2} + \int_0^t \tau v_1(\tau)\ d\tau + \int_0^1 v_1(\tau)\ d\tau\right\}\right\} \\ &= \frac{t^\alpha}{(4\alpha+1)\,\Gamma\left(1+\alpha\right)} - \frac{(1+\alpha)t^\alpha}{(4\alpha+1)(\Gamma\left(2+\alpha\right))^2} - \frac{(2+\alpha)t^{2+2\alpha}}{(32\alpha^2+24\alpha+4)\Gamma\left(2\alpha+2\right)}, \end{aligned}$$

$$\begin{aligned} p^3 : v_3(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{H_2(v(t,p))\right\}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-\frac{v_2(t)t^2}{4\alpha+2} + \int_0^t \tau v_2(\tau)\ d\tau + \int_0^1 v_2(\tau)\ d\tau\right\}\right\} \\ &= \frac{(1+\alpha)t^\alpha}{(4\alpha+1)(\Gamma(2+\alpha))^2} - \frac{(1+\alpha)t^\alpha}{(\Gamma(2+\alpha))^3(4\alpha+1)} - \frac{3(1+\alpha)t^\alpha}{(16\alpha^2+12\alpha+2)\Gamma(\alpha)\Gamma(4+2\alpha)} \\ &\quad + \frac{3\alpha(-1+\Gamma(2+\alpha))t^{2\alpha+2}}{(32\alpha^2+24\alpha+4)\Gamma(2+\alpha)\Gamma(2\alpha+2)} - \frac{3\alpha(16\alpha^5+36\alpha^4+4\alpha^3-33\alpha^2-20\alpha-3)t^{3\alpha+4}}{2(4\alpha+1)^2\Gamma(3\alpha+5)(2\alpha+1)^3}. \end{aligned}$$

By the process of the LHPM described above, the solution of IVP (4.12)-(4.13) is assumed in the following form

$$u(t) = \lim_{p\to 1}\sum_{i=0}^{\infty}p^i v_i(t) = v_0(t) + v_1(t) + v_2(t) + \ldots\ . \tag{4.20}$$

Because of the limitation of Maple software, here we only show, for instance, the symbolically analytical solution $u_3^H(t) = \lim_{p\to 1}\sum_{i=0}^{3}p^i v_i(t)$ as

$$\begin{aligned} u_3^H(t) = t^{4\alpha} &- \frac{\left((\alpha+2)\,\Gamma\left(4+2\alpha\right) + 3\left(\Gamma\left(\alpha\right)\right)^2\alpha^3\left(1+\alpha\right)^3\right)t^\alpha}{(4\alpha+2)\,\Gamma\left(4+2\alpha\right)(4\alpha+1)\left(\Gamma\left(\alpha\right)\right)^3\alpha^3\left(1+\alpha\right)^2} \\ &- \frac{3\alpha t^{2\alpha+2}}{(32\alpha^2+24\alpha+4)\,\Gamma\left(2+\alpha\right)\Gamma\left(2\alpha+2\right)} - \frac{3\alpha\left(2\alpha+3\right)\left(\alpha^2-1\right)t^{3\alpha+4}}{(8\alpha+2)\left(2\alpha+1\right)^2\Gamma\left(3\alpha+5\right)}. \end{aligned} \tag{4.21}$$

For the numerically computational convenience, it is possible to compute the twentieth-order approximate solution $u_{20}^H(t) = \lim_{p\to 1}\sum_{i=0}^{20}p^i v_i(t)$ of (4.12) when $\alpha = 1,\ 0.9,\ 0.7$. They are listed as:

$$\begin{aligned}
\alpha = 1 \quad &: \quad u^H_{20}(t) = -6.76\times10^{-7}t + t^4, \\
\alpha = 0.9 \quad &: \quad u^H_{20}(t) = -4.41\times10^{-6}t^{0.9} + t^{3.6} - 3.85\times10^{-7}t^{3.8} + 7.51\times10^{-10}t^{6.7} + \ldots \\
&\qquad\qquad +8.14\times10^{-41}t^{53.1} - 6.37\times10^{-43}t^{56}, \qquad (4.22)\\
\alpha = 0.7 \quad &: \quad u^H_{20}(t) = -1.46\times10^{-4}t^{0.7} + t^{2.8} - 1.45\times10^{-5}t^{3.4} + 1.39\times10^{-7}t^{6.1} + \ldots \\
&\qquad\qquad +1.81\times10^{-33}t^{49.3} - 3.32\times10^{-35}t^{52}.
\end{aligned}$$

As a result from (4.18) and (4.22), it can be noticed that the approximate solutions $u^V_{20}(t)$ and $u^H_{20}(t)$ are approaching t^4, $t^{3.6}$ and $t^{2.8}$ when $\alpha = 1,\ 0.9$ and 0.7, respectively because the coefficients of the remaining terms of $u^V_{20}(t)$ and $u^H_{20}(t)$ are infinitesimal. Therefore, the numerical solution graphs, acquired by the LVIM and LHPM, converge to the exact solution graphs for $\alpha = 1,\ 0.9,\ 0.7$ as portrayed in Figure 2. The arrow shown in Figure 2 points in the direction of increasing value of α.

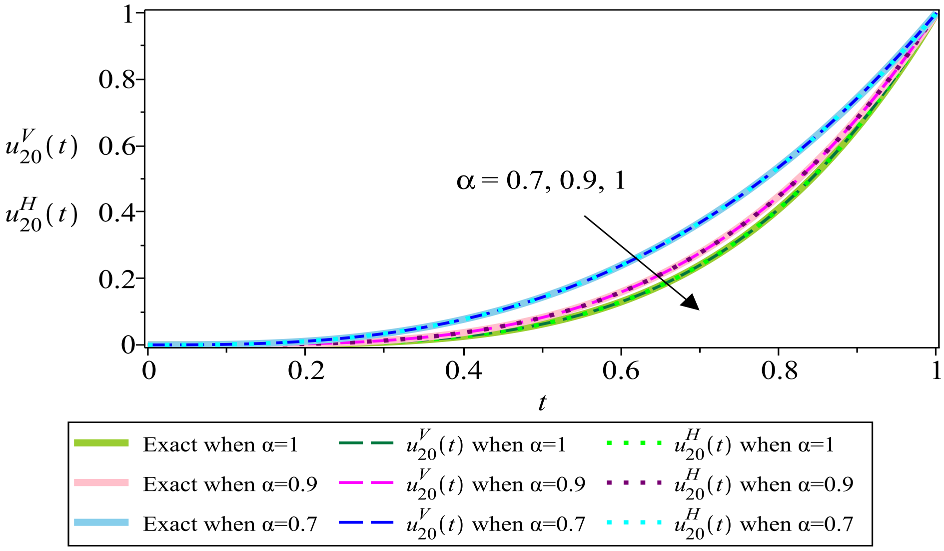


FIGURE 2. Comparison between the exact solution $u(t) = t^{4\alpha}$ and the approximate analytical solutions $u^V_{20}(t)$ and $u^H_{20}(t)$ for IVP (4.12)-(4.13) when $\alpha = 1,\ 0.9,\ 0.7$.

In addition, Table 3 and Table 4 list the numerical values of the exact solution $u(t) = t^{4\alpha}$, the approximate solutions $u^V_{20}(t)$, $u^H_{20}(t)$ and the absolute errors between the exact and the approximate solutions of problem (4.12) evaluated at some points of t when $\alpha = 1,\ 0.7$, respectively.

Example 3. Consider the nonlinear integro partial differential equation of the beam problem:

$$ {}_C\partial^\alpha_t u(x,t) = -\frac{\partial^4 u(x,t)}{\partial x^4} + \frac{\partial^2 u(x,t)}{\partial x^2} + 2\frac{\partial^2 u(x,t)}{\partial x^2}\int_0^1 \left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx + g(x,t),\ 1<\alpha\le 2, \qquad (4.23)$$

where $g(x,t) = e^{-x}\mathrm{E}_\alpha(t^\alpha)\left(1-\left(1-e^{-2}\right)\left(\mathrm{E}_\alpha(t^\alpha)\right)^2\right)$ subject to the initial conditions:

$$u(x,0) = e^{-x} \ \text{ and } \ u_t(x,0) = 0. \qquad (4.24)$$

It is not difficult to verify that $u^*(x,t) = e^{-x}\mathrm{E}_\alpha(t^\alpha)$ satisfies (4.23). Moreover, it is easy to see that $u^*(x,0) = e^{-x}$ and by (2.14) and the chain rule, we have $u^*_t(x,0) = 0$. Hence, $u(x,t) = e^{-x}\mathrm{E}_\alpha(t^\alpha)$ is the solution of the IVP comprising equations (4.23) and (4.24).

t	$u(t)$	$u_{20}^V(t)$	$u_{20}^H(t)$	Absolute error $\lvert u(t)-u_{20}^V(t)\rvert$	Absolute error $\lvert u(t)-u_{20}^H(t)\rvert$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.00160	0.001599865	0.001599865	1.35403×10^{-7}	1.35403×10^{-7}
0.4	0.02560	0.02559973	0.02559973	2.72031×10^{-7}	2.72031×10^{-7}
0.6	0.12960	0.1295996	0.1295996	4.13026×10^{-7}	4.13026×10^{-7}
0.8	0.40960	0.4095995	0.4095995	5.63630×10^{-7}	5.63630×10^{-7}
1.0	1.00000	0.9999992	0.9999992	7.31183×10^{-7}	7.31183×10^{-7}

TABLE 3. Exact solution $u(t) = t^{4\alpha}$, approximate solutions $u_{20}^V(t)$, $u_{20}^H(t)$ and their absolute errors for IVP (4.12)-(4.13) for $\alpha = 1$.

t	$u(t)$	$u_{20}^V(t)$	$u_{20}^H(t)$	Absolute error $\lvert u(t)-u_{20}^V(t)\rvert$	Absolute error $\lvert u(t)-u_{20}^H(t)\rvert$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.01103783729	0.01099017460	0.01099017453	4.766269×10^{-5}	4.766276×10^{-5}
0.4	0.07687196377	0.07679398916	0.07679398904	7.797461×10^{-5}	7.797473×10^{-5}
0.6	0.2392343301	0.2391290627	0.2391290625	1.052674×10^{-4}	1.052676×10^{-4}
0.8	0.5353674509	0.5352350425	0.5352350423	1.324084×10^{-4}	1.324086×10^{-4}
1.0	1.00000	0.9998387073	0.9998387071	1.612927×10^{-4}	1.612929×10^{-4}

TABLE 4. Exact solution $u(t) = t^{4\alpha}$, approximate solutions $u_{20}^V(t)$, $u_{20}^H(t)$ and their absolute errors for IVP (4.12)-(4.13) for $\alpha = 0.7$.

Although, the methods described above are for the fractional-order differential equation defined in (3.1), however, we will apply them to solve the beam problem (4.23) using the Laplace transform with respect to t as follows.

LVIM : Denoting $h_n(x,t) = N(u_n(x,t)) + g(x,t)$, where

$$\begin{aligned} N(u_n(x,t)) &= -\frac{\partial^4 u_n(x,t)}{\partial x^4} + \frac{\partial^2 u_n(x,t)}{\partial x^2} + 2\frac{\partial^2 u_n(x,t)}{\partial x^2}\int_0^1 \left|\frac{\partial u_n(x,t)}{\partial x}\right|^2 dx \quad \text{and} \\ g(x,t) &= e^{-x}\mathrm{E}_\alpha(t^\alpha)\left(1-\left(1-e^{-2}\right)\left(\mathrm{E}_\alpha(t^\alpha)\right)^2\right), \end{aligned} \tag{4.25}$$

and applying the successive relation (3.8) to the problem, we obtain

$$\begin{aligned} u_{n+1}(x,t) &= u_0(x,t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(x,t)\right\}\right\}, \\ &= \mathscr{L}^{-1}\left\{\frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_n(x,t)\right\}\right\}. \end{aligned} \tag{4.26}$$

For computational simplicity, we approximate $\mathrm{E}_\alpha(t^\alpha)$ in $g(x,t)$ with $\mathrm{E}_\alpha(t^\alpha) \approx \sum_{k=0}^{10} \dfrac{t^{\alpha k}}{\Gamma(\alpha k+1)}$ which is utilized for the following computations. Due to the symbolically computational restriction of Maple software for any $1 < \alpha \leq 2$, thus we can show the general terms for only $u_0(x,t)$ and $u_1(x,t)$ by using (4.26) as follows.

$$\begin{aligned} u_0(x,t) &= \mathscr{L}^{-1}\left\{\frac{u(x,0)}{s}+\frac{u_t(x,0)}{s^2}\right\} = e^{-x}, \\ u_1(x,t) &= e^{-x} + \frac{e^{-x}t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\left(3e^{-2-x}-2e^{-x}\right)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ \left(\frac{\left(3e^{-2-x}-2e^{-x}\right)}{\Gamma(3\alpha+1)} + \frac{12(e^{-2-x}-e^{-x})\Gamma(\alpha+\frac{1}{2})2^{\alpha}}{\sqrt{\pi}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} + \ldots + \omega(x;\alpha)t^{31\alpha}, \end{aligned} \tag{4.27}$$

where $\omega(x;\alpha) = \frac{\omega_1(x;\alpha)}{\omega_2(\alpha)}$ in which

$$\begin{aligned} \omega_1(x;\alpha) &= 4096\sqrt{6}\pi^{\frac{27}{2}}(205891132094649)^{\alpha}e^{-x}\Gamma\left(2\alpha+\frac{2}{15}\right)\Gamma\left(2\alpha+\frac{1}{3}\right)\Gamma\left(2\alpha+\frac{2}{3}\right)\Gamma\left(2\alpha+\frac{1}{15}\right) \\ &\times\Gamma\left(2\alpha+\frac{4}{15}\right)\Gamma\left(2\alpha+\frac{8}{15}\right)\Gamma\left(2\alpha+\frac{11}{15}\right)\Gamma\left(2\alpha+\frac{13}{15}\right)\Gamma\left(2\alpha+\frac{14}{15}\right)\Gamma\left(2\alpha+\frac{7}{15}\right), \\ \omega_2(\alpha) &= 1083179328015\Gamma(31\alpha+1)\left(\Gamma(2\alpha+1)\right)^2\left(\Gamma\left(2\alpha+\frac{1}{5}\right)\right)^2\left(\Gamma\left(2\alpha+\frac{3}{5}\right)\right)^2 \\ &\times\left(\Gamma\left(2\alpha+\frac{4}{5}\right)\right)^2\left(\Gamma\left(2\alpha+\frac{2}{5}\right)\right)^2. \end{aligned}$$

Next, we denote

$$u_3^V(x,t) = u_3(x,t) \tag{4.28}$$

as the approximate solution of IVP (4.23)-(4.24) from which the approximate analytical solutions are evaluated when $\alpha = 2,\ 1.9,\ 1.7$ as follows:

$$\begin{aligned} \alpha = 2 \quad &: \quad u_3^V(x,t) = e^{-x} + e^{-x}\frac{t^2}{2} + e^{-x}\frac{t^4}{24} + e^{-x}\frac{t^6}{720} + \ldots \\ &\qquad -2.91\times10^{-545}e^{-x}t^{564} - 2.57\times10^{-549}e^{-x}t^{566}, \\ \alpha = 1.9 \quad &: \quad u_3^V(x,t) = e^{-x} + 0.54e^{-x}t^{1.9} + 5.61\times10^{-2}e^{-x}t^{3.8} + 2.41\times10^{-3}e^{-x}t^{5.7} + \ldots \\ &\qquad -3.41\times10^{-507}e^{-x}t^{535.8} - 4.49\times10^{-511}e^{-x}t^{537.7}, \\ \alpha = 1.7 \quad &: \quad u_3^V(x,t) = e^{-x} + +0.64e^{-x}t^{1.7} + 9.86\times10^{-2}e^{-x}t^{3.4} + 7.02\times10^{-3}e^{-x}t^{5.1} + \ldots \\ &\qquad -5.63\times10^{-433}e^{-x}t^{479.4} - 1.61\times10^{-436}e^{-x}t^{481.1}. \end{aligned} \tag{4.29}$$

From the above results, the 3D-graphical approximate solutions of $u_3^V(x,t)$ evaluated at $\alpha = 2,\ 1.9$ and 1.7 are shown in Figure 3 (a).

LHPM : From (3.20) and (4.25), the first few terms of $H_n(v(x,t,p))$ are shown as

$$\begin{aligned} H_0(v(x,t,p)) &= -\left(v_0(x,t)\right)_{xxxx} + \left(v_0(x,t)\right)_{xx} + 2\left(v_0(x,t)\right)_{xx}\int_0^1\left(\left(v_0(x,t)\right)_x\right)^2\,dx, \\ H_1(v(x,t,p)) &= -\left(v_1(x,t)\right)_{xxxx} + \left(v_1(x,t)\right)_{xx} + 2\left(v_1(x,t)\right)_{xx}\int_0^1\left(\left(v_0(x,t)\right)_x\right)^2\,dx \\ &+ 4\left(v_0(x,t)\right)_{xx}\int_0^1\left(\left(v_0(x,t)\right)_x\left(v_1(x,t)\right)_x\right)\,dx, \\ H_2(v(x,t,p)) &= -\left(v_2(x,t)\right)_{xxxx} + \left(v_2(x,t)\right)_{xx} + 2\left(v_2(x,t)\right)_{xx}\int_0^1\left(\left(v_0(x,t)\right)_x\right)^2\,dx, \\ &+ 4\left(v_1(x,t)\right)_{xx}\int_0^1\left(\left(v_0(x,t)\right)_x\left(v_1(x,t)\right)_x\right)\,dx \\ &+ 2\left(v_0(x,t)\right)_{xx}\int_0^1\left(\left(v_1^2(x,t)\right)_x + 2\left(v_0(x,t)\right)_x\left(v_2(x,t)\right)_x\right)\,dx. \end{aligned} \tag{4.30}$$

In like manner as above, we use $\mathrm{E}_\alpha(t^\alpha) \approx \sum_{k=0}^{10} \dfrac{t^{\alpha k}}{\Gamma(\alpha k+1)}$ in the following calculations. With the symbolically computational limitation of Maple software for $1 < \alpha \leq 2$, so we can express the solution components for only $v_0(x,t)$ and $v_1(x,t)$ by using (3.22), (3.23) and (4.30) as follows.

$$\begin{aligned} p^0 &: v_0(x,t) = I(x,t) = e^{-x} + \frac{e^{-x}t^\alpha}{\Gamma(\alpha+1)}, \\ p^1 &: v_1(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-u_0 + H_0(v(x,t,p)) + g(x,t)\right\}\right\} \\ &\qquad = \frac{\left(0.13e^{-x} - e^{-2-x}\right)t^\alpha}{\Gamma(\alpha+1)} + \frac{\left(1.41e^{-x} - 3e^{-2-x}\right)t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots + \gamma(x;\alpha)t^{31\alpha}, \end{aligned}$$

where $\gamma(x;\alpha) = \frac{\gamma_1(x;\alpha)}{\gamma_2(\alpha)}$ in which

$$\begin{aligned} \gamma_1(x;\alpha) &= 4096\sqrt{6}\pi^{\frac{27}{2}}(205891132094649)^\alpha e^{-x}\Gamma\left(2\alpha+\frac{2}{15}\right)\Gamma\left(2\alpha+\frac{4}{15}\right)\Gamma\left(2\alpha+\frac{8}{15}\right)\Gamma\left(2\alpha+\frac{11}{15}\right) \\ &\quad \times\Gamma\left(2\alpha+\frac{1}{3}\right)\Gamma\left(2\alpha+\frac{13}{15}\right)\Gamma\left(2\alpha+\frac{2}{3}\right)\Gamma\left(2\alpha+\frac{14}{15}\right)\Gamma\left(2\alpha+\frac{7}{15}\right)\Gamma\left(2\alpha+\frac{1}{15}\right), \\ \gamma_2(\alpha) &= 1083179328015\Gamma(31\alpha+1)\left(\Gamma(2\alpha+1)\right)^2\left(\Gamma\left(2\alpha+\frac{1}{5}\right)\right)^2\left(\Gamma\left(2\alpha+\frac{3}{5}\right)\right)^2\left(\Gamma\left(2\alpha+\frac{4}{5}\right)\right)^2 \\ &\quad \times\left(\Gamma\left(2\alpha+\frac{2}{5}\right)\right)^2. \end{aligned}$$

By the process of the LHPM described above, the solution of IVP (4.23)-(4.24) can be written in the following form

$$u(x,t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i v_i(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \dots\ . \tag{4.31}$$

For the symbolically computational limitation, so the third-order approximate solution $u_3^H(x,t) = \lim_{p\to 1}\sum_{i=0}^{3} p^i v_i(t)$ is utilized to compute the approximate analytical solutions of (4.23) for $\alpha = 2,\ 1.9,\ 1.7$ as listed below.

$$\begin{aligned} \alpha = 2 \quad &: \quad u_3^H(x,t) = e^{-x} - 1.04\times 10^{-11}e^{-x}t^2 + e^{-x}\frac{t^2}{2} + e^{-x}\frac{t^4}{24} + \dots \\ &\qquad\qquad +5.06\times 10^{-119}e^{-x}t^{126} + 2.01\times 10^{-122}e^{-x}t^{128}, \\ \alpha = 1.9 \quad &: \quad u_3^H(x,t) = e^{-x} + 0.54e^{-x}t^{1.9} + 5.61\times 10^{-2}e^{-x}t^{3.8} + 2.42\times 10^{-3}e^{-x}t^{5.7} + \dots \\ &\qquad\qquad +1.18\times 10^{-110}e^{-x}t^{119.7} + 6.97\times 10^{-114}e^{-x}t^{121.6}, \qquad (4.32) \\ \alpha = 1.7 \quad &: \quad u_3^H(x,t) = e^{-x} + 0.64e^{-x}t^{1.7} + 9.86\times 10^{-2}e^{-x}t^{3.4} + 7.02\times 10^{-3}e^{-x}t^{5.1} + \dots \\ &\qquad\qquad +2.39\times 10^{-94}e^{-x}t^{107.1} + 3.08\times 10^{-97}e^{-x}t^{110.8}. \end{aligned}$$

From the above results, the 3D-graphical approximate solutions of $u_3^H(x,t)$ evaluated at $\alpha = 2$, 1.9 and 1.7 are portrayed in Figure 3 (b).

It is worth noting that the approximate analytical solutions obtained by the two methods are closer to the exact solutions $e^{-x}\mathrm{E}_2(t^2)$, $e^{-x}\mathrm{E}_{1.9}(t^{1.9})$ and $e^{-x}\mathrm{E}_{1.7}(t^{1.7})$ for $\alpha = 2$, 1.9 and 1.7, respectively, if the value of n and the number of expanded terms in $\mathrm{E}_\alpha(t^\alpha)$ are sufficiently larger. According to the convergence analysis for the two methods as referred above, $u_n(x,t)$ as $n\to\infty$ is the best approximation solution of (4.23) tending to the exact solution $u(x,t) = e^{-x}\mathrm{E}_\alpha(t^\alpha)$. At $x = 0$, for example, the 2D-graphs of the solutions $u_3^V(0,t)$ and $u_3^H(0,t)$ quite agree with those of the exact solutions for $\alpha = 2,\ 1.9,\ 1.7$ on the plotted domain as shown in Figure 4. The arrow appearing in Figure 4 points in the direction of increasing value of α.

Furthermore, Table 5 and Table 6 provide numerical values for the exact solution $u(0,t) = \mathrm{E}_\alpha(t^\alpha)$, the approximate solutions $u_3^V(0,t)$, $u_3^H(0,t)$ and their absolute errors

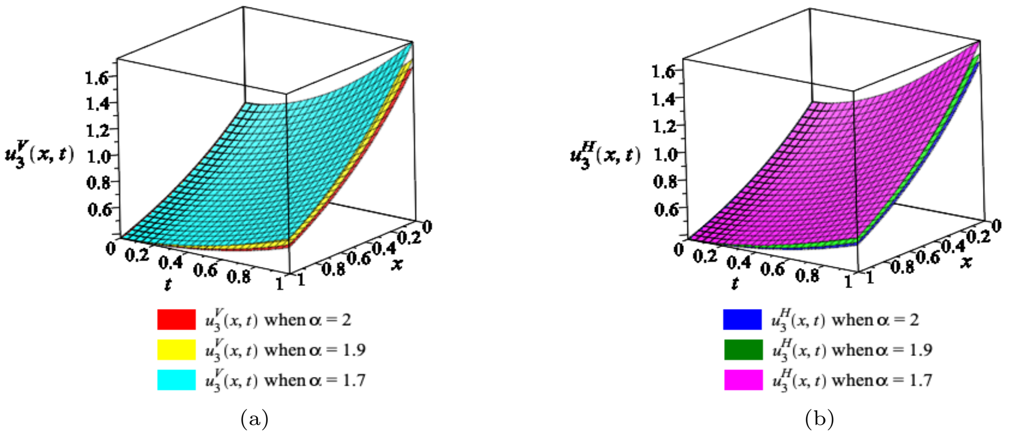


FIGURE 3. The 3D approximate solution graphs of IVP (4.23)-(4.24) for $\alpha = 2,\ 1.9,\ 1.7$: (a) $u_3^V(x,t)$ by the LVIM; (b) $u_3^H(x,t)$ by the LHPM.

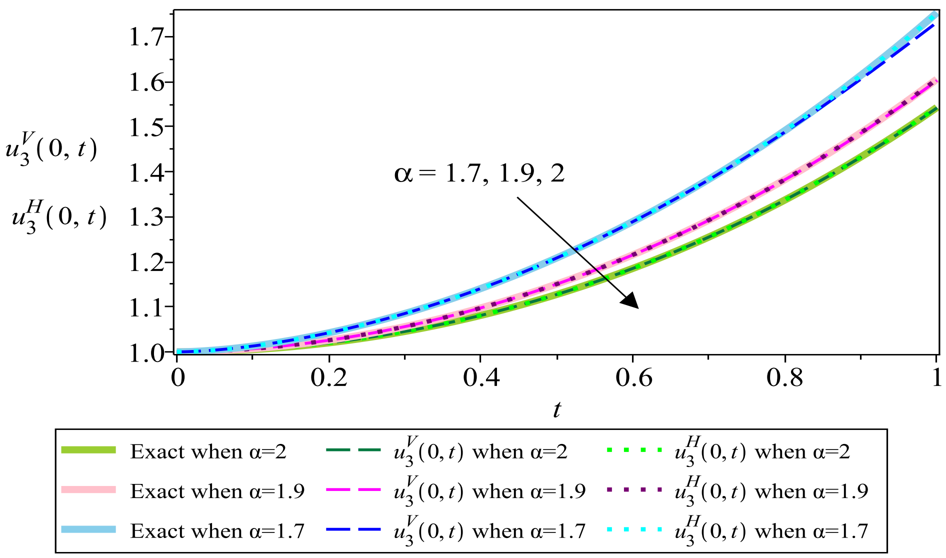


FIGURE 4. Comparison between the exact solution $u_3(0,t)$ and the approximate analytical solutions $u_3^V(0,t)$, $u_3^H(0,t)$ for IVP (4.23)-(4.24) when $\alpha = 2,\ 1.9,\ 1.7$.

of problem (4.23) evaluated at $x = 0$ and some interesting values of t when $\alpha = 2,\ 1.7$, respectively.

Example 4. Consider the following nonhomogeneous system of FIDEs for $\alpha \in (1,2]$ as [52]

$$\begin{aligned} {}_C D_t^{\alpha} u(t) &= 1 - \frac{t^3}{3} - \frac{(v'(t))^2}{2} + \frac{1}{2}\int_0^t \left(u^2(\tau) + v^2(\tau)\right)\, d\tau, \\ {}_C D_t^{\alpha} v(t) &= -1 + t^2 - tu(t) + \frac{1}{4}\int_0^t \left(u^2(\tau) - v^2(\tau)\right)\, d\tau, \end{aligned} \tag{4.33}$$

t	$u(0,t)$	$u_3^V(0,t)$	$u_3^H(0,t)$	Absolute error $\lvert u(0,t)-u_3^V(0,t)\rvert$	Absolute error $\lvert u(0,t)-u_3^H(0,t)\rvert$
0.0	1.00000	1.00000	1.00000	0.00000	0.00000
0.2	1.0200667556190759	1.02006675449	1.0200667556180516	1.13109×10^{-9}	1.02424×10^{-12}
0.4	1.08107237184	1.08107203759	1.08107237108	3.34247×10^{-7}	7.57714×10^{-10}
0.6	1.18546521824	1.18545473169	1.18546515176	1.04866×10^{-5}	6.64781×10^{-8}
0.8	1.33743494631	1.33729689580	1.33743300497	1.38051×10^{-4}	1.94133×10^{-6}
1.0	1.54308063482	1.54192223865	1.54304968688	1.15840×10^{-3}	3.09479×10^{-5}

TABLE 5. Exact solution $u(0,t)$, approximate solutions $u_3^V(0,t)$, $u_3^H(0,t)$ and their absolute errors for IVP (4.23)-(4.24) for $\alpha = 2$.

t	$u(0,t)$	$u_3^V(0,t)$	$u_3^H(0,t)$	Absolute error $\lvert u(0,t)-u_3^V(0,t)\rvert$	Absolute error $\lvert u(0,t)-u_3^H(0,t)\rvert$
0.0	1.00000	1.00000	1.00000	0.00000	0 .00000
0.2	1.04238379960	1.04238369884	1.04238379940	1.00766×10^{-7}	2.08140×10^{-10}
0.4	1.14079483338	1.14078083529	1.14079471711	1.39981×10^{-5}	1.16270×10^{-7}
0.6	1.28955424851	1.28925397767	1.28954808400	3.00271×10^{-4}	6.16450×10^{-6}
0.8	1.49152425979	1.48843338894	1.49139972102	3.09087×10^{-3}	1.24538×10^{-4}
1.0	1.75336267944	1.73196773038	1.75189582693	2.13949×10^{-2}	1.46685×10^{-3}

TABLE 6. Exact solution $u(0,t)$, approximate solutions $u_3^V(0,t)$, $u_3^H(0,t)$ and their absolute errors for IVP (4.23)-(4.24) for $\alpha = 1.7$.

with the initial conditions

$$\begin{aligned} u(0) &= 1,\ \ u'(0) = 2, \\ v(0) &= -1,\ \ v'(0) = 0. \end{aligned} \tag{4.34}$$

For $\alpha = 2$, the exact solution of this problem is $(u(t), v(t)) = (t + e^t, t - e^t)$.
LVIM : As described above, we define $h_{1n}(t) = N_1(u_n(t), v_n(t)) + g_1(t)$ and $h_{2n}(t) = N_2(u_n(t), v_n(t)) + g_2(t)$, where

$$\begin{aligned} N_1(u_n(t), v_n(t)) &= -\frac{(v_n'(t))^2}{2} + \frac{1}{2}\int_0^t \left(u_n^2(\tau) + v_n^2(\tau)\right) d\tau \text{ and } g_1(t) = 1 - \frac{t^3}{3}, \\ N_2(u_n(t), v_n(t)) &= \ - t u_n(t) + \frac{1}{4}\int_0^t \left(u_n^2(\tau) - v_n^2(\tau)\right) d\tau \text{ and } g_2(t) = -1 + t^2, \end{aligned} \tag{4.35}$$

and apply the iterative relation (3.8) to the system consisting of (4.33) and (4.34), we obtain

$$\begin{aligned} u_{n+1}(t) &= u_0(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_{1n}(t)\right\}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{u(0)}{s} + \frac{u'(0)}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_{1n}(t)\right\}\right\}, \\ v_{n+1}(t) &= v_0(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_{2n}(t)\right\}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{v(0)}{s} + \frac{v'(0)}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{h_{2n}(t)\right\}\right\}. \end{aligned} \tag{4.36}$$

Using (3.9) and (4.36), the first few approximate analytical solutions for $1 < \alpha \leq 2$ are shown as

$$
\begin{aligned}
u_0(t) &= \mathscr{L}^{-1}\left\{\frac{u(0)}{s}+\frac{u'(0)}{s^2}\right\} = 1+2t,\\
v_0(t) &= \mathscr{L}^{-1}\left\{\frac{v(0)}{s}+\frac{v'(0)}{s^2}\right\} = -1,\\
u_1(t) &= 1+2t+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{2t^{3+\alpha}}{\Gamma(4+\alpha)},\\
v_1(t) &= -1-\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}-\frac{t^{2+\alpha}}{\Gamma(\alpha+3)}+\frac{2t^{3+\alpha}}{\Gamma(4+\alpha)},\\
u_2(t) &= 1+2t+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{2t^{3+\alpha}}{\Gamma(4+\alpha)}+\frac{2t^{2\alpha+1}}{\Gamma(2+2\alpha)}\\
&\quad+\frac{2t^{2+2\alpha}}{\Gamma(3+2\alpha)}+\frac{3t^{3+2\alpha}}{\Gamma(4+2\alpha)}+...+\frac{5\Gamma(6+2\alpha)\,t^{5+3\alpha}}{2\Gamma(6+3\alpha)(5+2\alpha)(\Gamma(3+\alpha))^2},\\
v_2(t) &= 1+2t+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{2t^{3+\alpha}}{\Gamma(4+\alpha)}+\frac{2t^{2\alpha+1}}{\Gamma(2+2\alpha)}\\
&\quad-\frac{(1+\alpha)\,t^{2\alpha+1}}{\Gamma(2+2\alpha)}-\frac{t^{2+2\alpha}}{\Gamma(3+2\alpha)}-\frac{(7+2\alpha)\,t^{3+2\alpha}}{2\Gamma(4+2\alpha)}+...+\frac{3\Gamma(7+2\alpha)\,t^{6+3\alpha}}{2\Gamma(3\alpha+7)(\Gamma(4+\alpha))^2}.
\end{aligned}
$$

By the method, $(u_n(t), v_n(t))$ as $n \to \infty$ is the best approximation solution of IVP (4.33)-(4.34). Particularly, $(u_n(t), v_n(t))$ for $\alpha = 2$ tends to the exact solution $(t+e^t, t-e^t)$ when $n \to \infty$. With the symbolically computational limitation of Maple software, so it is extremely difficult to demonstrate the analytical solutions for $1 < \alpha \leq 2$ when n is higher. Consequently, we denote

$$u_5^V(t) = u_5(t) \text{ and } v_5^V(t) = v_5(t) \tag{4.37}$$

as the approximate solution of system (4.33) from which the approximate analytical solutions are evaluated when $\alpha = 2,\ 1.9,\ 1.7$ as follows:

$$
\begin{aligned}
&\alpha=2:\begin{cases} u_5^V(t) = 1+2t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+...+9.54\times10^{-90}t^{124}+3.05\times10^{-91}t^{125},\\ v_5^V(t) = -1-\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{24}-\frac{t^5}{120}+...+3.98\times10^{-90}t^{124}+1.27\times10^{-91}t^{125},\end{cases}\\
&\alpha=1.9:\begin{cases} u_5^V(t) = 1+2t+0.54t^{1.9}+0.18t^{2.9}+...+3.77\times10^{-86}t^{120.9}+1.23\times10^{-87}t^{121.9},\\ v_5^V(t) = -1-0.54t^{1.9}-0.18t^{2.9}-4.83\times10^{-2}t^{2.9}+...+1.57\times10^{-86}t^{120.9}\\ \qquad +5.17\times10^{-88}t^{121.9},\end{cases}\\
&\alpha=1.7:\begin{cases} u_5^V(t) = 1+2t+0.64t^{1.7}+0.23t^{2.7}+...+4.69\times10^{-79}t^{114.7}+1.62\times10^{-80}t^{115.7},\\ v_5^V(t) = -1-0.64t^{1.7}-0.23t^{2.7}-6.48\times10^{-2}t^{3.7}+...+1.96\times10^{-79}t^{114.7}\\ \qquad +6.77\times10^{-81}t^{115.7}.\end{cases}
\end{aligned}
\tag{4.38}
$$

$LHPM$: From (3.20) and (4.35), the following relations for $n = 0,\ 1,\ 2,\ \dots$,

$$
\begin{aligned}
\mathcal{H}_n(\bar{u}(t,p),\bar{v}(t,p)) &= \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N_1\left(\bar{u}(t,p),\bar{v}(t,p)\right)\right)_{p=0}\\
&= \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N_1\left(\sum_{i=0}^{\infty}p^i\bar{u}_i(t),\sum_{i=0}^{\infty}p^i\bar{v}_i(t)\right)\right)_{p=0},\\
\mathscr{H}_n(\bar{u}(t,p),\bar{v}(t,p)) &= \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N_2\left(\bar{u}(t,p),\bar{v}(t,p)\right)\right)_{p=0}\\
&= \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N_2\left(\sum_{i=0}^{\infty}p^i\bar{u}_i(t),\sum_{i=0}^{\infty}p^i\bar{v}_i(t)\right)\right)_{p=0},
\end{aligned}
\tag{4.39}
$$

are He's polynomials for the recursive formulas of $u(t)$ and $v(t)$, respectively with $\bar{u}(t,p) = \sum_{i=0}^{\infty}p^i\bar{u}_i(t)$ and $\bar{v}(t,p) = \sum_{i=0}^{\infty}p^i\bar{v}_i(t)$. Using (4.39), one can obtain the first few iterations of $\mathcal{H}_n(\bar{u}(t,p),\bar{v}(t,p))$ as follows:

$$
\begin{aligned}
\mathcal{H}_0(\bar{u}(t,p),\bar{v}(t,p)) &= -\frac{(\bar{v}_0'(t))^2}{2}+\frac{1}{2}\int_0^t\Big((\bar{u}_0(\tau))^2+(\bar{v}_0(\tau))^2\Big)d\tau,\\
\mathcal{H}_1(\bar{u}(t,p),\bar{v}(t,p)) &= -\bar{v}_0'(t)\bar{v}_1'(t)+\int_0^t\Big(\bar{u}_0(\tau)\bar{u}_1(\tau)+\bar{v}_0(\tau)\bar{v}_1(\tau)\Big)d\tau,\\
\mathcal{H}_2(\bar{u}(t,p),\bar{v}(t,p)) &= -\frac{(\bar{v}_1'(t))^2}{2}-\bar{v}_0'(t)\bar{v}_2'(t)\\
&\quad+\frac{1}{2}\int_0^t\Big((\bar{u}_1(\tau))^2+2\bar{u}_0(\tau)\bar{u}_2(\tau)+(\bar{v}_1(\tau))^2+2\bar{v}_0(\tau)\bar{v}_2(\tau)\Big)d\tau,\\
\mathcal{H}_3(\bar{u}(t,p),\bar{v}(t,p)) &= -\bar{v}_1'(t)\bar{v}_2'(t)-\bar{v}_0'(t)\bar{v}_3'(t)\\
&\quad+\int_0^t\Big(\bar{u}_1(\tau)\bar{u}_2(\tau)+\bar{u}_0(\tau)\bar{u}_3(\tau)+\bar{v}_1(\tau)\bar{v}_2(\tau)+\bar{v}_0(\tau)\bar{v}_3(\tau)\Big)d\tau,
\end{aligned}
\tag{4.40}
$$

and the first few terms of $\mathscr{H}_n(\bar{u}(t,p),\bar{v}(t,p))$ as follows:

$$
\begin{aligned}
\mathscr{H}_0(\bar{u}(t,p),\bar{v}(t,p)) &= -t\bar{u}_0(t)+\frac{1}{4}\int_0^t\Big((\bar{u}_0(\tau))^2-(\bar{v}_0(\tau))^2\Big)d\tau,\\
\mathscr{H}_1(\bar{u}(t,p),\bar{v}(t,p)) &= -t\bar{u}_1(t)+\frac{1}{2}\int_0^t\Big(\bar{u}_0(\tau)\bar{u}_1(\tau)-\bar{v}_0(\tau)\bar{v}_1(\tau)\Big)d\tau,\\
\mathscr{H}_2(\bar{u}(t,p),\bar{v}(t,p)) &= -t\bar{u}_2(t)+\frac{1}{4}\int_0^t\Big((\bar{u}_1(\tau))^2+2\bar{u}_0(\tau)\bar{u}_2(\tau)-(\bar{v}_1(\tau))^2-2\bar{v}_0(\tau)\bar{v}_2(\tau)\Big)d\tau,\\
\mathscr{H}_3(\bar{u}(t,p),\bar{v}(t,p)) &= -t\bar{u}_3(t)+\frac{1}{2}\int_0^t\Big(\bar{u}_1(\tau)\bar{u}_2(\tau)+\bar{u}_0(\tau)\bar{u}_3(\tau)-\bar{v}_1(\tau)\bar{v}_2(\tau)-\bar{v}_0(\tau)\bar{v}_3(\tau)\Big)d\tau.
\end{aligned}
\tag{4.41}
$$

Applying relation (3.23) along with (3.22), (4.40) and (4.41) to the problem, then the first few recursive formulas are obtained as:

$$p^0 : \bar{u}_0(t) = \mathcal{I}(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\bar{u}(0,p)s^{\alpha-1} + \bar{u}'(0,p)s^{\alpha-2} + \mathscr{L}\{u_0\}\right]\right\} = 1 + 2t + \frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$\bar{v}_0(t) = \mathscr{I}(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\bar{v}(0,p)s^{\alpha-1} + \bar{v}'(0,p)s^{\alpha-2} + \mathscr{L}\{v_0\}\right]\right\} = -1 - \frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$\begin{aligned} p^1 : \bar{u}_1(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-u_0 + \mathcal{H}_0(\bar{u}(t,p), \bar{v}(t,p)) + g_1(t)\right\}\right\} \\ &= \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{2t^{3+\alpha}}{\Gamma(4+\alpha)} + \frac{2t^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{2(1+\alpha)t^{2+2\alpha}}{\Gamma(3+2\alpha)} \\ &\quad + \frac{\Gamma(1+2\alpha)t^{1+3\alpha}}{(\Gamma(1+\alpha))^2\Gamma(2+3\alpha)} - \frac{\alpha^2\Gamma(-1+2\alpha)t^{3\alpha-2}}{2\left(\Gamma(1+\alpha)\right)^2\Gamma(3\alpha-1)}, \\ \bar{v}_1(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-v_0 + \mathscr{H}_0(\bar{u}(t,p), \bar{v}(t,p)) + g_2(t)\right\}\right\} \\ &= -\frac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{2t^{3+\alpha}}{\Gamma(4+\alpha)} - \frac{(1+\alpha)\,t^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{(1+\alpha)\,t^{2+2\alpha}}{\Gamma(3+2\alpha)}, \\ p^2 : \bar{u}_2(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{\mathcal{H}_1(\bar{u}(t,p), \bar{v}(t,p))\right\}\right\} \\ &= \frac{2t^{2+2\alpha}}{\Gamma(3+2\alpha)} + \frac{(7+2\alpha)\,t^{3+2\alpha}}{\Gamma(4+2\alpha)} + \frac{4(3+\alpha)\,t^{4+2\alpha}}{\Gamma(5+2\alpha)} + \frac{4(4+\alpha)\,t^{5+2\alpha}}{\Gamma(2\alpha+6)} + \dots \\ &\quad + \frac{4^\alpha\Gamma(2+4\alpha)\,\Gamma\left(\alpha+\frac{1}{2}\right)t^{2+5\alpha}}{\sqrt{\pi}\left(\Gamma(1+\alpha)\right)^2\Gamma(2+3\alpha)\,\Gamma(3+5\alpha)}, \\ \bar{v}_2(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{\mathscr{H}_1(\bar{u}(t,p), \bar{v}(t,p))\right\}\right\} \\ &= -\frac{(2+\alpha)\,t^{2+2\alpha}}{\Gamma(3+2\alpha)} - \frac{(7+2\alpha)\,t^{3+2\alpha}}{2\Gamma(4+2\alpha)} + \frac{2(4+\alpha)\,t^{5+2\alpha}}{\Gamma(2\alpha+6)} - \frac{(9\alpha+7)\,t^{2+3\alpha}}{2\Gamma(3+3\alpha)} + \dots \\ &\quad - \frac{4^\alpha\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma(1+4\alpha)\,t^{5\alpha-1}}{\sqrt{\pi}\,(-64+256\alpha)\,\Gamma(5\alpha)\left(\Gamma(1+\alpha)\right)^2\Gamma(3\alpha-1)}. \end{aligned}$$

The above results are obtained by replacing $\bar{u}(0,p) = u_0 = u(0) = 1$, $\bar{v}(0,p) = v_0 = v(0) = -1$ and $\bar{u}'(0,p) = u'(0) = 2$, $\bar{v}'(0,p) = v'(0) = 0$. By the LHPM as described above, the solution of IVP (4.33)-(4.34) can be expressed as

$$u(t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i\bar{u}_i(t) = \bar{u}_0(t) + \bar{u}_1(t) + \bar{u}_2(t) + \dots, \tag{4.42}$$

$$v(t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i\bar{v}_i(t) = \bar{v}_0(t) + \bar{v}_1(t) + \bar{v}_2(t) + \dots. \tag{4.43}$$

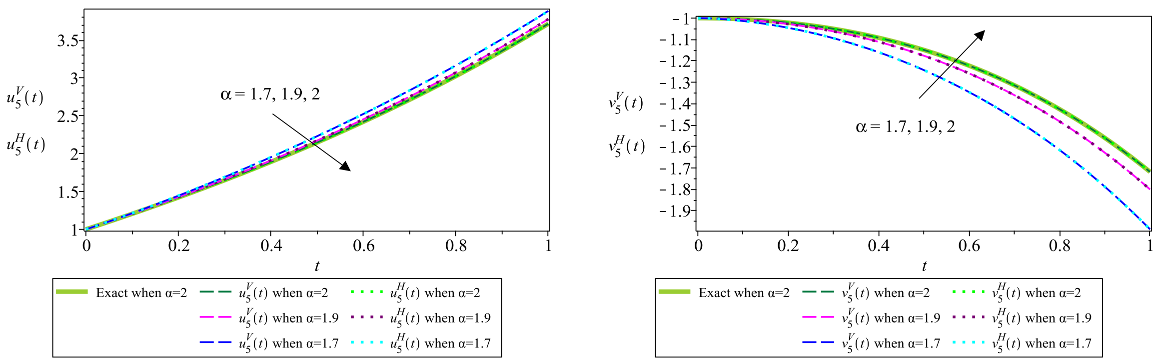
For the numerically computational convenience, we employ the following fifth-order approximate solution

$$\left(u_5^H(t), v_5^H(t)\right) = \left(\lim_{p\to 1}\sum_{i=0}^{5} p^i\bar{u}_i(t), \lim_{p\to 1}\sum_{i=0}^{5} p^i\bar{v}_i(t)\right) \tag{4.44}$$

to estimate the solution of IVP (4.33)-(4.34) for $\alpha = 2,\ 1.9,\ 1.7$ as follows:

$$\alpha=2:\begin{cases}u_5^H(t)=1+2t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+...+6.11\times10^{-16}t^{26}+2.15\times10^{-17}t^{27},\\ v_5^H(t)=-1-\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{24}-\frac{t^5}{120}+...+2.31\times10^{-16}t^{26}+1.04\times10^{-17}t^{27},\end{cases}$$
$$\alpha=1.9:\begin{cases}u_5^H(t)=1+2t+0.54t^{1.9}+0.18t^{2.9}+...+5.22\times10^{-15}t^{25}+2.16\times10^{-16}t^{25.9},\\ v_5^H(t)=-1-0.54t^{1.9}-0.18t^{2.9}-4.83\times10^{-2}t^{3.9}+...+1.99\times10^{-15}t^{25}\\ \qquad\qquad +1.05\times10^{-16}t^{25.9},\end{cases}\tag{4.45}$$
$$\alpha=1.7:\begin{cases}u_5^H(t)=1+2t+0.64t^{1.7}+0.23t^{2.7}+...+3.42\times10^{-13}t^{23}+1.92\times10^{-14}t^{23.7},\\ v_5^H(t)=-1-0.64t^{1.7}-0.23t^{2.7}-6.48\times10^{-2}t^{3.7}+...+1.32\times10^{-13}t^{23}\\ \qquad\qquad +9.53\times10^{-15}t^{23.7}.\end{cases}$$

From (4.38) and (4.45), it is worth noting that the approximate analytical solutions $\left(u_5^V(t),\ v_5^V(t)\right)$ and $\left(u_5^H(t),\ v_5^H(t)\right)$ for $\alpha=2$ are nearly identical to the exact solution $(u(t),v(t))=(t+e^t,t-e^t)$ as observed in Figure 5. Moreover, the simulation graphs of $\left(u_5^V(t),\ v_5^V(t)\right)$ and $\left(u_5^H(t),\ v_5^H(t)\right)$ for $\alpha=1.9,1.7$ are also plotted in Figure 5. The arrows emerging in Figure 5 aim in the direction of increasing value of α.



(a) (b)

FIGURE 5. Comparison between the exact solution $(u(t),v(t))=(t+e^t,t-e^t)$ for $\alpha=2$ and the approximate analytical solutions $\left(u_5^V(t),\ v_5^V(t)\right)$ and $\left(u_5^H(t),\ v_5^H(t)\right)$ when $\alpha=2,\ 1.9,\ 1.7$ for IVP (4.33)-(4.34).

Additionally, Table 7 and Table 8 give numerical values evaluated at some values of t for the exact solution $(u(t),v(t))=(t+e^t,t-e^t)$, the approximate solutions $\left(u_5^V(t),\ v_5^V(t)\right)$, $\left(u_5^H(t),\ v_5^H(t)\right)$ and their absolute errors for IVP (4.33)-(4.34) when $\alpha=2$.

5. Discussions and Conclusions

In this article, the LVIM and LHPM have been successfully applied to obtain approximate analytical solutions for certain fractional integro-differential equations (FIDEs) with the Caputo time-derivative, for instance, the nonhomogeneous Volterra-Fredholm FIDE, the fractional-order beam problem and the nonhomogeneous system of FIDEs as shown in section 4. The LVIM is a powerful and reliable technique for solving FIDEs. The method uses the Laplace transform to identify the Laplace-Lagrange multiplier which is incorporated with the process of VIM. By the LVIM, the Lagrange multiplier can be easily discovered without complicated computation and new variation iteration schemes

t	$u(t)$	$u_5^V(t)$	$u_5^H(t)$	Absolute error $\lvert u(t)-u_5^V(t)\rvert$	Absolute error $\lvert u(t)-u_5^H(t)\rvert$
0.0	1.00000	1.00000	1.00000	0.00000	0.00000
0.2	1.42140275816017	1.42140275816046	1.42140275815959	2.89958×10^{-13}	5.73049×10^{-13}
0.4	1.89182469764127	1.89182469764396	1.89182469759574	2.67756×10^{-12}	4.55390×10^{-11}
0.6	2.42211880039051	2.42211880044338	2.42211879411346	5.28629×10^{-11}	6.27705×10^{-9}
0.8	3.02554092849247	3.02554093113008	3.02554069262813	2.63762×10^{-9}	2.35864×10^{-7}
1.0	3.71828182845904	3.71828189066379	3.71827773529571	6.22048×10^{-8}	4.093168×10^{-6}

Table 7. Exact solution $u(t)$, approximate solutions $u_5^V(t)$, $u_5^H(t)$ and their absolute errors for IVP (4.33)-(4.34) for $\alpha = 2$.

t	$v(t)$	$v_5^V(t)$	$v_5^H(t)$	Absolute error $\lvert v(t)-v_5^V(t)\rvert$	Absolute error $\lvert v(t)-v_5^H(t)\rvert$
0.0	-1.00000	-1.00000	-1.00000	0.00000	0.00000
0.2	-1.02140275816017	-1.02140275816926	-1.02140275816044	9.09507×10^{-12}	2.72207×10^{-13}
0.4	-1.09182469764127	-1.09182469768238	-1.09182469764331	4.10887×10^{-11}	2.02785×10^{-12}
0.6	-1.22211880039051	-1.22211880048690	-1.22211880034257	9.63910×10^{-11}	4.79438×10^{-11}
0.8	-1.42554092849247	-1.42554092804186	-1.42554092554570	4.50612×10^{-10}	2.94677×10^{-9}
1.0	-1.71828182845905	-1.71828180798636	-1.71828176551006	2.04727×10^{-8}	6.29490×10^{-8}

Table 8. Exact solution $v(t)$, approximate solutions $v_5^V(t)$, $v_5^H(t)$ and their absolute errors for IVP (4.33)-(4.34) for $\alpha = 2$.

for solving FIDEs can be established. Generally, the more updated solutions obtained by means of the LVIM are closer to the exact solutions of problems. On the other hand, the LHPM is a hybrid approach for solving nonlinear FIDEs and combines the Laplace transform and the homotopy perturbation method. The LHPM can be applied to solve FIDEs for approximate analytical solutions written in terms of a series form converging very fast to its exact solution. By the use of He's polynomials in the method, the nonlinear terms arising in nonlinear problems can be conveniently implemented. It is worth mentioning that the LHPM can reduce the considerable amount of computational work as compared with other existing methods while still maintain the high accuracy of results.

By the mentioned advantages of the two methods along with the use of Maple software, the obtained approximate analytical solutions for each proposed problems are simply accessible, highly accurate and quite agree with the existing exact solutions on the considered domains. The approximate solutions are displayed in terms of their explicit analytical forms, numerical graphs and numerical values at some interesting points for different values of the fractional-order to explore the effects of the fractional-order variation on the solution behaviors. In summary, the LVIM and LHPM could be applied to other real-world problems occurring in applied sciences and engineering for their precisely analytical solutions with less computational work.

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