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The Partial Operation of Formulas with Applications to Formulas Generated by Order-decreasing Mappings

Khwancheewa Wattanatripop[1] and Thodsaporn Kumduang[2,*]

[1] *Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50300, Thailand*
e-mail : khwancheewa.wat@rmutl.ac.th

[2] *Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Rattanakosin, Nakhon Pathom 73170, Thailand*
e-mail : thodsaporn.kum@rmutr.ac.th

Abstract Based on algebras of type τ and algebraic systems of type (τ, τ'), in this work, the partial superposition operation on the union of the set of all terms consisting of expressions and the set of all formulas consisting of equations of terms and expressions connected by logical connectors and a quantifier is defined. This allows us to form a partial algebra of type $(n+1)$ satisfying the axiom of superassociativity as a weak identity where n is an arbitrary positive integer. In particular, using the concept of order-decreasing full terms of type τ_n, terms in which each component is constructed from a variable in the alphabet X_n having some conditions and an order-decreasing transformation defined on a finite chain $\bar{n} = \{1, \ldots, n\}$ ordered in a canonical way, a mapping α in the full transformation semigroup on $\bar{n}$ satisfying the inequality $\alpha(x) \leq x$ for all $x \in \bar{n}$, the set of all order-decreasing full formulas of a given type is further studied. An algebra of type $(n+1)$ consisting of the set of all order-decreasing full formulas and a superposition operation which satisfies the axiom of superassociativity is proved.

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1. Introduction and preliminaries

The wide use of terms or trees as a natural structure in computer science allows us to consider its theoretical basics [1]. In sense of the study of logic, terms can be regared as one of the important tools in both first and second-order languages [2]. Consider a family $\{f_i \mid i \in I\}$ of operation symbols, indexed by the set I. The type is the sequence $\tau = (n_i)_{i \in I}$ of the natural number arities of the symbols f_i. For a set $X = \{x_1, x_2, x_3, \ldots\}$, we denote a countably infinite set of alphabet whose elements are called *variables*, particularly, we let $X_n = \{x_1, \ldots, x_n\}$. Notationally, the set $W_\tau(X_n)$ of all n-ary terms of type τ over the

*Corresponding author.

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alphabet X_n consist of expressions of two forms: $x_j \in X_n$ for all $1 \leq j \leq n$ and a term $f_i(t_1, \ldots, t_{n_i})$ of the inductive term $t_1, \ldots, t_{n_i}$ and the n_i-ary operation symbol f_i for all $i \in I$. Moreover, $W_\tau(X)$ denotes the set of all terms of type τ. For other basic facts and recent contributions on terms, one can refer the reader to [3–9].

We now illustrate some examples of terms. Let us consider a set $I = \{1, 2\}$ and the type $\tau = (3, 3)$ with two ternary operation symbols f_1 and f_2. Then we have

$$x_1, x_2, x_3, f_1(x_2, x_1, x_3), f_2(x_2, f_1(x_2, x_1, x_3), x_1) \in W_{(3,3)}(X_3),$$

$$x_4, f_2(x_4, x_1, x_3), f_1(x_2, x_5, x_1) \notin W_{(3,3)}(X_3).$$

The *variety of all generalized clones* [10] is a family of algebras satisfying the following four identities:

(C1) $\tilde{S}^n(\tilde{S}^n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n), \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{S}^n(\tilde{Z}, \tilde{S}^n(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n(\tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n))$;
(C2) $\tilde{S}^n(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_j$, for $1 \leq j \leq n$;
(C3) $\tilde{S}^n(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \lambda_j$, for $j > n$;
(C4) $\tilde{S}^n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}$;

where $\tilde{S}^n$ is an operation symbol, $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}$ are variables, and λ_j are symbols for variables. In general, (C1) is said to be the *superassociative law* since it generalizes the associative law. In fact, if we set $n = 1$, one can reduce it to the associative law of the form $\cdot(\cdot(a, b), c) = \cdot(a, \cdot(b, c))$. For more on the study of varieties and clones, see [2, 11–13]. Recall from [14, 15] that a nonempty set G with operation defined on G satisfying (C1) is called an *superassociative algebra*. Current developments in such algebras may be found in [16, 17]. Generally, an algebra of type $(n + 1)$ is said to be a *superassociative algebra with infinitely many nullary operations* if it satisfies (C1)-(C4).

One of the most important operations on terms is a generalized superposition operation. Basically, a new term is obtained after substituting all variables occuring in a former term by the other terms. This can be described by the $(n + 1)$-generalized superpostion S^n_g, $n \geq 1$,

$$S^n_g : W_\tau(X)^{n+1} \to W_\tau(X)$$

defined inductively by the following steps: for $t, t_1, \ldots, t_n \in W_\tau(X)$

(1) If $t = x_i$; $1 \leq i \leq n$, then $S^n_g(x_i, t_1, \ldots, t_n) := t_i$.
(2) If $t = x_i$; $n < i$, then $S^n_g(x_i, t_1, \ldots, t_n) := x_i$.
(3) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^n_g(t, t_1, \ldots, t_n) := f_i(S^n_g(s_1, t_1, \ldots, t_n), \ldots, S^n_g(s_{n_i}, t_1, \ldots, t_n))$.

We can form the algebra $(W_\tau(X), S^n_g, (x_j)_{j\geq 1})$ of type $(n + 1, 0, 0, 0, \ldots)$ consisting the universe $W_\tau(X)$ together with one $(n+1)$-ary operation S^n_g and the variable terms as infinitely many nullary operations. According to [10], the superassociative algebra of terms with infinitely many nullary operations was constructed. Hence, the superassociative algebra $(W_\tau(X_n), S^n_g)$ satisfying (C1) is formed. Adding a variable x_i which act as a nullary operation, the algebra $(W_\tau(X_n), S^n_g, (x_i)_{i\geq 1})$ which satisfies (C1)-(C4) is obtained.

One of the outstanding structures that plays a vital role in the first and second order languages considered in theoretical computer science is an algebraic system. It is a triplet consisting of a nonempty set A, a sequence of n_i-ary operations defined on A, and a sequence of n_j-ary relations on A. Normally, we may write $\mathcal{A} = (A, (f^{\mathcal{A}}_i)_{i\in I}, (\gamma^{\mathcal{A}}_j)_{j\in J})$ for an algebraic system of type (τ, τ') where $\tau = (n_i)_{i\in I}$ and $f^{\mathcal{A}}_i : A^{n_i} \to A$ for each $i \in I$ and $\tau' = (n_j)_{j\in J}$ and $\gamma^{\mathcal{A}}_j \subseteq A^{n_j}$ for each $j \in J$. We remark here that if a sequence of n_j-ary relations on A is not defined, this structure is reduced to an original algebra of type τ, i.e., $\mathcal{A} = (A, (f_i^{\mathcal{A}})_{i\in I})$. Clearly, any ordered semigroup is a basic example of algebraic systems of type $((2),(2))$. For extensive information on algebraic systems, the reader is referred to the monograph of A.I. Malcev [18]. Among recent contributions in algebraic systems are [19–22]. To investigate several properties of algebraic systems of type (τ, τ'), we need the concept of formulas. Recall from [23, 24] that for $n \in \mathbb{N}$ an *n-ary formula of type* (τ, τ') is defined in the following way:

(1) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ').
(2) If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary formula of type (τ, τ').
(3) If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type (τ, τ').
(4) If F_1 and F_2 are n-ary formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ').
(5) If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ, τ') and the set of all formulas of type (τ, τ'), respectively. By an *atomic formula of type* (τ, τ'), we refer to the formula of the form (1) and (2).

Example 1.1. Let $(\tau, \tau') = ((3),(2))$ be the type with a ternary operation symbol f and a binary relation symbol γ. We provide lists of some elements in $\mathcal{F}_{((3),(2))}(W_{(3)}(X_3))$. For this, some atomic formulas are determined as follows: $x_1 \approx x_3, x_2 \approx x_2, f(x_1, x_2, x_3) \approx x_1, f(x_2, x_2, x_2) \approx f(x_1, x_3, x_1), \gamma(x_1, x_2), \gamma(x_3, x_3), \gamma(x_2, x_3), \gamma(f(x_3, x_3, x_2), f(x_1, x_3, x_1))$. Apart form these are obtained by using the following three logical connectors, say $\neg, \exists, \vee$.

In this paper, we mainly focus on terms of type τ of all arities and formulas of an arbitrary type. Applying the generalized superposition operation S_g^n of terms, the partial operation of formulas is determined. The presentation of this paper is organized as follows: First, in Section 2, the partial operation defined on the union of the set of all terms and the set of all formulas is considered. The fact that this partial operation satisfies the superassociativity as a weak identity is mentioned. In section 3, based on some classes of transformation semigroups on a finite chain and order-decreasing full terms of type τ_n, a particular class of formulas arising from order-decreasing full terms is introduced and some properties of such formulas with one operation of the arity $(n + 1)$ are studied.

2. Partial Superassociative Algebras of Terms and Formulas

In this section, we mainly focus on a type (τ, τ') with the corresponding arbitrary operation symbols and relation symbols of all arities indexed by I and J, respectively. To acheive this aim, the operation defined on the set of all formulas is recalled. The operation

$$R_g^n : \big(W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))\big) \times (W_\tau(X))^n \to W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

are defined in the following way:

(1) If $t \in W_\tau(X)$, then $R_g^n(t, s_1, \ldots, s_n) := S_g^n(t, s_1, \ldots, s_n)$.
(2) If $t_1 \approx t_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R_g^n(t_1 \approx t_2, s_1, \ldots, s_n)$ is the formula

$$R_g^n(t_1, s_1, \ldots, s_n) \approx R_g^n(t_2, s_1, \ldots, s_n).$$

(3) If $\gamma_j(t_1,\ldots,t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R^n_g(\gamma_j(t_1,\ldots,t_{n_j}),s_1,\ldots,s_n)$ is the formula $\gamma_j(R^n_g(t_1,s_1,\ldots,s_n),\ldots,R^n_g(t_{n_j},s_1,\ldots,s_n))$.

(4) If $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R^n_g(\neg F,s_1,\ldots,s_n)$ is the formula

$$\neg R^n_g(F,s_1,\ldots,s_n).$$

(5) If $F_1,F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R^n_g(F_1\vee F_2,s_1,\ldots,s_n)$ is the formula

$$R^n_g(F_1,s_1,\ldots,s_n)\vee R^n_g(F_2,s_1,\ldots,s_n).$$

(6) If $\exists x_i(F) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R^n_g(\exists x_i(F),s_1,\ldots,s_n)$ is the formula

$$\exists x_i(R^n_g(F,s_1,\ldots,s_n)).$$

Now we let

$$W\mathcal{F}_{(\tau,\tau')} := W_\tau(X)\cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X)).$$

The partial superposition operation of type $(n+1)$ which is a partial mapping

$$\overline{R}^n_g : (W\mathcal{F}_{(\tau,\tau')})^{n+1} \multimap W\mathcal{F}_{(\tau,\tau')}$$

can be defined by

$$\overline{R}^n_g(a,b_1,\ldots,b_n) = \begin{cases} R^n_g(a,b_1,\ldots,b_n) & \text{if } a\in W\mathcal{F}_{(\tau,\tau')}, b_1,\ldots,b_n\in W_\tau(X),\\ \text{not defined} & \text{otherwise.}\end{cases}$$

Some examples that demonstrate the process of this partial operation are now mentioned. Let $|I| = 1 = |J|$ and let $(\tau,\tau') = ((2),(2))$ be a type with the corresponding operation symbol and relation symbol, say f and γ, respectively. The set $W\mathcal{F}_{(\tau,\tau')}$ consist of all terms of type (2) and all formulas of type $((2),(2))$. Prepare the following tools:

a_1 is a variable x_1, a_2 is a term $f(x_1,x_3)$,
a_3 is a term $f(f(x_4,x_1),x_2)$, a_4 is a formula $f(x_5,x_1)\approx x_1$,
a_5 is a formula $\gamma(x_6,f(x_2,x_1))$, a_6 is a formula $\gamma(x_3,f(x_2,x_1))\vee\neg(f(x_4,x_1)\approx x_7)$,
b_1 is a term $f(x_2,x_1)$, b_2 is a variable x_3.

Obviously, $a_1,a_2,\ldots,a_6,b_1,b_2$ are elements in $W\mathcal{F}_{((2),(2))}$. Furthermore, We have

$$\overline{R}^2_g(a_1,b_1,b_2) = S^2_g(a_1,b_1,b_2) = b_1 = f(x_2,x_1),$$

$$\overline{R}^2_g(a_2,b_1,b_2) = S^2_g(a_2,b_1,b_2) = f(b_1,x_3) = f(f(x_2,x_1),x_3),$$

$$\overline{R}^3_g(a_4,b_1,b_2,a_3) = R^3_g(a_4,b_1,b_2,a_3) \text{ which equals to } f(x_5,b_1)\approx b_1, \text{ and thus } f(x_5,f(x_2,x_1))\approx f(x_2,x_1),$$

$$\overline{R}^2_g(a_5,b_1,b_2) = R^2_g(a_5,b_1,b_2) = \gamma(x_6,f(b_2,b_1)), \text{ which equals to } \gamma(x_6,f(x_2,f(x_2,x_1))).$$

On the other hand, $\overline{R}^2_g(a_6,a_3,a_4)$ and $\overline{R}^4_g(a_1,a_5,b_2,a_3,a_4)$ are not defined.

As a result, we can form the following two partial algebras. The first one is the partial algebra $(W\mathcal{F}_{(\tau,\tau')},\overline{R}^n_g)$ of type $(n+1)$ and the second one is the partial algebra $(W\mathcal{F}_{(\tau,\tau')},\overline{R}^n_g,(x_j)_{j\geq 1})$ of type $(n+1,0,0,0,\ldots)$. We show that $(W\mathcal{F}_{(\tau,\tau')},\overline{R}^n_g)$ satisfies (C1) as a weak identity. For this, the concept of weak identities is needed. We recall from [25] that an equation $s\approx t$ is said to be a *weak identity* in an algebra $\mathcal{A}$ if one side is defined then another side is also defined and both sides are equal.

Theorem 2.1. *$(W\mathcal{F}_{(\tau,\tau')},\overline{R}^n_g)$ is a partial superassociative algebra.*

Proof. Assume that $a, b_1, \ldots, b_n, d_1, \ldots, d_n$ are elements in $W\mathcal{F}_{(\tau,\tau')}$. To show that, $\overline{R}_g^n$ satisfies (C1) as a weak identity, we replace an operation symbol $\tilde{S}^n$ by $\overline{R}_g^n$, replace variable symbols $\tilde{Z}, \tilde{Y}_j, \tilde{X}_j$ for all $1 \leq j \leq n$ by terms a, b_j, d_j, respectively. This means that we have $\overline{R}_g^n(\overline{R}_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n) \approx \overline{R}_g^n(a, \overline{R}_g^n(b_1, d_1, \ldots, d_n), \ldots, \overline{R}_g^n(b_n, d_1, \ldots, d_n))$. Suppose that the left-hand side of this identity is defined. Then we have the following two cases: $a, b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ in the first case and a is a formula of type (τ, τ') but $b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ in the second case.

Consider the first case when $a, b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ. We obtain that $\overline{R}_g^n(\overline{R}_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$ equals to $S_g^n(S_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$. Additionally, for each $j = 1 \ldots, n$, $\overline{R}_g^n(b_j, d_1, \ldots, d_n)$ is defined and equals to $S_g^n(b_j, d_1, \ldots, d_n)$. This implies that $\overline{R}_g^n(a, \overline{R}_g^n(b_1, d_1, \ldots, d_n), \ldots, \overline{R}_g^n(b_n, d_1, \ldots, d_n))$ is defined and equals to $S_g^n(a, S_g^n(b_1, d_1, \ldots, d_n), \ldots, S_g^n(b_n, d_1, \ldots, d_n))$. It was mentioned in [10] that the generalized superposition S_g^n satisfies (C1) which means $S_g^n(S_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n) = S_g^n(a, S_g^n(b_1, d_1, \ldots, d_n), \ldots, S_g^n(b_n, d_1, \ldots, d_n))$.

We now consider the case when a is a formula F of type (τ, τ') and $b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ. It imples that the left-hand side $\overline{R}_g^n(\overline{R}_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$ equals to $R_g^n(R_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$. For each $j = 1, \ldots, n$, $\overline{R}_g^n(b_j, d_1, \ldots, d_n)$ is also defined and equals to $R_g^n(b_j, d_1, \ldots, d_n)$. Because we know that, for each $j = 1 \ldots, n$, $R_g^n(b_j, d_1, \ldots, d_n)$ is a term of type τ, we further obtain that the right-hand side is defined and equals to $R_g^n(a, R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n))$. Finally, we prove that $R_g^n(R_g^n(a, b_1, \ldots, b_n), d_1, \ldots, d_n)$ and $R_g^n(a, R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n))$ are identical. For this, a proof by a definition of a formula a is given. If a is a formula of the form $\gamma_j(a_1, \ldots, a_{n_j})$, it was mentioned in [26] that $R_g^n(R_g^n(\gamma_j(a_1, \ldots, a_{n_j}), b_1, \ldots, b_n), d_1, \ldots, d_n)$ and $R_g^n(\gamma_j(a_1, \ldots, a_{n_j}), R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n))$ are equal. If a is an equation $s_1 \approx s_2$ for any terms s_1 and s_2, then we have that $R_g^n(R_g^n(s \approx t, b_1, \ldots, b_n), d_1, \ldots, d_n)$ equals to $R_g^n(S_g^n(s_1, b_1, \ldots, b_n) \approx S_g^n(s_2, b_1, \ldots, b_n), d_1, \ldots, d_n)$ which equals the expression $S_g^n(S_g^n(s_1, b_1, \ldots, b_n), d_1, \ldots, d_n) \approx S_g^n(S_g^n(s_2, b_1, \ldots, b_n), d_1, \ldots, d_n)$. Since we known from the first case that for each $j = 1, 2$, $S_g^n(S_g^n(s_j, b_1, \ldots, b_n), d_1, \ldots, d_n)$ equals to $S_g^n(s_j, S_g^n(b_1, d_1, \ldots, d_n), \ldots, S_g^n(b_n, d_1, \ldots, d_n))$, it is not hard to continue the process that $S_g^n(S_g^n(s_1, b_1, \ldots, b_n), d_1, \ldots, d_n) \approx S_g^n(S_g^n(s_2, b_1, \ldots, b_n), d_1, \ldots, d_n)$ equals to $R_g^n(s_1 \approx s_2, S_g^n(b_1, d_1, \ldots, d_n), \ldots, S_g^n(b_n, d_1, \ldots, d_n))$. Suppose now that a formula a satisfies (C1). By the definition of the generalized superposition operation R_g^n, we have

$$R_g^n(R_g^n(\neg a, b_1, \ldots, b_n), d_1, \ldots, d_n) = R_g^n(\neg a, R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n))$$

and $R_g^n(R_g^n(\exists x_i(a), b_1, \ldots, b_n), d_1, \ldots, d_n) = R_g^n(\exists x_i(a), R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n))$. Finally, in the case a is a formula $F_1 \vee F_2$, the proof is directly obtained. ∎

On the partial algebra $(W\mathcal{F}_{(\tau,\tau')}, \overline{R}_g^n)$, one can derive the following partial binary operation as follows: For any $a, b \in W\mathcal{F}_{(\tau,\tau')}$, we define $\oplus : (W\mathcal{F}_{(\tau,\tau')})^2 \multimap W\mathcal{F}_{(\tau,\tau')}$ by

$$a \oplus b = \overline{R}_g^1(a, b).$$

As a consequence, the binary partial algebra $(W\mathcal{F}_{(\tau,\tau')}, \oplus)$ is obtained. Obviously, from Theorem 2.1, by taking $n = 1$, the binary partial algebra $(W\mathcal{F}_{(\tau,\tau')}, \oplus)$ is a partial semigroup.

We now explain a subsemigroup of the partial semigroup $(W\mathcal{F}_{(\tau,\tau')}, \oplus)$ in a specific type by the following multiplicative table.

Example 2.2. Consider the type $(\tau, \tau') = ((2),(2))$ with one binary operation symbol f one binary relation symbol ρ and a subset

$$A = \{x_1, f(x_2, x_4), x_3 \approx x_5, \rho(x_3, f(x_2, x_2))\}$$

of $W\mathcal{F}_{((2),(2))}$ with respect to a partial binary operation $\oplus$ which is defined by the following table.

$\oplus$	x_1	$f(x_2,x_4)$	$x_3 \approx x_5$	$\rho(x_3, f(x_2,x_2))$
x_1	x_1	$f(x_2,x_4)$	not defined	not defined
$f(x_2,x_4)$	$f(x_2,x_4)$	$f(x_2,x_4)$	not defined	not defined
$x_3 \approx x_5$	$x_3 \approx x_5$	$x_3 \approx x_5$	not defined	not defined
$\rho(x_3, f(x_2,x_2))$	$\rho(x_3, f(x_2,x_2))$	$\rho(x_3, f(x_2,x_2))$	not defined	not defined

It is not difficult to show that the binary operation $\oplus$ defined on A is associative. To illustrate some examples, we consider elelemts $x_1, f(x_2, x_4)$ and $\rho(x_3, f(x_2, x_2))$ in A. To show that an equation

$$(\rho(x_3, f(x_2, x_2)) \oplus x_1) \oplus f(x_2, x_4) \approx \rho(x_3, f(x_2, x_2)) \oplus (x_1 \oplus f(x_2, x_4))$$

is a weak identity, assume that the left-hand side is defined. We have $(\rho(x_3, f(x_2, x_2)) \oplus x_1) \oplus f(x_2, x_4) = \rho(x_3, f(x_2, x_2)) \oplus f(x_2, x_4) = \rho(x_3, f(x_2, x_2))$. Then the right-hand side is defined and equals to $\rho(x_3, f(x_2, x_2)) \oplus f(x_2, x_4)$, subsequently, $\rho(x_3, f(x_2, x_2))$. This shows that the above equation satisfies an associative law as a weak identity.

Consequently, $(A, \oplus)$ froms a partial semigroup. Furthermore, it is also a partial subsemigroup of $(W\mathcal{F}_{((2),(2))}, \oplus)$.

Considering a variable from an alphabet X, the following theorem is stated.

Theorem 2.3. *$(W\mathcal{F}_{(\tau,\tau')}, \overline{R}^n_g, (x_j)_{j\geq 1})$ is a partial superassociative algebra with infinitely many nullary operations.*

Proof. The proof of (C1) follows by a direct verification of Theorem 2.1. To prove (C2) we substitute a symbol $\tilde{S}^n$ by a partial operation $\overline{R}^n_g$, a symbol λ_j by a variable x_j for $1 \leq j \leq n$, and $\tilde{X}_k$ by an element $b_k \in W\mathcal{F}_{(\tau,\tau')}$. Then we have $\overline{R}^n_g(x_j, b_1, \ldots, b_n)$. Suppose that the left-habd side of (C2) is defined. We have that $b_1, \ldots, b_n$ are terms of type τ. Thus $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = R^n_g(x_j, b_1, \ldots, b_n) = b_j$. Therefore, (C2) is proved. To show that (C3) holds as a weak identity, we continue from (C2) but replace a symbol λ_j by a variable x_j for $j > n$. Assume that the left-hand side of (C3) is defined. We obtain that $b_1, \ldots, b_n \in W_\tau(X)$ and thus $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = R^n_g(x_j, b_1, \ldots, b_n) = x_j$. Hence, (C3) is also verified. Finally, in (C4), we replace $\tilde{Y}$ by an element $a \in W\mathcal{F}_{((\tau),(\tau'))}$, a symbol λ_j by a variable x_j for $1 \leq k \leq n$. Then we have $\overline{R}^n_g(a, x_1, \ldots, x_n) \approx a$. It is not hard to see that the left-hand side is defined and thus $\overline{R}^n_g(a, x_1, \ldots, x_n) = R^n_g(a, x_1, \ldots, x_n)$. If a is a term of type τ, from [10], we have $R^n_g(a, x_1, \ldots, x_n) = S^n_g(a, x_1, \ldots, x_n) = a$. For a formula a, we give a proof by the following way: If a is an equation $s \approx t$, then $R^n_g(s \approx t, x_1, \ldots, x_n) = S^n_g(s, x_1, \ldots, x_n) \approx S^n_g(t, x_1, \ldots, x_n) = s \approx t$. If a has a form $\gamma_j(t_1, \ldots, t_{n_j})$, we have $R^n_g(\gamma_j(t_1, \ldots, t_{n_j}), x_1, \ldots, x_n) = \gamma_j(S^n_g(t_1, x_1, \ldots, x_n), \ldots, S^n_g(t_{n_j}, x_1, \ldots, x_n)) = \gamma_j(t_1, \ldots, t_{n_j})$. Assume that a is satisfied C_4 as a weak identity. Then we obtain $R^n_g(\neg a, x_1, \ldots, x_n) = \neg R^n_g(a, x_1, \ldots, x_n) = \neg a$ and $R^n_g(\exists x_i(a), x_1, \ldots, x_n) = \exists x_i(R^n_g(a, x_1, \ldots, x_n)) = \exists x_i(a)$.

Finally, suppose that F_1 and F_2 are satisfied. Then we have $R_g^n(F_1 \vee F_2, x_1, \ldots, x_n) = R_g^n(F_1, x_1, \ldots, x_n) \vee R_g^n(F_2, x_1, \ldots, x_n) = F_1 \vee F_2$. The proof is completed. ∎

3. Formulas Defined by Order-decreasing Transformations

Based on the theory of transformations semigroups, especially transformations of order-decreasing on a finite chain, the first aim of this section is to apply one of the outstanding classes of terms of type τ_n, which is called order-decreasing full term, to introduce othe classes of formulas.

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under the usual composition of mappings. By the symbol $\bar{n}$, we mean a chain $\{1, \ldots, n\}$ with a natural order $\leq$. If $X = \bar{n}$ we write T_n instead of $T(X)$. We consider the set

$$OD_n = \{\alpha \in T_n \mid \forall k \in \{1, \ldots, n\}, \alpha(k) \leq k\}$$

of all *order-decreasing full transformations* which is a submonoid of T_n. More details on order-decreasing transformations may be found in [27, 28]

For a positive integer n, let $\tau_n = (\underbrace{n, \ldots, n}_{|I|})$ be a type indexed by a nonempty set I, with each operation symbol f_i of arity n for every $i \in I$.

Definition 3.1. *[29]* For each $i \in I$, let f_i be an n-ary operation symbol and $\alpha \in OD_n$. An n-ary order-decreasing full term of type τ_n is inductively defined by

(1) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary order-decreasing full term of type τ_n,
(2) if $t_1, \ldots, t_n$ are n-ary order-decreasing full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary order-decreasing full term of type τ_n.

Let $W_{\tau_n}^{OD_n}(X_n)$ be the set of all n-ary order-decreasing full terms of type τ_n.

Now we present some concrete examples of order-decreasing full terms of some types.

Example 3.2. Let $\tau_3 = (3, 3, 3)$ be a type with three ternary operation symbols f, g, and h. Then we have

$$f(x_1, x_1, x_1), g(x_1, x_2, x_3), h(x_1, x_2, x_2) \in W_{\tau_3}^{OD_3}(X_3),$$

because $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \in OD_3$ and

$$g(f(x_1, x_1, x_1), h(x_1, x_2, x_2), g(x_1, x_2, x_3)) \in W_{\tau_3}^{OD_3}(X_3).$$

On the other hand,

$$f(x_2, x_3, x_1), g(x_1, x_3, x_2), f(x_3, x_3, x_1), h(x_2, x_2, x_2) \notin W_{\tau_3}^{OD_3}(X_3),$$

since $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \notin OD_3$.

The set $W_{\tau_n}^{OD_n}(X_n)$ of all n-ary order-decreasing full terms of type τ_n is closed under the following superposition

$$S^n : (W_{\tau_n}^{OD_n}(X_n))^{n+1} \to W_{\tau_n}^{OD_n}(X_n)$$

given by:

(1) $S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n) := f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})$; and
(2) $S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$.

As a result, in [29], the algebra $(W_{\tau_n}^{OD_n}(X_n), S^n)$ was formed. Furthermore, it was shown that this algebra satisfied the equation (C1).

Let us consider the type $\tau'_m = (\underbrace{m, \ldots, m}_{|J|})$ of all arities of relation symbols and all $j \in J$. We define the new concept of full formulas of type (τ_n, τ'_m) for natural numbers $n, m \geq 1$.

Definition 3.3. Let $n \in \mathbb{N}$. *An n-ary order-decreasing quantifier free full formula* of type (τ_n, τ'_m) (for short, order-decreasing full formula) is defined in the following way:

(1) If s, t are n-ary order-decreasing full terms of type τ_n, then the equation $s \approx t$ is an n-ary order-decreasing full formula of type (τ_n, τ'_m).
(2) If $t_1, \ldots, t_m$ are n-ary order-decreasing full terms of type τ_n and γ_j is a relation symbol of type τ'_m, then $\gamma_j(t_1, \ldots, t_m)$ is an n-ary order-decreasing full formula of type (τ_n, τ'_m).
(3) If F is an n-ary order-decreasing full formula of type (τ_n, τ'_m), then $\neg F$ is also an n-ary order-decreasing full formula of type (τ_n, τ'_m).
(4) If F_1 and F_2 are n-ary order-decreasing full formulas of type (τ_n, τ'_m), then $F_1 \vee F_2$ is an n-ary order-decreasing full formula of type (τ_n, τ'_m).

Let $\mathcal{F}_{(\tau_n, \tau'_m)}^{OD_n}(W_{\tau_n}^{OD_n}(X_n))$ be the set of all order-decreasing full formulas of type (τ_n, τ'_m). We now give some examples.

Example 3.4. Let $(\tau_4, \tau'_3) = ((4,4,4,4),(3,3,3))$ be a type of algebraic system, i.e., we have four quaternary operation symbols f_1, f_2, f_3, f_4 and three ternary relation symbols $\gamma_1, \gamma_2, \gamma_3$. We see that the following terms

$$\begin{array}{ll} t_1 = f_1(x_1,x_1,x_1,x_1), & t_2 = f_2(x_1,x_1,x_2,x_2), \\ t_3 = f_4(x_1,x_2,x_3,x_4), & t_4 = f_4(x_1,x_2,x_1,x_3), \\ t_5 = f_3(x_1,x_2,x_2,x_1), & t_6 = f_2(x_1,x_1,x_1,x_2), \\ t_7 = f_1(x_1,x_2,x_3,x_3), & t_8 = f_4(x_1,x_2,x_2,x_4) \end{array}$$

are elements $W_{\tau_4}^{OD_4}(X_4)$ because

$$\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&2&1&3\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4\\1&2&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&2&3&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&2&2&4\end{pmatrix}$$

are order-decreasing transformations on $\bar{4}$. As a result, we have that

$$t_1 \approx t_1, t_3 \approx t_2, \gamma_1(t_4, t_5, f_3(t_6, t_7, t_8, t_1)) \in \mathcal{F}_{(\tau_4, \tau'_3)}^{OD_4}(W_{\tau_4}^{OD_4}(X_4)).$$

We also obtain other full formulas by applying the logical connectors $\approx, \neg$ and $\vee$, for example,

$$\neg(t_1 \approx t_1), \neg(t_2 \approx t_1), \neg(\gamma_3(t_7, t_6, t_5)), (t_4 \approx t_3) \vee (t_1 \approx f_4(t_3, t_6, t_1, t_2))$$

and

$$(t_5 \approx t_7) \vee \neg(t_4 \approx t_2), (\neg(\gamma_3(t_4, t_1, t_7)) \vee (t_7 \approx t_4)) \vee \gamma_2(t_5, t_6, t_2).$$

Now, we extend the definition of superposition operation of order-decreasing full terms to order-decreasing full formulas by replacing variables occuring in an order-decreasing full formula by order-decreasing full terms.

Definition 3.5. Let m, n be natural numbers. The operation

$$R^n : \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n)) \times (W^{OD_n}_{\tau_n}(X_n))^n \to \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$$

can be defined by the following inductive steps:

(1) If $s, t \in W^{OD_n}_{\tau_n}(X_n)$, then $R^n(s \approx t, s_1, \ldots, s_n)$ is an order-decreasing full formula $S^n(s, s_1, \ldots, s_n) \approx S^n(t, s_1, \ldots, s_n)$.

(2) If $t_1, \ldots, t_m \in W^{OD_n}_{\tau_n}(X_n)$, then $R^n(\gamma_j(t_1, \ldots, t_m), s_1, \ldots, s_n)$ is an order-decreasing full formula $\gamma_j(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_m, s_1, \ldots, s_n))$.

(3) If $F \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$, then $R^n(\neg F, s_1, \ldots, s_n)$ is an order-decreasing full formula $\neg R^n(F, s_1, \ldots, s_n)$.

(4) If $F_1, F_2 \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$, then $R^n(F_1 \vee F_2, s_1, \ldots, s_n)$ is an order-decreasing full formula $R^n(F_1, s_1, \ldots, s_n) \vee R^n(F_2, s_1, \ldots, s_n)$.

Below we present an example of Definition 3.5 that demonstrates a method for substituting order-decreasing full formulas by order-decreasing full on some finite set.

Example 3.6. Let $(\tau_3, \tau'_3) = ((3,3,3),(3,3,3))$ be a type of algebraic systems with three ternary operation symbols h_1, h_2, h_3 and three ternary relation symbols η_1, η_2, η_3. Consider the superposition

$$R^3 : \mathcal{F}^{OD_3}_{(\tau_3,\tau'_3)}(W^{OD_3}_{\tau_3}(X_3)) \times (W^{OD_3}_{\tau_3}(X_3))^3 \to \mathcal{F}^{OD_3}_{(\tau_3,\tau'_3)}(W^{OD_3}_{\tau_3}(X_3))$$

and these three ternary order-decreasing full terms say $s_1 = h_1(x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)}), s_2 = h_2(x_{\kappa(1)}, x_{\kappa(2)}, x_{\kappa(3)})$ and $s_3 = h_3(s_2, s_1, s_1)$ in $W^{OD_3}_{\tau_3}(X_3)$ where γ and κ are mappings in OD_3 defined by $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ and $\kappa = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$. Then we have the following:

(1) If $s \approx t$ has the form $g_1(x_1, x_1, x_1) \approx g_2(x_1, x_2, x_2)$, then $R^3(g_1(x_1, x_1, x_1) \approx g_2(x_1, x_2, x_2), s_1, s_2, s_3)$ is an order-decreasing full formula $S^3(g_1(x_1, x_1, x_1), s_1, s_2, s_3) \approx S^3(g_2(x_1, x_2, x_2), s_1, s_2, s_3)$. By the superposition S^3, we further have

$$g_1(s_1, s_1, s_1) \approx g_2(s_1, s_2, s_2).$$

(2) If $\eta_3(s_3, s_2, s_1)$ is an order-decreasing full formula in $\mathcal{F}^{OD_3}_{(\tau_3,\tau'_3)}(W^{OD_3}_{\tau_3}(X_3))$, then we obtain that
$R^3(\eta_3(s_3, s_2, s_1), s_1, s_2, s_3)$. By Definition 3.5 and S^3, it is equal to

$$\eta_3(S^3(s_3, s_1, s_2, s_3), S^3(s_2, s_1, s_2, s_3), S^3(s_1, s_1, s_2, s_3)).$$

Moreover, applying logical connectors $\approx, \neg$ and $\vee$, we obtain other order-decreasing full formulas on a set $\bar{3}$.

Then we can form the many-sorted algebra

$$(W^{OD_n}_{\tau_n}(X_n), \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n)), S^n, R^n)$$

which is called the *clone of order-decreasing full formulas of type* (τ_n, τ'_m). We may use the notation $\text{Formclone}^{OD_n}(\tau_n, \tau'_m)$ for the clone of order-decreasing full formulas of type (τ_n, τ'_m).

Furthermore we prove

Theorem 3.7. *The algebra* $\text{Formclone}^{OD_n}(\tau_n, \tau'_m)$ *satisfies the axiom of superassociativity.*

Proof. Firstly, we replace $\tilde{Z}$ by an order-decreasing full formula F in $\mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$ and substitute all symbols $\tilde{Y}_1,\dots,\tilde{Y}_n$ and $\tilde{X}_1,\dots,\tilde{X}_n$ by order-decreasing full terms $p_1,\dots,p_n,q_1,\dots,q_n$. Firstly, if F is an order-decreasing full formula $s\approx t$, since the satisfaction in (C1) of S^n, we have $R^n(R^n(s\approx t,p_1,\dots,p_n),q_1,\dots,q_n)$

$$\begin{aligned} &= S^n(S^n(s,p_1,\dots,p_n),q_1,\dots,q_n)\approx S^n(S^n(t,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= S^n(s,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n))\\ &\quad \approx S^n(t,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n))\\ &= R^n(s\approx t,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)).\end{aligned}$$

Secondly, if F is an order-decreasing full formula $\gamma_j(u_1,\dots,u_m)$, where $u_1,\dots,u_m\in W^{OD_n}_{\tau_n}(X_n)$, it follows from the satisfaction of (C1) that

$$\begin{aligned} &R^n(R^n(\gamma_j(u_1,\dots,u_m),p_1,\dots,p_n),q_1,\dots,q_n)\\ &= R^n(\gamma_j(S^n(u_1,p_1,\dots,p_n),\dots,S^n(u_m,p_1,\dots,p_n)),q_1,\dots,q_n)\\ &= \gamma_j(S^n(u_1,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)),\dots,\\ &\quad S^n(u_m,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)))\\ &= R^n(\gamma_j(u_1,\dots,u_m),S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)).\end{aligned}$$

Assume that an order-decreasing full formula F satisfies the statement of the theorem. Then

$$\begin{aligned} &R^n(R^n(\neg F,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= R^n(\neg R^n(F,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= \neg R^n(R^n(F,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= \neg R^n(F,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n))\\ &= R^n(\neg F,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)).\end{aligned}$$

Suppose that order-decreasing full formulas F_1 and F_2 satisfy the statement of the theorem. Then

$$\begin{aligned} &R^n(R^n(F_1\vee F_2,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= R^n(R^n(F_1,p_1,\dots,p_n),q_1,\dots,q_n)\vee R^n(R^n(F_2,p_1,\dots,p_n),q_1,\dots,q_n)\\ &= R^n(F_1,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n))\\ &\quad \vee R^n(F_2,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n))\\ &= R^n(F_1\vee F_2,S^n(p_1,q_1,\dots,q_n),\dots,S^n(p_n,q_1,\dots,q_n)).\end{aligned}$$

This finishes the proof. ∎

To investigate some properties of the algebra Formclone$^{OD_n}(\tau_n,\tau'_m)$, a concept of order-decreasing full formulas generated by $\beta\in OD_n$ is introduced.

For $t\in W^{OD_n}_{\tau_n}(X_n)$ and $\beta\in OD_n$, we define a term t that arises from β as follows:

(1) $t_\beta := f_i(x_{\beta(\alpha(1))},\dots,x_{\beta(\alpha(n))})$ if $t=f_i(x_{\alpha(1)},\dots,x_{\alpha(n)})$.

(2) $t_\beta := f_i((t_1)_\beta,\dots,(t_n)_\beta)$ if $t=f_i(t_1,\dots,t_n)$.

It is observed that for all $\beta\in W^{OD_n}_{\tau_n}(X_n)$, t_β is again an n-ary order-decreasing full term of type τ_n if t is.

Proposition 3.8. *[29] Let $t\in W^{OD_n}_{\tau_n}(X_n),\beta,\gamma\in OD_n$. Then*

$$t_{\gamma\circ\beta}=(t_\beta)_\gamma.$$

Definition 3.9. An order-decreasing full formula F of type (τ_n,τ'_m) arising from a mapping $\beta\in OD_n$, denoted by F_β, can be inductively defined by the following steps:

(1) If F is $s\approx t$, then $F_\beta=(s\approx t)_\beta := s_\beta\approx t_\beta$.

(2) If F is $\gamma_j(t_1,\dots,t_m)$, then $F_\beta=(\gamma_j(t_1,\dots,t_m))_\beta := \gamma_j((t_1)_\beta,\dots,(t_m)_\beta)$.

(3) $(\neg F)_\beta=\neg(F_\beta)$ where F_β is already defined.

(4) $(F_1\vee F_2)_\beta=(F_1)_\beta\vee(F_2)_\beta$ where $(F_1)_\beta,(F_2)_\beta$ are already defined.

For instance, consider all preparations of Example 3.4 and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 4 \end{pmatrix} \in OD_4$. Then a formula $(t_3 \approx t_5)_\beta$ is equal to

$$(f_4(x_1, x_2, x_3, x_4) \approx f_3(x_1, x_2, x_2, x_1))_\beta$$

and thus $f_4(x_1, x_2, x_3, x_4)_\beta \approx f_3(x_1, x_2, x_2, x_1)_\beta$, which implies

$$f_4(x_1, x_2, x_2, x_4) \approx f_3(x_1, x_2, x_2, x_1) \in \mathcal{F}^{OD_4}_{(\tau_4,\tau'_3)}(W^{OD_4}_{\tau_4}(X_4)).$$

Similarly, if $\gamma_2(t_4, t_6, t_8) \in \mathcal{F}^{OD_4}_{(\tau_4,\tau'_3)}(W^{OD_4}_{\tau_4}(X_4))$, then $\gamma_2(t_4, t_6, t_8)_\beta$ is a formula

$$\gamma_2(f_4(x_1, x_2, x_1, x_3)_\beta, f_2(x_1, x_1, x_1, x_2)_\beta, f_4(x_1, x_2, x_2, x_4)_\beta),$$

which means

$$\gamma_2(f_4(x_1, x_2, x_1, x_2), f_2(x_1, x_1, x_1, x_2), f_4(x_1, x_2, x_2, x_4)).$$

The following statement establishes a relationship between $F \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$ and two mappings in OD_n.

Proposition 3.10. *For any F in $\mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$, we have*

$$F_{\gamma\circ\beta} = (F_\beta)_\gamma$$

for all $\gamma, \beta \in OD_n$.

Proof. Let $F \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$. We give a proof by the following steps. If F has a form $s \approx t$, then by Proposition 3.8, we get

$$F_{\gamma\circ\beta} = (s \approx t)_{\gamma\circ\beta} = s_{\gamma\circ\beta} \approx t_{\gamma\circ\beta} = (s_\beta)_\gamma \approx (t_\beta)_\gamma = (s_\beta \approx t_\beta)_\gamma = ((s \approx t)_\beta)_\gamma = (F_\beta)_\gamma.$$

If F is a full formula $\gamma_j(s_1, \dots, s_m)$, where $s_1, \dots, s_m \in W^{OD_n}_{\tau_n}(X_n)$, and $(s_k)_{\gamma\circ\beta} = ((s_k)_\beta)_\gamma$ for all $1 \leq k \leq n$, then by Proposition 3.8, we have

$$\begin{aligned} F_{\gamma\circ\beta} &= \gamma_j(s_1, \dots, s_m)_{\gamma\circ\beta} \\ &= \gamma_j((s_1)_{\gamma\circ\beta}, \dots, (s_m)_{\gamma\circ\beta}) \\ &= \gamma_j(((s_1)_\beta)_\gamma, \dots, ((s_m)_\beta)_\gamma) \\ &= (\gamma_j((s_1)_\beta, \dots, (s_m)_\beta))_\gamma \\ &= (\gamma_j(s_1, \dots, s_m)_\beta)_\gamma \\ &= (F_\beta)_\gamma. \end{aligned}$$

Assume now that F satisfies $F_{\gamma\circ\beta} = (F_\beta)_\gamma$. Then

$$(\neg F)_{\gamma\circ\beta} = \neg(F_{\gamma\circ\beta}) = \neg((F_\beta)_\gamma) = \neg(F_\beta)_\gamma = (\neg F_\beta)_\gamma.$$

Finally, suppose that F_1 and F_2 satisfy the statement. Then $(F_1 \vee F_2)_{\gamma\circ\beta} = (F_1)_{\gamma\circ\beta} \vee (F_2)_{\gamma\circ\beta} = ((F_1)_\beta)_\gamma \vee ((F_2)_\beta)_\gamma = ((F_1)_\beta \vee (F_2)_\beta)_\gamma = ((F_1 \vee F_2)_\beta)_\gamma$. We conclude that $F_{\gamma\circ\beta} = (F_\beta)_\gamma$ for all $F \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$. ∎

Combining Definitions 3.5 and 3.9 and Proposition 3.10, the following corollary is obtained.

Corollary 3.11. *On the algebra* $\text{Formclone}^{OD_n}(\tau_n, \tau'_m)$*, the equations*

$$R^n(F_\beta, s_1, \dots, s_n) = R^n(F, s_{\beta(1)}, \dots, s_{\beta(n)}) = R^n(F, s_1, \dots, s_n)_\beta$$

are satisfied for all $F \in \mathcal{F}^{OD_n}_{(\tau_n,\tau'_m)}(W^{OD_n}_{\tau_n}(X_n))$ and $s_1, \dots, s_n \in W^{OD_n}_{\tau_n}(X_n)$.

4. Conclusion

This paper presents a connection among terms in universal algebra, formulas in algebraic systems, and semigroups of full transformations on a finite chain. A partial operation on the union set between terms and formulas is introduced. We notice that the results obtained extend the construction of full terms presented in the paper [29] and full formulas mentioned in the paper [24]. In closing this paper, we give some open problems.

(1) Characterize idempotency, regularity, and Green's relations on a partial semigroup $(W\mathcal{F}_{(\tau,\tau')}, \oplus)$.
(2) Determine a generating system of the algebra Formclone$^{OD_n}(\tau_n, \tau'_m)$.
(3) Apply a concept of formula languages introduced in [30] to study a particular class of formulas arising from order-decreasing mappings.

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