Thai Journal of Mathematics
Special Issue (2023) :Annual Meeting in Mathematics 2022
Pages 54 - 77

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

An application of the novel (G'/G^2)-expansion method for solving the conformable space-time breaking soliton equation

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Abstract The core objective of this article is to employ the novel (G'/G^2)-expansion method, proposed here for the first time, for constructing exact traveling wave solutions of the $(2+1)$-dimensional conformable space-time breaking soliton equation. To the best of the authors' knowledge, the equation has not been solved for its exact solutions by means of the used technique. As a result, the obtained exact solutions of the problem are expressed in terms of hyperbolic, trigonometric and rational function solutions. Along with the help of a symbolic software package, the method can be simply and efficiently utilized to solve the equation for acquiring accurate and trustworthy exact traveling wave solutions. Consequently, the method could be used to determine some new exact solutions for other nonlinear conformable partial differential equations occurring in physics and engineering.

MSC: 35A25; 35G20; 35C07

Keywords: novel (G'/G^2)-expansion method; exact solutions; conformable derivative; (2+1)-dimensional conformable space-time breaking soliton equation; kink-wave solution

Submission date: 31.05.2022 / Acceptance date: 15.12.2022

1. Introduction

Nonlinear partial differential equations (NPDEs) are used to describe complex physical phenomena in scientific fields such as mechanics [1], electrostatics [2], fluid mechanics [3], quantum mechanics [4], optical fibers [5], plasma [6], oceanology [7], finance [8], applied physics [9], chemistry [10] and biology [11]. As is well known, exact traveling wave solutions of NPDEs play a significant role in exactly portraying wave and other physical phenomena because they are analytical solutions without any errors, unlike numerical solutions. Recently, exact traveling wave solutions of NPDEs have been consequently investigated through accurate and reliable techniques. The powerful approaches used to find exact traveling wave solutions of NPDEs are the exp-function method [12, 13],

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the ansatz method [14, 15], the F-expansion method [16, 17], the Jacobi elliptic function method [18, 19], the first integral method [20, 21], the improved Bernoulli sub-equation function method [22], the modified extended tanh-function method [23, 24], the Riemann-Hilbert method [25, 26], the Lie symmetry method [27], the (G'/G)-expansion method [28, 29], the $(G'/G, 1/G)$-expansion method [30, 31] and the novel (G'/G)-expansion method [32].

The following interesting wave problem which fascinated us is the $(2+1)$-dimensional breaking soliton equation. The equation reads [33, 34]

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0, \tag{1.1}$$

where $u(x, y, t)$ is a traveling wave solution of the equation. Equation (1.1), initially established by Calogero and Degasperis [35, 36], describes the $(2 + 1)$-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave propagated along the x-axis [37]. If we let $y = x$ in (1.1) and integrate the resulting equation, then the breaking soliton equation (1.1) is transformed to the KdV equation, which is a crucial mathematical model for special waves called solitons on shallow water surfaces. The main characteristic feature of breaking soliton equations is that the spectral parameter utilized in the Lax representations possesses so-called breaking behavior [38]. Finding exact solutions of (1.1) and its modified Riemann-Liouville derivative and stochastic versions by some existing methods can be reviewed in [39–41].

In this article, we develop equation (1.1) by substituting its classical partial derivatives with the conformable partial derivatives as shown below. The $(2 + 1)$-dimensional conformable space-time breaking soliton equation can be expressed as

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta u}{\partial x^\beta} - 4\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\left(\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right) - 2\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right)\frac{\partial^\beta u}{\partial y^\beta} \\ + \frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right)\right) = 0, \end{aligned} \tag{1.2}$$

where $0 < \alpha, \beta \leq 1$ are the fractional-orders of the conformable partial derivatives $\frac{\partial^\alpha}{\partial t^\alpha}$, $\frac{\partial^\beta}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial y^\beta}$ which will be defined in section 2. A recent literature review for extracting explicit exact solutions for equation (1.2) using the simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method can be found in [42]. Furthermore, we will establish a new method called the novel (G'/G^2)-expansion method, which is modified from the novel (G'/G)-expansion method [32]. This new method has not been studied and used by any scholar researchers before. Therefore, it is quite interesting to solve (1.2) by utilizing the novel (G'/G^2)-expansion method for its exact traveling wave solutions.

The outline of this paper is as follows. The definition and some important properties of the conformable derivative are provided in section 2. In section 3, we summarize steps of the novel (G'/G^2)-expansion which is first proposed here. The application of the method for extracting analytical exact solutions of (1.2) is explored in section 4, followed by some graphical representations in section 5. Finally, we give some conclusions in section 6.

2. Definition and properties of conformable derivative

In this section, we provide fundamental concepts of the conformable derivative including its definition and important properties as follows.

Definition 2.1. Suppose f is a function such that $f:[0,\infty)\to\mathbb{R}$. Then the conformable derivative of f of order α where $0<\alpha\leq 1$ is defined as [30, 43–48]

$$D_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{2.1}$$

for all $t>0$. If the limit in (2.1) exists, then we can state that f is α-conformable differentiable at a point $t>0$. Moreover, if f is α-conformable differentiable in some $(0,a)$, $a>0$ and $\lim_{t\to 0^+} D_t^\alpha f(t)$ exists, then we define $D_t^\alpha f(0)=\lim_{t\to 0^+} D_t^\alpha f(t)$.

The fundamental properties of the conformable derivative, for instance, the conformable derivatives of a sum or a division of two functions and the relationship between the conformable and classical derivatives, can be found in [30, 43, 44, 46, 47, 49, 50]. Also, the conformable derivatives of some interesting functions were formulated in [30, 43, 44, 46, 47].

Theorem 2.2. *[46, 47, 49, 51, 52] Suppose that the functions $f,g:(0,\infty)\to\mathbb{R}$ are differentiable and also α-conformable differentiable. Further, assume that g is a function defined in the range of f. Then, we have*

$$D_t^\alpha(f\circ g)(t)=t^{1-\alpha}f'(g(t))g'(t),$$

where the prime symbol ($'$) represents the ordinary derivative.

Utilizing the definition (2.1), the definition of the conformable partial derivative of a function such as $u=u(x,t)$ with respect to t of order $\gamma\in(0,1]$ can be defined as [30]

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}=\lim_{\varepsilon\to 0}\frac{u(x,t+\varepsilon t^{1-\gamma})-u(x,t)}{\varepsilon},\ t>0. \tag{2.2}$$

The higher-order definition of the conformable derivative and its properties can be discovered in [43, 50, 53].

3. Algorithm of the novel (G'/G^2)-expansion method

In this section, we are proposing the new method for finding exact traveling wave solutions of classical and conformable NPDEs named the novel (G'/G^2)-expansion method. The description of the approach is concisely given as follows. Consider a nonlinear conformable partial differential equation in an unknown function $u=u(x_1,x_2,...,x_n,t)$ of the independent variables $x_1,x_2,...,x_n$ and t as

$$P\left(u,\frac{\partial^\alpha}{\partial t^\alpha}u,\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}u,...,\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}u,u_{tt},u_{x_1x_1},...,u_{x_nx_n},\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}u\right),...\right)=0, \tag{3.1}$$

where $0<\alpha,\beta_1,\beta_2,...,\beta_n\leq 1$. The symbol $\frac{\partial^\gamma}{\partial v^\gamma}u$ is a generic term for the conformable partial derivative of u with respect to v of order $\gamma\in(0,1]$ and the subscript symbols denote the classical partial derivatives, for example, $u_{tt}=\frac{\partial^2}{\partial t^2}u$. The function P in (3.1) is a polynomial of u and its various partial derivatives.

Step 1: Transform the nonlinear conformable partial differential equation (3.1) into an ordinary differential equation (ODE) using the fractional complex traveling wave transformation in a variable ξ as shown below

$$u\left(x_1,x_2,...,x_n,t\right)=U\left(\xi\right),\ \xi=\frac{c_1x_1^{\beta_1}}{\beta_1}+\frac{c_2x_2^{\beta_2}}{\beta_2}+...+\frac{c_nx_n^{\beta_n}}{\beta_n}+\frac{kt^\alpha}{\alpha}, \tag{3.2}$$

where $c_1, c_2, ..., c_n, k$ are nonzero constants which will be determined at a later step. Applying transformation (3.2) to (3.1) and then integrating the resulting equation with respect to ξ as much as possible, we obtain the following ODE in $U = U(\xi)$ as

$$Q\left(U, U^{'}, U^{''}, U^{'''}, ...\right) = 0, \tag{3.3}$$

where Q is a polynomial function of $U(\xi)$ and its various integer-order derivatives. The prime notation $(')$ denotes the ordinary derivative with respect to ξ.

Step 2: Suppose that a solution of the ODE (3.3) can be expressed in terms of $\psi(\xi)$ as

$$U(\xi) = \sum_{i=-N}^{N} a_i \left(\psi\left(\xi\right)\right)^i, \tag{3.4}$$

where

$$\psi(\xi) = d + \phi(\xi) \;\text{ with }\; \phi(\xi) = \frac{G'(\xi)}{G^2(\xi)}. \tag{3.5}$$

The coefficients $a_i\,(i = 0, \pm 1, \pm 2, ..., \pm N)$ and d are unknown constants, which are determined later under the condition that $(a_{-N})^2 + (a_N)^2 \neq 0$. The function $G = G(\xi)$ satisfies the following nonlinear second-order ODE:

$$G^2 G'' = \mu G^4 + \lambda G^2 G' + (2G + v - 1)\left(G'\right)^2, \tag{3.6}$$

where the prime notation $(')$ denotes the ordinary derivative with respect to ξ and where λ, μ, and v are real parameters.

The following transformation $\phi(\xi) = \frac{(\ln(G^2(\xi)))_\xi}{2G(\xi)} = \frac{G'(\xi)}{G^2(\xi)}$ reduces equation (3.6) into the generalized Riccati equation [54, 55]:

$$\phi'(\xi) = \mu + \lambda\phi(\xi) + (v - 1)\phi^2(\xi). \tag{3.7}$$

It has been discovered that equation (3.7) has thirty nine solutions (see [54, 55] and Appendix in [56]).

Step 3: The value of the positive integer N in (3.4) can be calculated by utilizing the homogeneous balance principle, i.e., balancing between the highest-order linear terms and the highest-order nonlinear terms occurring in (3.3). More precisely, if the degree of $U(\xi)$ is $\text{Deg}[U(\xi)] = N$, then the degree of the following terms can be determined as follows [47]:

$$\text{Deg}\left[\frac{d^q U(\xi)}{d\xi^q}\right] = N + q, \;\text{Deg}\left[(U(\xi))^p\left(\frac{d^q U(\xi)}{d\xi^q}\right)^s\right] = Np + s(N + q). \tag{3.8}$$

Step 4: Substituting equation (3.4) along with equations (3.5) and (3.6) into equation (3.3), we get a polynomial in $\psi(\xi) = d + \phi(\xi)$. Collecting all coefficients of like-power of the resulting polynomial to zero, we obtain an over-determined set of algebraic equations for the unknowns a_i $(i = 0, \pm 1, \pm 2, ..., \pm N)$, d, c_1, $c_2, ...,$ c_n and k.

Step 5: Assuming that the algebraic equations in Step 4 can be solved for the unknowns via Maple software, we substitute the values of the unknowns together with the solutions of (3.7) into (3.4) to obtain exact traveling wave solutions of the nonlinear conformable partial differential equation (3.1) with ξ defined in (3.2).

4. Application of the method

In this section, we demonstrate an application of the novel (G'/G^2)-expansion method for finding exact solutions of the $(2+1)$-dimensional conformable space-time breaking soliton equation (1.2). Before extracting exact traveling wave solutions of (1.2) via the novel (G'/G^2)-expansion method, one must convert the conformable equation into an ordinary differential equation using the following transformations

$$u(x,y,t) = U(\xi) \text{ and } \xi = k_1\left(\frac{x^\beta}{\beta} + k_2\frac{y^\beta}{\beta} - c\frac{t^\alpha}{\alpha}\right), \tag{4.1}$$

where k_1, k_2 are c are constants determined at a later step. Applying (4.1) to (1.2), we obtain the ordinary differential equation in the variable $U = U(\xi)$ as

$$k_1^4 k_2 U^{(4)} - 6k_1^3 k_2 U' U'' - ck_1^2 U'' = 0, \tag{4.2}$$

where the prime notation ($'$) represents the ordinary derivative with respect to ξ. Integrating equation (4.2) with respect to ξ and letting the constant of integration to be zero, one finally get the following ODE

$$k_1^4 k_2 U''' - 3k_1^3 k_2 \left(U'\right)^2 - ck_1^2 U' = 0. \tag{4.3}$$

Using the solution form (3.4) and balancing the highest-order derivative U''' with the highest-order nonlinear term $\left(U'\right)^2$, we obtain $N = 1$ via the formulas in (3.8). Hence, the solution of (4.3) can be expressed as

$$U(\xi) = a_{-1}\left(\psi\left(\xi\right)\right)^{-1} + a_0 + a_1\left(\psi\left(\xi\right)\right)^1, \tag{4.4}$$

where a_{-1}, a_0, a_1 are unknown constants and $\psi(\xi) = d + \phi(\xi)$ with $\phi(\xi) = \frac{G'(\xi)}{G^2(\xi)}$ satisfying the generalized Riccati equation (3.7). Substituting (4.4) into (4.3), the left hand side of (4.3) is transformed into polynomials of $\left(\psi\left(\xi\right)\right)^j = \left(d + \phi\left(\xi\right)\right)^j$ where $j = 0, \pm 1, \pm 2, \pm 3, \pm 4$. Equating the coefficient of each power of $\psi(\xi)$ to zero, we obtain the following set of nonlinear algebraic equations:

$$\begin{aligned}
\psi^{-4}\left(\xi\right) :\ & -6k_1^4 k_2 d^6 v^3 a_{-1} + 18k_1^4 k_2 d^6 v^2 a_{-1} + 18k_1^4 k_2 d^5 \lambda v^2 a_{-1} - 18k_1^4 k_2 d^6 v a_{-1} \\
& - 36k_1^4 d^5 \lambda v a_{-1} k_2 - 18k_1^4 d^4 \lambda^2 v a_{-1} k_2 - 18k_1^4 d^4 \mu v^2 a_{-1} k_2 + 6k_1^4 d^6 a_{-1} k_2 \\
& + 18k_1^4 d^5 \lambda a_{-1} k_2 + 18k_1^4 k_2 a_{-1} \lambda^2 d^4 + 36k_1^4 d^4 \mu v a_{-1} k_2 - 3k_1^3 k_2 a_{-1}^2 v^2 d^4 \\
& + 6k_1^4 k_2 a_{-1} \lambda^3 d^3 + 36k_1^4 k_2 a_{-1} \lambda v \mu d^3 - 18k_1^4 k_2 a_{-1} \mu d^4 + 6k_1^3 k_2 a_{-1}^2 v d^4 \\
& - 36k_1^4 d^3 \lambda \mu a_{-1} k_2 + 6k_1^3 k_2 a_{-1}^2 \lambda v d^3 - 18k_1^4 d^2 \lambda^2 \mu a_{-1} k_2 - 3k_1^3 k_2 a_{-1}^2 d^4 \\
& - 18k_1^4 d^2 \mu^2 v a_{-1} k_2 - 6k_1^3 k_2 a_{-1}^2 \lambda d^3 - 3k_1^3 k_2 a_{-1}^2 \lambda^2 d^2 + 18k_1^4 d^2 \mu^2 a_{-1} k_2 \\
& - 6k_1^3 k_2 a_{-1}^2 \mu v d^2 + 18k_1^4 d \lambda \mu^2 a_{-1} k_2 + 6k_1^3 k_2 a_{-1}^2 \mu d^2 + 6k_1^3 k_2 a_{-1}^2 \lambda \mu d \\
& - 6k_1^4 k_2 a_{-1} \mu^3 - 3k_1^3 k_2 a_{-1}^2 \mu^2 = 0,
\end{aligned}$$

$$
\begin{aligned}
\psi^{-3}(\xi)\ :\ & 24k_1^4d^5v^3a_{-1}k_2-72k_1^4d^5v^2a_{-1}k_2-60k_1^4d^4\lambda v^2a_{-1}k_2+72k_1^4k_2d^5va_{-1}\\
&+120k_1^4d^4\lambda va_{-1}k_2+48k_1^4d^3\lambda^2va_{-1}k_2+48k_1^4d^3\mu v^2a_{-1}k_2-6k_1^3k_2a_{-1}^2\lambda\mu\\
&-24k_1^4k_2d^5a_{-1}-60k_1^4d^4\lambda a_{-1}k_2-48k_1^4k_2a_{-1}\lambda^2d^3-96k_1^4d^3\mu va_{-1}k_2\\
&+12k_1^3k_2a_{-1}^2v^2d^3-12k_1^4k_2a_{-1}\lambda^3d^2-72k_1^4k_2a_{-1}\lambda v\mu d^2+48k_1^4k_2a_{-1}\mu d^3\\
&-24k_1^3k_2a_{-1}^2vd^3+72k_1^4d^2\lambda\mu a_{-1}k_2-18k_1^3k_2a_{-1}^2\lambda vd^2+24k_1^4d\lambda^2\mu a_{-1}k_2\\
&+24k_1^4d\mu^2va_{-1}k_2+12k_1^3k_2a_{-1}^2d^3+18k_1^3k_2a_{-1}^2\lambda d^2+6k_1^3k_2a_{-1}^2\lambda^2d\\
&-24k_1^4k_2a_{-1}\mu^2d+12k_1^3k_2a_{-1}^2\mu vd-12k_1^4k_2a_{-1}\lambda\mu^2-12k_1^3k_2a_{-1}^2\mu d=0,\\
\psi^{-2}(\xi)\ :\ & -36k_1^4d^4v^3a_{-1}k_2+6k_1^3d^4v^2a_{-1}a_1k_2+108k_1^4d^4v^2a_{-1}k_2\\
&+72k_1^4d^3\lambda v^2a_{-1}k_2-12k_1^3d^4va_{-1}a_1k_2-108k_1^4d^4va_{-1}k_2\\
&-12k_1^3\lambda va_{-1}a_1k_2d^3-144k_1^4d^3\lambda va_{-1}k_2-43k_1^4d^2\lambda^2va_{-1}k_2\\
&-44k_1^4d^2\mu v^2a_{-1}k_2+6k_1^3a_{-1}a_1k_2d^4+36k_1^4d^4a_{-1}k_2+12k_1^3d^3\lambda a_{-1}a_1k_2\\
&+72k_1^4d^3\lambda a_{-1}k_2+6k_1^3d^2\lambda^2a_{-1}a_1k_2+43d^2\lambda^2a_{-1}k_1^4k_2\\
&+12d^2\mu va_{-1}a_1k_1^3k_2+88d^2\mu va_{-1}k_1^4k_2-18d^2v^2a_{-1}^2k_1^3k_2+7d\lambda^3a_{-1}k_1^4k_2\\
&+44d\lambda\mu va_{-1}k_1^4k_2-12k_1^3d^2\mu a_{-1}a_1k_2-44k_1^4k_2a_{-1}\mu d^2+36k_1^3k_2a_{-1}^2vd^2\\
&-12k_1^3\lambda\mu a_{-1}a_1k_2d-44k_1^4d\lambda\mu a_{-1}k_2+18k_1^3k_2a_{-1}^2\lambda vd-7k_1^4k_2a_{-1}\lambda^2\mu\\
&-8k_1^4k_2a_{-1}v\mu^2-18k_1^3k_2a_{-1}^2d^2-18k_1^3k_2a_{-1}^2\lambda d-3k_1^3k_2a_{-1}^2\lambda^2\\
&+6k_1^3\mu^2a_{-1}a_1k_2+8k_1^4k_2a_{-1}\mu^2-6k_1^3k_2a_{-1}^2\mu v+k_1^2cva_{-1}d^2\\
&+6k_1^3k_2a_{-1}^2\mu-k_1^2ca_{-1}d^2-k_1^2c\lambda a_{-1}d+k_1^2c\mu a_{-1}=0,\\
\psi^{-1}(\xi)\ :\ & 24k_1^4d^3v^3a_{-1}k_2-24k_1^3d^3v^2a_{-1}a_1k_2-72k_1^4d^3v^2a_{-1}k_2-36k_1^4d^2\lambda v^2a_{-1}k_2\\
&+48k_1^3d^3va_{-1}a_1k_2+72k_1^4d^3va_{-1}k_2+36k_1^3\lambda va_{-1}a_1k_2d^2+72k_1^4d^2\lambda va_{-1}k_2\\
&+14k_1^4d\lambda^2va_{-1}k_2+16k_1^4d\mu v^2a_{-1}k_2-24k_1^3a_{-1}a_1k_2d^3-24k_1^4d^3a_{-1}k_2\\
&-36k_1^3d^2\lambda a_{-1}a_1k_2-36k_1^4d^2\lambda a_{-1}k_2-12k_1^3d\lambda^2a_{-1}a_1k_2-14k_1^4k_2a_{-1}\lambda^2d\\
&-24k_1^3\mu va_{-1}a_1k_2d-32k_1^4d\mu va_{-1}k_2+12k_1^3k_2a_{-1}^2v^2d-k_1^4k_2a_{-1}\lambda^3\\
&-8k_1^4k_2a_{-1}\lambda v\mu+24k_1^3d\mu a_{-1}a_1k_2+16k_1^4k_2a_{-1}\mu d-24k_1^3k_2a_{-1}^2vd\\
&+12k_1^3\lambda\mu a_{-1}a_1k_2+8k_1^4k_2a_{-1}\lambda\mu-6k_1^3k_2a_{-1}^2\lambda v+12k_1^3k_2a_{-1}^2d+6k_1^3k_2a_{-1}^2\lambda\\
&-2k_1^2cva_{-1}d+2k_1^2ca_{-1}d+k_1^2c\lambda a_{-1}=0,\\
\psi^{0}(\xi)\ :\ & 6k_1^4v^3a_1k_2d^4-3k_1^3v^2a_1^2k_2d^4-18k_1^4v^2a_1k_2d^4-12k_1^4d^3\lambda v^2a_1k_2+6k_1^3va_1^2k_2d^4\\
&+18k_1^4va_1k_2d^4+6k_1^3d^3\lambda va_1^2k_2+24k_1^4d^3\lambda va_1k_2+7k_1^4d^2\lambda^2va_1k_2\\
&+8k_1^4d^2\mu v^2a_1k_2-6k_1^4d^2v^3a_{-1}k_2-3k_1^3a_1^2k_2d^4-6k_1^4a_1k_2d^4-6k_1^3\lambda a_1^2k_2d^3\\
&-12k_1^4\lambda a_1k_2d^3-3k_1^3\lambda^2a_1^2k_2d^2-7k_1^4\lambda^2a_1k_2d^2-6k_1^3d^2\mu va_1^2k_2-16k_1^4d^2\mu va_1k_2\\
&+36k_1^3d^2v^2a_{-1}a_1k_2+18k_1^4d^2v^2a_{-1}k_2-k_1^4\lambda^3a_1k_2d-8k_1^4\lambda\mu va_1k_2d\\
&+6k_1^4d\lambda v^2a_{-1}k_2+6k_1^3\mu a_1^2k_2d^2+8k_1^4\mu a_1k_2d^2-72k_1^3d^2va_{-1}a_1k_2\\
&-18k_1^4d^2va_{-1}k_2+6k_1^3d\lambda\mu a_1^2k_2+8k_1^4d\lambda\mu a_1k_2-36k_1^3\lambda va_{-1}a_1k_2d
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
& -12k_1^4 d\lambda v a_{-1}k_2 + k_1^4 k_2 a_1 \lambda^2 \mu - k_1^4 k_2 a_{-1}\lambda^2 v + 2k_1^4 k_2 a_1 v \mu^2 - 2k_1^4 k_2 a_{-1}\mu v^2 \\
& + 36k_1^3 a_{-1}a_1 k_2 d^2 + 6k_1^4 d^2 a_{-1}k_2 + 36k_1^3 d\lambda a_{-1}a_1 k_2 + 6k_1^4 d\lambda a_{-1}k_2 - k_1^2 c a_{-1} \\
& + 6k_1^3 \lambda^2 a_{-1}a_1 k_2 + k_1^4 k_2 a_{-1}\lambda^2 - 3k_1^3 k_2 a_1^2 \mu^2 - 2k_1^4 k_2 a_1 \mu^2 + 12k_1^3 \mu v a_{-1}a_1 k_2 \\
& + 4k_1^4 k_2 a_{-1}\mu v - 3k_1^3 k_2 a_{-1}^2 v^2 - k_1^2 c v a_1 d^2 - 12k_1^3 \mu a_{-1}a_1 k_2 - 2k_1^4 k_2 a_{-1}\mu \\
& + 6k_1^3 k_2 a_{-1}^2 v + k_1^2 c a_1 d^2 + k_1^2 c\lambda a_1 d - 3k_1^3 k_2 a_{-1}^2 - c k_1^2 a_1 \mu + k_1^2 c v a_{-1} = 0, \\
\psi(\xi) \ : \ & -24k_1^4 v^3 a_1 k_2 d^3 + 12k_1^3 v^2 a_1^2 k_2 d^3 + 72k_1^4 v^2 a_1 k_2 d^3 + 36k_1^4 d^2 \lambda v^2 a_1 k_2 \\
& - 24k_1^3 v a_1^2 k_2 d^3 - 72k_1^4 v a_1 k_2 d^3 - 18k_1^3 d^2 \lambda v a_1^2 k_2 - 72k_1^4 d^2 \lambda v a_1 k_2 \\
& - 14k_1^4 d\lambda^2 v a_1 k_2 - 16k_1^4 d\mu v^2 a_1 k_2 + 12k_1^3 a_1^2 k_2 d^3 + 24k_1^4 a_1 k_2 d^3 + 18k_1^3 \lambda a_1^2 k_2 d^2 \\
& + 36k_1^4 \lambda a_1 k_2 d^2 + 6k_1^3 \lambda^2 a_1^2 k_2 d + 14k_1^4 \lambda^2 a_1 k_2 d + 12k_1^3 d\mu v a_1^2 k_2 + 32k_1^4 d\mu v a_1 k_2 \\
& - 24k_1^3 d v^2 a_{-1}a_1 k_2 + k_1^4 \lambda^3 a_1 k_2 + 8k_1^4 \lambda\mu v a_1 k_2 - 12k_1^3 \mu a_1^2 k_2 d - 16k_1^4 \mu a_1 k_2 d \\
& + 48k_1^3 d v a_{-1}a_1 k_2 - 6k_1^3 \lambda\mu a_1^2 k_2 - 8k_1^4 \lambda\mu a_1 k_2 + 12k_1^3 \lambda v a_{-1}a_1 k_2 \\
& - 24k_1^3 a_{-1}a_1 k_2 d - 12k_1^3 \lambda a_{-1}a_1 k_2 + 2k_1^2 c v a_1 d - 2k_1^2 c a_1 d - k_1^2 c\lambda a_1 = 0, \\
\psi^2(\xi) \ : \ & 36k_1^4 v^3 a_1 k_2 d^2 - 18k_1^3 v^2 a_1^2 k_2 d^2 - 108k_1^4 v^2 a_1 k_2 d^2 - 36k_1^4 d\lambda v^2 a_1 k_2 + 36k_1^3 v a_1^2 k_2 d^2 \\
& + 108k_1^4 v a_1 k_2 d^2 + 18k_1^3 d\lambda v a_1^2 k_2 + 72k_1^4 d\lambda v a_1 k_2 + 7k_1^4 \lambda^2 v a_1 k_2 + 8k_1^4 \mu v^2 a_1 k_2 \\
& - 18k_1^3 a_1^2 k_2 d^2 - 36k_1^4 a_1 k_2 d^2 - 18k_1^3 \lambda a_1^2 k_2 d - 36k_1^4 \lambda a_1 k_2 d - 3k_1^3 \lambda^2 a_1^2 k_2 \\
& - 7k_1^4 \lambda^2 a_1 k_2 - 6k_1^3 \mu v a_1^2 k_2 - 16k_1^4 \mu v a_1 k_2 + 6k_1^3 v^2 a_{-1}a_1 k_2 + 6k_1^3 \mu a_1^2 k_2 \\
& + 8k_1^4 \mu a_1 k_2 - 12k_1^3 v a_{-1}a_1 k_2 + 6k_1^3 a_{-1}a_1 k_2 - k_1^2 c v a_1 + k_1^2 c a_1 = 0, \\
\psi^3(\xi) \ : \ & -24k_1^4 v^3 a_1 k_2 d + 12k_1^3 v^2 a_1^2 k_2 d + 72k_1^4 v^2 a_1 k_2 d + 12k_1^4 \lambda v^2 a_1 k_2 - 24k_1^3 v a_1^2 k_2 d \\
& - 72k_1^4 v a_1 k_2 d - 6k_1^3 \lambda v a_1^2 k_2 - 24k_1^4 \lambda v a_1 k_2 + 12k_1^3 a_1^2 k_2 d + 24k_1^4 a_1 k_2 d \\
& + 6k_1^3 \lambda a_1^2 k_2 + 12k_1^4 \lambda a_1 k_2 = 0, \\
\psi^4(\xi) \ : \ & 6k_1^4 v^3 a_1 k_2 - 3k_1^3 v^2 a_1^2 k_2 - 18k_1^4 v^2 a_1 k_2 + 6k_1^3 v a_1^2 k_2 + 18k_1^4 v a_1 k_2 - 3k_1^3 a_1^2 k_2 \\
& - 6k_1^4 a_1 k_2 = 0.
\end{aligned}
$$

Using the symbolic computation software such as Maple 17 to solve system (4.5), one can obtain the following three cases of the unknown constants $a_{-1}, a_0, a_1, d, k_1, k_2, c$.

Case 1 :

$$
a_{-1} = -2d^2 v k_1 + 2d^2 k_1 + 2d\lambda k_1 - 2\mu k_1 \ , \ a_1 = 0 \ , \ c = k_1^2 k_2 \left(\lambda^2 - 4\mu v + 4\mu\right), \tag{4.6}
$$

where $a_0, k_1 \neq 0, k_2 \neq 0, \mu, \lambda, v, d$ are arbitrary constants.

Case 2 :

$$
a_{-1} = 0 \ , \ a_1 = 2vk_1 - 2k_1 \ , \ c = k_1^2 k_2 \left(\lambda^2 - 4\mu v + 4\mu\right), \tag{4.7}
$$

where $a_0, k_1 \neq 0, k_2 \neq 0, \mu, \lambda, v, d$ are arbitrary constants.

Case 3 :

$$
\begin{aligned}
& a_{-1} = \frac{k_1 \left(\lambda^2 - 4\mu v + 4\mu\right)}{2(v-1)}, \ a_1 = 2vk_1 - 2k_1, \ d = \frac{\lambda}{2(v-1)}, \\
& c = 4k_1^2 k_2 \left(\lambda^2 - 4\mu v + 4\mu\right),
\end{aligned} \tag{4.8}
$$

where $a_0, k_1 \neq 0, k_2 \neq 0, \mu, \lambda, v \neq 1$ are arbitrary constants.

For computational convenience, we denote

$$\Delta = \lambda^2 - 4\mu\,(v-1)\,. \tag{4.9}$$

All of the following exact traveling wave solutions $u\,(x,y,t)$ for (1.2) are simplified and expressed in terms of the solutions of the generalized Riccati equation, which are separated into four families as mentioned in the Appendix of [56], the change of variables in (4.1) and the symbol Δ in (4.9). All of them have been substituted into (4.3) with the aid of the Maple package program to confirm that they correctly satisfy the equation.

Exact solutions of (1.2) using the unknown constants of Case 1: The exact traveling wave solutions of (1.2), obtained using the solution form (4.4), the unknown constants in (4.6) together with (4.1) and (4.9), are expressed as follows.

Family 1. When $\Delta > 0$ and $\lambda(v-1) \neq 0$ (or $\mu(v-1) \neq 0$), the hyperbolic function solutions of (1.2) are as follows:

$$u^1_{1,1}(x,y,t) = \frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)}{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - 2dv + 2d + \lambda} + a_0, \tag{4.10}$$

$$u^1_{1,2}(x,y,t) = \frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)}{\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right) - 2dv + 2d + \lambda} + a_0, \tag{4.11}$$

$$u^1_{1,3-4}(x,y,t) = -\frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)\cosh(\sqrt{\Delta}\xi)}{(2dv - 2d - \lambda)\cosh(\sqrt{\Delta}\xi) - \sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi) \mp i)} + a_0, \tag{4.12}$$

$$u^1_{1,5-6}(x,y,t) = -\frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)\sinh\left(\sqrt{\Delta}\xi\right)}{(2dv - 2d - \lambda)\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi) \mp 1)} + a_0, \tag{4.13}$$

$$u^1_{1,7}(x,y,t) = \frac{8k_1(d^2v - d^2 - d\lambda + \mu)(v-1)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}{2(-2dv + 2d + \lambda)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \sqrt{\Delta}\left(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right) - 1\right)} + a_0, \tag{4.14}$$

$$u^1_{1,8-9}(x,y,t) = -\frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)(A\sinh(\sqrt{\Delta}\xi) + B)}{(2dv - 2d - \lambda)(A\sinh(\sqrt{\Delta}\xi) + B) - A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \pm \sqrt{(A^2 + B^2)\Delta}} + a_0, \tag{4.15}$$

$$u^1_{1,10-11}(x,y,t) = -\frac{4k_1(d^2v - d^2 - d\lambda + \mu)(v-1)(A\cosh(\sqrt{\Delta}\xi) + B)}{(2dv - 2d - \lambda)(A\cosh(\sqrt{\Delta}\xi) + B) - A\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) \mp \sqrt{(B^2 - A^2)\Delta}} + a_0, \tag{4.16}$$

where A and B are two nonzero real constants and satisfy the condition that $B^2 - A^2 > 0$,

$$u^1_{1,12}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(\lambda\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{(d\lambda - 2\mu)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - d\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + a_0, \tag{4.17}$$

$$u^1_{1,13}(x,y,t) = -\frac{2k_1(d^2v-d^2-d\lambda+\mu)\left(\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{d\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+(2\mu-d\lambda)\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}+a_0, \tag{4.18}$$

$$u^1_{1,14-15}(x,y,t) = -\frac{2k_1(d^2v-d^2-d\lambda+\mu)\left(\sqrt{\Delta}(i\pm\sinh(\sqrt{\Delta}\xi))\mp\lambda\cosh(\sqrt{\Delta}\xi)\right)}{d\sqrt{\Delta}(i\pm\sinh(\sqrt{\Delta}\xi))\mp(d\lambda-2\mu)\cosh(\sqrt{\Delta}\xi)}+a_0, \tag{4.19}$$

$$u^1_{1,16-17}(x,y,t) = -\frac{2k_1(d^2v-d^2-d\lambda+\mu)\left(\sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi)\pm 1)-\lambda\sinh(\sqrt{\Delta}\xi)\right)}{d\sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi)\pm 1)-(d\lambda-2\mu)\sinh(\sqrt{\Delta}\xi)}+a_0, \tag{4.20}$$

$$u^1_{1,18}(x,y,t) = -\frac{2k_1(d^2v-d^2-d\lambda+\mu)\left(-2\lambda\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)\right)}{2d\sqrt{\Delta}(\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2(d\lambda+2\mu)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}+a_0. \tag{4.21}$$

Family 2. When $\Delta<0$ and $\lambda(v-1)\neq 0$ (or $\mu(v-1)\neq 0$), the trigonometric function solutions of (1.2) are as follows:

$$u^1_{2,1}(x,y,t) = \frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)}{\sqrt{-\Delta}\tanh\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-2dv+2d+\lambda}+a_0, \tag{4.22}$$

$$u^1_{2,2}(x,y,t) = -\frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)}{\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-2\,dv+2\,d+\lambda}+a_0, \tag{4.23}$$

$$u^1_{2,3-4}(x,y,t) = -\frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)\cos(\sqrt{-\Delta}\xi)}{(2dv-2d-\lambda)\cos(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)}+a_0, \tag{4.24}$$

$$u^1_{2,5-6}(x,y,t) = -\frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)\sin(\sqrt{-\Delta}\xi)}{(2dv-2d-\lambda)\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(-\cos(\sqrt{-\Delta}\xi)\mp 1)}+a_0, \tag{4.25}$$

$$u^1_{2,7}(x,y,t) = \frac{8k_1(d^2v-d^2-d\lambda+\mu)(v-1)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}{2(-2dv+2d+\lambda)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)+\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1)}+a_0, \tag{4.26}$$

$$u^1_{2,8-9}(x,y,t) = \frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)(A\sin(\sqrt{-\Delta}\xi)+B)}{(-2dv+2d+\lambda)(A\sin(\sqrt{-\Delta}\xi)+B)+A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-(A^2-B^2)\Delta}}+a_0, \tag{4.27}$$

$$u^1_{2,10-11}(x,y,t) = \frac{4k_1(d^2v-d^2-d\lambda+\mu)(v-1)(A\sin(\sqrt{-\Delta}\xi)+B)}{(-2dv+2d+\lambda)(A\sin(\sqrt{-\Delta}\xi)+B)+A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\mp\sqrt{-(A^2-B^2)\Delta}}+a_0, \tag{4.28}$$

where A and B are two nonzero real constants and satisfy the condition that $A^2 - B^2 > 0$,

$$u^1_{2,12}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + \sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{(d\lambda - 2\mu)\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + d\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0, \tag{4.29}$$

$$u^1_{2,13}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - \lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{d\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + (2\mu - d\lambda)\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0, \tag{4.30}$$

$$u^1_{2,14-15}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi) \pm 1) + \lambda\cos(\sqrt{-\Delta}\xi)\right)}{d\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi) \pm 1) + (d\lambda - 2\mu)\cos(\sqrt{-\Delta}\xi)} + a_0, \tag{4.31}$$

$$u^1_{2,16-17}(x,y,t) = \frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi \mp 1)) + \lambda\sin(\sqrt{-\Delta}\xi)\right)}{d\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi) \pm 1) - (d\lambda - 2\mu)\sin(\sqrt{-\Delta}\xi)} + a_0, \tag{4.32}$$

$$u^1_{2,18}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda + \mu)\left(-2\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right) + 2\sqrt{-\Delta}\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right) - \sqrt{-\Delta}\right)}{d\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right) - 1) - 2(d\lambda - 2\mu)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)} + a_0. \tag{4.33}$$

Family 3. When $\mu = 0$ and $\lambda(v-1) \neq 0$, the hyperbolic function solutions of (1.2) are expressed as:

$$u^1_{3,1}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda)\left(\cosh(\lambda\xi) - \sinh(\lambda\xi) + c_1v - c_1\right)}{d(\cosh(\lambda\xi) - \sinh(\lambda\xi)) + (dv - d - \lambda)c_1} + a_0, \tag{4.34}$$

$$u^1_{3,2}(x,y,t) = -\frac{2k_1(d^2v - d^2 - d\lambda)(v-1)(2\cosh(\lambda\xi)c_1 + c_1^2 + 1)}{2c_1d(v-1)\cosh(\lambda\xi) - c_1\lambda(\cosh(\lambda\xi) + \sinh(\lambda\xi)) + dc_1^2v - dc_1^2 + dv - d - \lambda} + a_0, \tag{4.35}$$

where c_1 is an arbitrary constant.

Family 4. When $\mu = \lambda = 0$ and $v - 1 \neq 0$, the rational function solution of (1.2) is shown below:

$$u^1_{4,1}(x,y,t) = -\frac{2k_1(d^2v - d^2)(v\xi - \xi + c_2)}{dv\xi - d\xi + dc_2 - 1} + a_0, \tag{4.36}$$

where c_2 is an arbitrary constant.

Exact solutions of (1.2) using the unknown constants of Case 2: The exact traveling wave solutions of (1.2), obtained using the solution form (4.4), the unknown constants in (4.7) together with (4.1) and (4.9), are listed as follows.

Family 1. When $\Delta > 0$ and $\lambda(v-1) \neq 0$ (or $\mu(v-1) \neq 0$), the hyperbolic function solutions of (1.2) are as follows:

$$u^2_{1,1}(x,y,t) = a_0 - k_1\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - 2dv + 2d + \lambda\right), \tag{4.37}$$

$$u^2_{1,2}(x,y,t) = a_0 - k_1\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right) - 2dv + 2d + \lambda\right), \tag{4.38}$$

$$u^2_{1,3-4}(x,y,t) = a_0 - \frac{k_1(v-1)((-2dv+2d+\lambda)\cosh(\sqrt{\Delta}\xi) + \sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta})}{\cosh(\sqrt{\Delta}\xi)(v-1)}, \tag{4.39}$$

$$u^2_{1,5-6}(x,y,t) = a_0 - \frac{k_1(v-1)((-2dv+2d+\lambda)\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta})}{\sinh(\sqrt{\Delta}\xi)(v-1)}, \tag{4.40}$$

$$u^2_{1,7}(x,y,t) = a_0 - \frac{k_1\left(2(-2dv+2d+\lambda)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \sqrt{\Delta}\left(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right) - 1\right)\right)}{2\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}, \tag{4.41}$$

$$u^2_{1,8-9}(x,y,t) = a_0 - \frac{k_1((-2dv+2d+\lambda)(A\sinh(\sqrt{\Delta}\xi)+B) + A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \mp \sqrt{(A^2+B^2)\Delta})}{A\sinh(\sqrt{\Delta}\xi)+B}, \tag{4.42}$$

$$u^2_{1,10-11}(x,y,t) = a_0 - \frac{k_1((-2dv+2d+\lambda)(A\cosh(\sqrt{\Delta}\xi)+B) + A\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) \pm \sqrt{(B^2-A^2)\Delta})}{A\cosh(\sqrt{\Delta}\xi)+B}, \tag{4.43}$$

where A and B are two nonzero real constants and satisfy the condition that $B^2 - A^2 > 0$,

$$u^2_{1,12}(x,y,t) = a_0 + \frac{2k_1(v-1)\left(d\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + (2\mu - d\lambda)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \lambda\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}, \tag{4.44}$$

$$u^2_{1,13}(x,y,t) = a_0 + \frac{2k_1(v-1)\left(d\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + (2\mu - d\lambda)\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}, \tag{4.45}$$

$$u^2_{1,14-15}(x,y,t) = a_0 + \frac{2k_1(v-1)(d\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi) \pm i) + (2\mu - d\lambda)\cosh(\sqrt{\Delta}\xi))}{\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi) \pm i) - \lambda\cosh(\sqrt{\Delta}\xi)}, \tag{4.46}$$

$$u^2_{1,16-17}(x,y,t) = a_0 + \frac{2k_1(v-1)(d\sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi) \pm 1) + (2\mu - d\lambda)\sinh(\sqrt{\Delta}\xi))}{-\lambda\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi) \pm 1)}, \tag{4.47}$$

$$u^2_{1,18}(x,y,t)=$$
$$a_0+\frac{2k_1(v-1)\left(d\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)+(4\mu-2d\lambda)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)}{\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}. \quad (4.48)$$

Family 2. When $\Delta<0$ and $\lambda(v-1)\neq 0$ (or $\mu(v-1)\neq 0$), the trigonometric function solutions of (1.2) are listed below:

$$u^2_{2,1}(x,y,t)=a_0+\frac{k_1(v-1)\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+2dv-2d-\lambda\right)}{v-1}, \quad (4.49)$$

$$u^2_{2,2}(x,y,t)=a_0-\frac{k_1(v-1)\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-2dv+2d+\lambda\right)}{v-1}, \quad (4.50)$$

$$u^2_{2,3-4}(x,y,t)=a_0+\frac{k_1\left((2dv-2d-\lambda)\cos(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)\right)}{\cos(\sqrt{-\Delta}\xi)}, \quad (4.51)$$

$$u^2_{2,5-6}(x,y,t)=a_0-\frac{k_1\left((-2dv+2d+\lambda)\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi)\pm 1)\right)}{\sin(\sqrt{-\Delta}\xi)}, \quad (4.52)$$

$$u^2_{2,7}(x,y,t)=$$
$$a_0-\frac{k_1(2(-2dv+2d+\lambda)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)+\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1))}{2\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}, \quad (4.53)$$

$$u^2_{2,8-9}(x,y,t)=$$
$$a_0-\frac{k_1((-2dv+2d+\lambda)(A\sin(\sqrt{-\Delta}\xi)+B)+A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\mp\sqrt{-(A^2-B^2)\Delta})}{A\sin(\sqrt{-\Delta}\xi)+B}, \quad (4.54)$$

$$u^2_{2,10-11}(x,y,t)=$$
$$a_0-\frac{k_1((-2dv+2d+\lambda)(A\sin(\sqrt{-\Delta}\xi)+B)+A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm(\sqrt{-(A^2-B^2)\Delta}))}{A\sin(\sqrt{-\Delta}\xi)+B}, \quad (4.55)$$

where A and B are two nonzero real constants and satisfy the condition that $A^2-B^2>0$,

$$u^2_{2,12}(x,y,t)=a_0+\frac{2k_1(v-1)\left(d\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+(d\lambda-2\mu)\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}, \quad (4.56)$$

$$u^2_{2,13}(x,y,t)=a_0+\frac{2k_1(v-1)\left(d\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+(2\mu-d\lambda)\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{-\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}, \quad (4.57)$$

$$u^2_{2,14-15}(x,y,t)=$$
$$a_0+\frac{2k_1(v-1)(d\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)+(d\lambda-2\mu)\cos(\sqrt{-\Delta}\xi))}{\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)+\lambda\cos(\sqrt{-\Delta}\xi)}, \quad (4.58)$$

$$u^2_{2,16-17}(x,y,t) = a_0 + \frac{2k_1(v-1)\left(d\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi)\pm 1) + (2\mu - d\lambda)\sin(\sqrt{-\Delta}\xi)\right)}{\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi)\pm 1) - \lambda\sin(\sqrt{-\Delta}\xi)}, \tag{4.59}$$

$$u^2_{2,18}(x,y,t) = a_0 + \frac{2k_1(v-1)\left(d\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right) - 1) + 2(2\mu - d\lambda)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)}{-2\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right) + \sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right) - 1)}. \tag{4.60}$$

Family 3. When $\mu = 0$ and $\lambda(v-1) \neq 0$, the hyperbolic function solutions of (1.2) are as follows:

$$u^2_{3,1}(x,y,t) = a_0 + \frac{2k_1(v-1)(dvc_1 - dc_1 - c_1\lambda)(d\cosh(\lambda\xi) - d\sinh(\lambda\xi))}{\cosh(\lambda\xi) - \sinh(\lambda\xi) + c_1 v - c_1}, \tag{4.61}$$

$$u^2_{3,2}(x,y,t) = a_0 + \frac{2k_1((dv - d - \lambda)(\cosh(\lambda\xi) + \sinh(\lambda\xi)) - dc_1 + dvc_1)}{\cosh(\lambda\xi) + \sinh(\lambda\xi) + c_1}, \tag{4.62}$$

where c_1 is an arbitrary constant.

Family 4. When $\mu = \lambda = 0$ and $v - 1 \neq 0$, the rational function solution of (1.2) is expressed below:

$$u^2_{4,1}(x,y,t) = a_0 + \frac{2k_1(v-1)(dv\xi - d\xi + dc_2 - 1)}{v\xi - \xi + c_2}, \tag{4.63}$$

where c_2 is an arbitrary constant.

Exact solutions of (1.2) using the unknown constants of Case 3: The exact traveling wave solutions of (1.2), obtained using the solution form (4.4), the unknown constants in (4.8) together with (4.1) and (4.9), are shown as follows.

Family 1. When $\Delta > 0$ and $\lambda(v-1) \neq 0$ (or $\mu(v-1) \neq 0$), the hyperbolic function solutions of (1.2) are as follows:

$$u^3_{1,1}(x,y,t) = -\frac{k_1\sqrt{\Delta}}{\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + a_0 - k_1\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right), \tag{4.64}$$

$$u^3_{1,2}(x,y,t) = -\frac{k_1\sqrt{\Delta}}{\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + a_0 - k_1\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right), \tag{4.65}$$

$$u^3_{1,3-4}(x,y,t) = -\frac{k_1\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{\sinh(\sqrt{\Delta}\xi)\pm i} + a_0 - \frac{k_1\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi)\pm i)}{\cosh(\sqrt{\Delta}\xi)}, \tag{4.66}$$

$$u^3_{1,5-6}(x,y,t) = -\frac{k_1\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)}{\cosh(\sqrt{\Delta}\xi)\pm 1} + a_0 - \frac{k_1\sqrt{\Delta}(\cosh(\sqrt{\Delta}\xi)\pm 1)}{\sinh(\sqrt{\Delta}\xi)}, \tag{4.67}$$

$$u^3_{1,7}(x,y,t) = -\frac{2k_1\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}{2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right) - 1} + a_0 - \frac{k_1\sqrt{\Delta}\left(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right) - 1\right)}{2\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}, \tag{4.68}$$

$$\begin{aligned}u^3_{1,8-9}(x,y,t) =& -\frac{k_1\Delta(A\sinh(\sqrt{\Delta}\xi)+B)}{A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\mp\sqrt{(A^2+B^2)\Delta}}+a_0\\ &-\frac{k_1(A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\mp\sqrt{(A^2+B^2)\Delta})}{A\sinh(\sqrt{\Delta}\xi)+B},\end{aligned} \tag{4.69}$$

$$\begin{aligned}u^3_{1,10-11}(x,y,t) =& -\frac{k_1\Delta(A\cosh(\sqrt{\Delta}\xi)+B)}{A\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)\pm\sqrt{(B^2-A^2)\Delta}}+a_0\\ &-\frac{k_1(A\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)\pm\sqrt{(B^2-A^2)\Delta})}{A\cosh(\sqrt{\Delta}\xi)+B},\end{aligned} \tag{4.70}$$

where A and B are two nonzero real constants and satisfy the condition that $B^2-A^2>0$,

$$\begin{aligned}u^3_{1,12}(x,y,t) =& \frac{k_1\Delta\left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\lambda\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\lambda\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\Delta\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}+a_0\\ &+\frac{k_1\lambda\left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\Delta\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\lambda\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)},\end{aligned} \tag{4.71}$$

$$\begin{aligned}u^3_{1,13}(x,y,t) =& \frac{k_1\Delta\left(\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\lambda\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\Delta\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}+a_0\\ &+\frac{k_1\left(\lambda\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\Delta\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)-\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)},\end{aligned} \tag{4.72}$$

$$\begin{aligned}u^3_{1,14-15}(x,y,t) =& \frac{k_1\Delta(\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi)\pm i)-\lambda\cosh(\sqrt{\Delta}\xi))}{\lambda\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi)\pm i)-\Delta\cosh(\sqrt{\Delta}\xi)}+a_0\\ &+\frac{k_1\left(\lambda\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi)\pm i)-\Delta\cosh(\sqrt{\Delta}\xi)\right)}{\sqrt{\Delta}(\sinh(\sqrt{\Delta}\xi)\pm i)-\lambda\cosh(\sqrt{\Delta}\xi)},\end{aligned} \tag{4.73}$$

$$\begin{aligned}u^3_{1,16-17}(x,y,t) =& \frac{k_1\Delta\left(\lambda\sinh(\sqrt{\Delta}\xi)-\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}\right)}{\Delta\sinh(\sqrt{\Delta}\xi)+\lambda\sqrt{\Delta}(-\cosh(\sqrt{\Delta}\xi)\pm 1)}+a_0\\ &+\frac{k_1\left(\Delta\sinh(\sqrt{\Delta}\xi)+\lambda\sqrt{\Delta}(-\cosh(\sqrt{\Delta}\xi)\pm 1)\right)}{\lambda\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}(-\cosh(\sqrt{\Delta}\xi)\pm 1)},\end{aligned} \tag{4.74}$$

$$\begin{aligned}u^3_{1,18}(x,y,t) =& \frac{k_1\Delta\left(\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)}{\lambda\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2\Delta\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}+a_0\\ &+\frac{k_1\left(\lambda\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2\Delta\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)}{\sqrt{\Delta}(2\cosh^2\left(\frac{\sqrt{\Delta}\xi}{4}\right)-1)-2\lambda\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)}.\end{aligned} \tag{4.75}$$

Family 2. When $\Delta < 0$ and $\lambda(v-1) \neq 0$ (or $\mu(v-1) \neq 0$), the trigonometric function solutions of (1.2) are shown below:

$$u_{2,1}^3(x,y,t) = -\frac{k_1\sqrt{-\Delta}}{\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0 + k_1\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right), \tag{4.76}$$

$$u_{2,2}^3(x,y,t) = \frac{k_1\sqrt{-\Delta}}{\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0 - k_1\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right), \tag{4.77}$$

$$u_{2,3-4}^3(x,y,t) = -\frac{k_1\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{\sin(\sqrt{-\Delta}\xi)\pm 1} + a_0 + \frac{k_1\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)}{\cos(\sqrt{-\Delta}\xi)}, \tag{4.78}$$

$$u_{2,5-6}^3(x,y,t) = \frac{k_1\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)}{\cos(\sqrt{-\Delta}\xi)\pm 1} + a_0 - \frac{k_1\sqrt{-\Delta}(\cos(\sqrt{-\Delta}\xi)\pm 1)}{\sin(\sqrt{-\Delta}\xi)}, \tag{4.79}$$

$$\begin{aligned} u_{2,7}^3(x,y,t) &= \\ &\frac{2k_1\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}{2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1} + a_0 - \frac{k_1\sqrt{-\Delta}\left(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1\right)}{2\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}, \end{aligned} \tag{4.80}$$

$$\begin{aligned} u_{2,8-9}^3(x,y,t) = &-\frac{k_1\Delta(A\sin(\sqrt{-\Delta}\xi)+B)}{A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\mp\sqrt{-(A^2-B^2)\Delta}} + a_0 \\ &-\frac{k_1(A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\mp\sqrt{-(A^2-B^2)\Delta})}{A\sin(\sqrt{-\Delta}\xi)+B}, \end{aligned} \tag{4.81}$$

$$\begin{aligned} u_{2,10-11}^3(x,y,t) = &-\frac{k_1\Delta(A\sin(\sqrt{-\Delta}\xi)+B)}{A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-(A^2-B^2)\Delta}} + a_0 \\ &-\frac{k_1(A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-(A^2-B^2)\Delta})}{A\sin(\sqrt{-\Delta}\xi)+B}, \end{aligned} \tag{4.82}$$

where A and B are two nonzero real constants and satisfy the condition that $A^2-B^2>0$,

$$\begin{aligned} u_{2,12}^3(x,y,t) = &\frac{k_1\Delta\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\sqrt{-\Delta}\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\Delta\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0 \\ &+\frac{k_1\left(\lambda\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\Delta\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}, \end{aligned} \tag{4.83}$$

$$\begin{aligned} u_{2,13}^3(x,y,t) = &\frac{k_1\Delta\left(\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\Delta\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\lambda\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} + a_0 \\ &+\frac{k_1\left(\Delta\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\sqrt{-\Delta}\lambda\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}, \end{aligned} \tag{4.84}$$

$$
\begin{aligned}
u^3_{2,14-15}(x,y,t) =& \frac{k_1\Delta\left(\lambda\cos(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)\right)}{\Delta\cos(\sqrt{-\Delta}\xi)+\lambda\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)}+a_0\\
&+\frac{k_1\left(\Delta\cos(\sqrt{-\Delta}\xi)+\lambda\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)\right)}{\lambda\cos(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(\sin(\sqrt{-\Delta}\xi)\pm 1)},
\end{aligned}
\tag{4.85}
$$

$$
\begin{aligned}
u^3_{2,16-17}(x,y,t) =& \frac{k_1\Delta\left(\lambda\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(-\cos(\sqrt{-\Delta}\xi)\pm 1)\right)}{\Delta\sin(\sqrt{-\Delta}\xi)+\lambda\sqrt{-\Delta}(-\cos(\sqrt{-\Delta}\xi)\pm 1)}+a_0\\
&+\frac{k_1\left(\Delta\sin(\sqrt{-\Delta}\xi)+\lambda\sqrt{-\Delta}(-\cos(\sqrt{-\Delta}\xi)\pm 1)\right)}{\lambda\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}(-\cos(\sqrt{-\Delta}\xi)\pm 1)},
\end{aligned}
\tag{4.86}
$$

$$
\begin{aligned}
u^3_{2,18}(x,y,t) =& \frac{k_1\Delta\left(\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1)-2\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)}{\lambda\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1)-2\Delta\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}+a_0\\
&+\frac{k_1\left(\lambda\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1)-2\Delta\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)}{\sqrt{-\Delta}(2\cos^2\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-1)-2\lambda\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}.
\end{aligned}
\tag{4.87}
$$

Family 3. When $\mu=0$ and $\lambda(v-1)\neq 0$, the hyperbolic function solutions of (1.2) are as follows:

$$
\begin{aligned}
u^3_{3,1}(x,y,t) =& \frac{k_1\Delta(\cosh(\lambda\xi)-\sinh(\lambda\xi)+c_1v-c_1)}{\lambda(\cosh(\lambda\xi)-\sinh(\lambda\xi)+c_1-c_1v)}+a_0\\
&+\frac{k_1\lambda(\cosh(\lambda\xi)-\sinh(\lambda\xi)+c_1-c_1v)}{\cosh(\lambda\xi)-\sinh(\lambda\xi)+c_1v-c_1},
\end{aligned}
\tag{4.88}
$$

$$
\begin{aligned}
&u^3_{3,2}(x,y,t) =\\
&-\frac{k_1\Delta(\cosh(\lambda\xi)+\sinh(\lambda\xi)+c_1)}{\lambda(\cosh(\lambda\xi)+\sinh(\lambda\xi)-c_1)}+a_0-\frac{k_1\lambda(\cosh(\lambda\xi)+\sinh(\lambda\xi)-c_1)}{\cosh(\lambda\xi)+\sinh(\lambda\xi)+c_1},
\end{aligned}
\tag{4.89}
$$

where c_1 is an arbitrary constant.

Family 4. When $\mu=\lambda=0$ and $v-1\neq 0$, the rational function solution of (1.2) is expressed as:

$$
u^3_{4,1}(x,y,t)=\frac{k_1\Delta(v\xi-\xi+c_2)}{-2v+2}+a_0+\frac{k_1(-2v+2)}{v\xi-\xi+c_2},
\tag{4.90}
$$

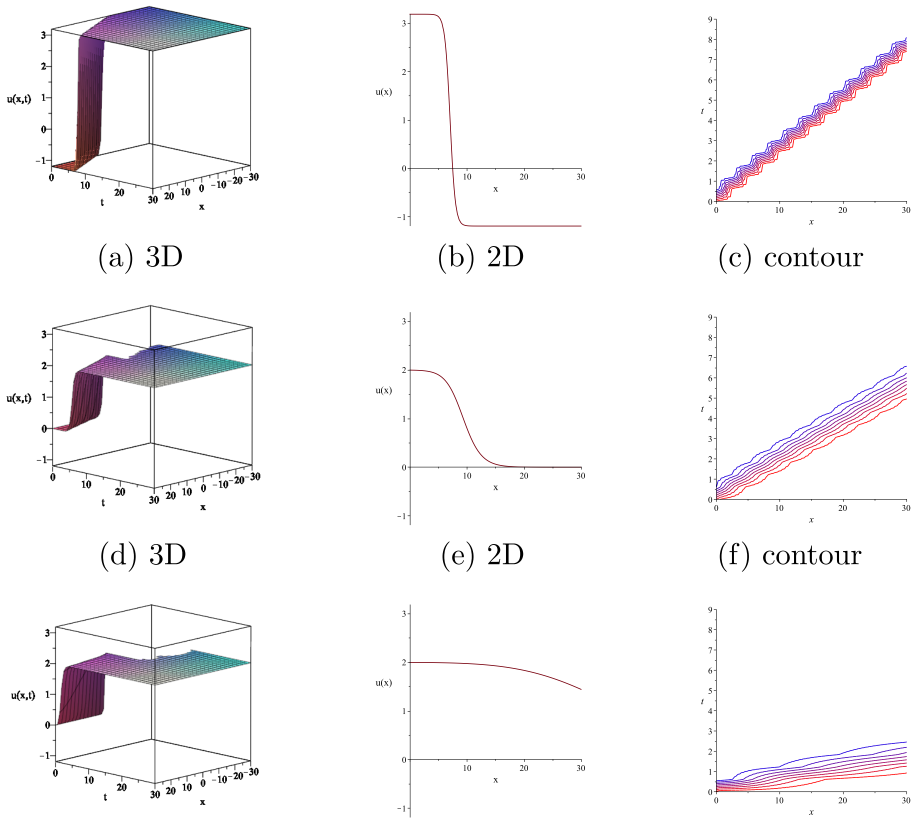
where c_2 is an arbitrary constant.

5. Graphical representations and discussions

In this section, interesting graphical representations of some selected exact solutions of the $(2+1)$-dimensional conformable space-time breaking soliton equation (1.2), obtained via the novel (G'/G^2)-expansion method, are depicted as 3D, 2D and contour graphs. On the following domains: $D_1=\{(x,y,t)\mid -30\leq x\leq 30,\ y=1 \text{ and } 0\leq t\leq 30\}$ for all 3D and contour graphs and $D_2=\{(x,y,t)\mid -30\leq x\leq 30,\ y=1 \text{ and } t=2\}$ for all 2D plots, the exact traveling wave solutions $u^3_{1,3}(x,y,t)$ in (4.66), $u^3_{2,3}(x,y,t)$ in (4.78), $u^3_{2,10}(x,y,t)$ in (4.82) and $u^3_{3,1}(x,y,t)$ in (4.88) are chosen to graphically portray on the domains for solution behaviors when values of the fractional-orders α, β are varied.

Particularly, $\alpha = \beta = 1$, $\alpha = \beta = 0.8$ and $\alpha = \beta = 0.4$ are inserted into (1.2) to investigate effects of the fractional-orders on the solution simulations.

In Figure 1, various graphs of the exact solution $u_{1,3}^3(x, y, t)$ in (4.66) are plotted on the mentioned domains using the parameter values $a_0 = 1$, $k_1 = k_2 = 1$, $\lambda = 1$, $\mu = 0.1$, $v = 0.5$. Particularly, Figure 1 (a)-(c), 1 (d)-(f) and 1 (g)-(i) demonstrate the 3D, 2D and contour plots for the exact solution (4.66) evaluated at $\alpha = \beta = 1$, $\alpha = \beta = 0.8$ and $\alpha = \beta = 0.4$, respectively. As observed from the 3D graphs of Figure 1, solution (4.66) is characterized as a solitary wave soliton of kink type.



(a) 3D (b) 2D (c) contour

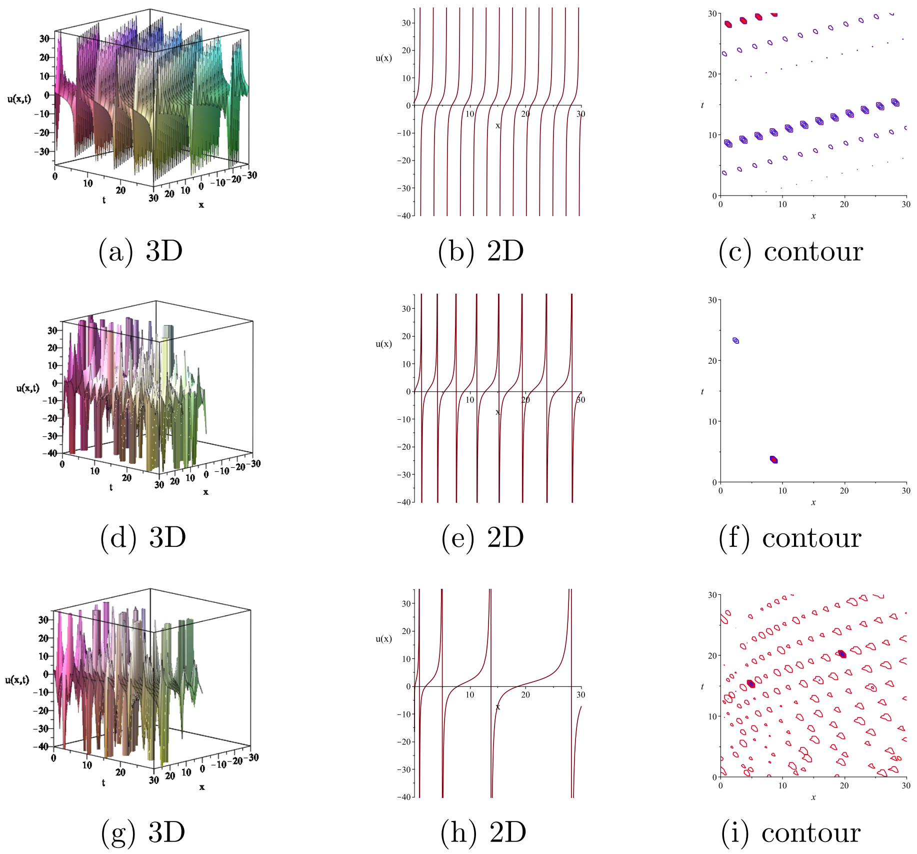
(d) 3D (e) 2D (f) contour

(g) 3D (h) 2D (i) contour

FIGURE 1. Solution plots for $u_{1,3}^3(x, y, t)$ in (4.66) obtained using the novel (G'/G^2)-expansion method: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

Associated graphs of $u_{2,3}^3(x, y, t)$ in (4.78) are drawn on the domain, as shown in Figure 2, by employing the parameter values $a_0 = 1$, $k_1 = k_2 = 1$, $\lambda = 0.5$, $\mu = 1$, $v = 1.5$. Especially, Figure 2 (a)-(c), 2 (d)-(f) and 2 (g)-(i) present the 3D, 2D and contour plots for the exact solution (4.78) evaluated at $\alpha = \beta = 1$, $\alpha = \beta = 0.8$ and $\alpha = \beta = 0.4$, respectively. By classifying a shape of the 3D graphs in Figure 2, solution (4.78) can be identified as a singularly periodic wave solution.

Figure 3 displays the associated graphs of $u_{2,10}^3(x, y, t)$ in (4.82), which are computed using $a_0 = 1$, $k_1 = k_2 = 1$, $\lambda = 0.5$, $\mu = 1$, $v = 1.5$, $A = 1$, $B = 0.5$ and plotted on the



(a) 3D (b) 2D (c) contour

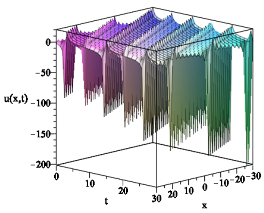
(d) 3D (e) 2D (f) contour

(g) 3D (h) 2D (i) contour

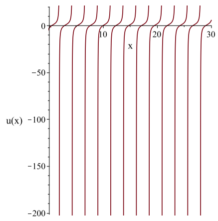
FIGURE 2. Solution plots for $u^3_{2,3}(x, y, t)$ in (4.78) obtained using the novel (G'/G^2)-expansion method: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

specified domains. In particular, Figure 3 (a)-(c), 3 (d)-(f) and 3 (g)-(i) describe the 3D, 2D and contour graphs for the exact solution (4.82) evaluated at $\alpha = \beta = 1$, $\alpha = \beta = 0.8$ and $\alpha = \beta = 0.4$, respectively. From the 3D graphs of Figure 3, the physical behavior of solution (4.82) is considered as a singularly periodic wave solution.

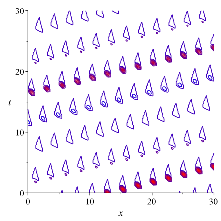
Various plots of $u^3_{3,1}(x, y, t)$ in (4.88) are depicted on the specified domain, as shown in Figure 4, by using the parameter values $a_0 = 1$, $k_1 = k_2 = 1$, $\lambda = 0.5$, $\mu = 0$, $v = 2$, $c_1 = 2$. Moreover, Figure 4 (a)-(c), 4 (d)-(f) and 4 (g)-(i) provide the 3D, 2D and contour graphs for the exact solution (4.88) calculated utilizing $\alpha = \beta = 1$, $\alpha = \beta = 0.8$ and $\alpha = \beta = 0.4$, respectively. By identifying a shape of the 3D graphs in Figure 4, solution (4.88) can be characterized as a singular kink wave solution.



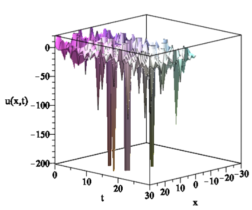
(a) 3D



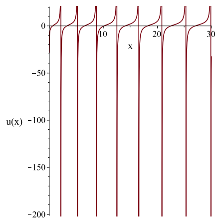
(b) 2D



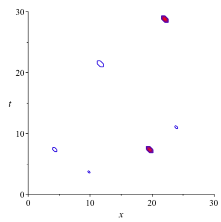
(c) contour



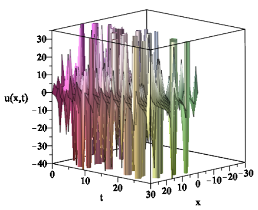
(d) 3D



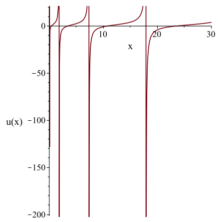
(e) 2D



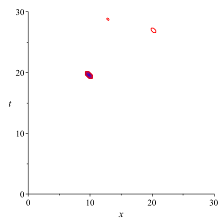
(f) contour



(g) 3D



(h) 2D

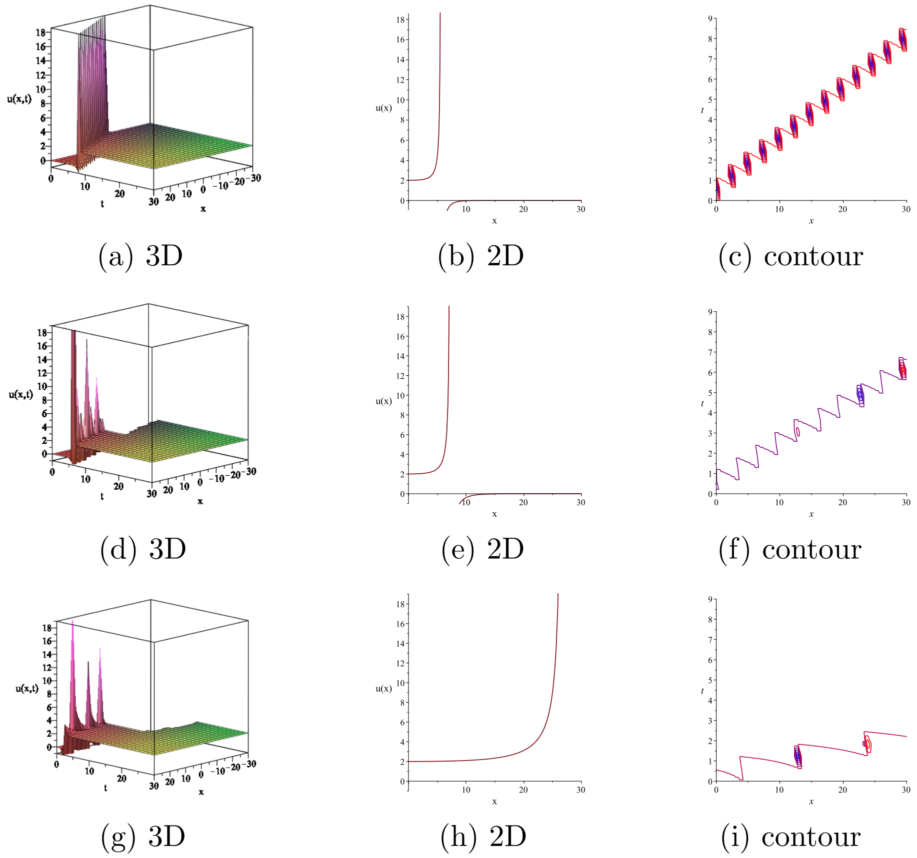


(i) contour

FIGURE 3. Solution plots for $u^3_{2,10}(x, y, t)$ in (4.82) obtained using the novel (G'/G^2)-expansion method: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

6. Conclusions

In this paper, the novel (G'/G^2)-expansion method, established for the first time, is developed from the novel (G'/G)-expansion method. The proposed technique differs from the original one by using the new auxiliary equation (3.6), which still satisfies the generalized Riccati equation through a certain transformation. The new method is applied to the $(2 + 1)$-dimensional conformable space-time breaking soliton equation (1.2) for constructing various solitary wave solutions. We successfully obtain abundant new exact traveling wave solutions, based on the plentiful solutions of the generalized Riccati equation as expressed in [56], such as hyperbolic function, trigonometric function and rational function solutions as reported in section 4. In addition, the variation of the fractional-orders α, β in equation (1.2) is studied in terms of the solution graphs including the 3D, 2D and contour plots as shown in section 5. The selected exact solutions are plotted to show their graphical behaviors and to disclose their distinct physical characteristics, for instance, a cross kink-wave solution, a singularly periodic wave solution and a singular kink wave solution. However, all of the exact solutions, obtained by the method, are verified by substituting them back into the relevant ODE (4.3) with the help of Maple. According to the reported results in this investigation, it has been noticed that the novel



(a) 3D (b) 2D (c) contour

(d) 3D (e) 2D (f) contour

(g) 3D (h) 2D (i) contour

FIGURE 4. Solution plots for $u^3_{3,1}(x, y, t)$ in (4.88) obtained using the novel (G'/G^2)-expansion method: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

(G'/G^2)-expansion method along with the aid of the Maple package program is a powerful, efficient and reliable mathematical tool which may be used to scrutinize a wide range of nonlinear PDEs including the conformable derivative cases because it provides several types of exact traveling wave solutions and generally gives much more explicit solutions than other existing methods.

Acknowledgements

The authors are grateful to anonymous referees for the valuable comments, which have significantly improved this article. In addition, the first author would like to acknowledge the partial support from the Graduate College, King Mongkut's University of Technology North Bangkok.

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