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An Invariant-Preserving Three-Level Linear Finite Difference Method for the Viscous Fornberg-Whitham Equation

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Abstract In this research, we design a family of finite difference θ-schemes to approximate solutions of the viscous Fornberg-Whithem equation which is a shallow-water wave model describing waves breaking. The model admits two invariants: momentum and energy. Although the model is nonlinear, our present schemes are linear and use information from three time steps. For $\theta = 1/3$, it can be shown that the conservation of invariants is still maintained. The methods have second order of accuracy in time and space. Moreover, we can prove that the convergence of the numerical solutions is uniform. Finally, numerical examples are used to demonstrate effectiveness and also to prove our theoretical results.

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1. Introduction

In applied sciences and engineering, partial differential equations (PDEs) are commonly employed to simulate a variety of physical phenomena. The solutions to these PDEs play a crucial role in determining a physical behavior of various physical phenomena. Although accurate analytical solutions to these PDEs are desirable, exact analytical solutions are only achievable for simple problems with simple boundary conditions owing to mathematical complexities.

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In mathematical physics, one of the well-known nonlinear equations is the Burgers–Poisson (BP)-system:

$$u_t + uu_x = \phi_x, \tag{1.1}$$

$$\phi_{xx} = \phi + u, \tag{1.2}$$

which is a shallow-water model where u and ϕ depend on $(x, t) \in (0, \infty) \times \mathbb{R}$. The subscripts t and x are partial derivatives with respect to the temporal and spatial variables respectively. The variable u represents the fluid velocity along the x-axis.

The numerical approximation of the systems (1.1) and (1.2) with zero boundary is what we are interested in. Not only will this help us to better understand shallow-water waves, but it will also better explain many other natural phenomena with similar models, such as the Two-Species-Euler-Poisson system [1]. We also add the viscous term to the Burgers equation in order to cover the one-dimensional version of the Navier-Stokes-Poisson system as mentioned in [2]. The traveling and periodic solutions are found in [1], where Schmeiser and Fellner investigate the analytic properties of (1.1) and (1.2). The smooth solution's local existence and global existence for the weak entropy solution are also confirmed. In [3], the classification of group invariant solutions for systems (1.1) and (1.2) is determined by using the classical Lie method. In [4], another global existence for the entropy solution is investigated under particular regularity conditions on the initial data.

One may alternatively rewrite the system (1.1) and (1.2) as a single-equation form to explore it in a different way. It is called the Whitham equation, which is a non-local model for non-linear dispersive waves. The Whitham equation has previously been recognized to have wave breaking (bounded solutions with unbounded derivatives) [5]. It becomes the Fornberg–Whitham equation:

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

which was first proposed for studying the qualitative behaviour of wave breaking [5]. It also has the form:

$$u_t - u_{xxt} - u_x + \frac{3}{2}uu_x = \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx}, \tag{1.4}$$

which was derived by Fornberg and Whitham in [6] as a model for shallow-water waves describing wave breaking.

In [7], both finite difference and variational iteration approaches are used to investigate the numerical solution of the Burgers-Poisson Equation (1.3). However the invariant-preserving property and convergence analysis are not explored. The homotopy perturbation approach and the Adomian decomposition method are used to find both precise and approximate explicit solutions of the fractional-order Burgers-Poisson problem in [8] and [9]. The efficiency of the presented methods is demonstrated by comparing numerical and exact solutions, although no convergence and stability analysis is provided. In [2] and [10], the Local Discontinuous Galerkin (LDG) method is applied to the inviscous problem and the viscous problem, respectively. In [11], Darayon et al. studied a new finite difference scheme for a viscous Burgers-Poisson system with periodic boundary conditions. The scheme belongs to a family of three-level linearized finite difference methods, which are proved to preserve both momentum and energy in the discrete sense, and it is also proved that the method converges uniformly and has a second order of accuracy in space. In [12], Ploymaklam studied a numerical approximation of the solution of the reduced

Burgers-Poisson equation using the local discontinuous Galerkin (LDG) method. In [13], Ploymaklam and Chaturantabut studied the proper orthogonal decomposition (POD) and the discrete empirical interpolation method (DEIM) to approximate the solution of the Burgers-Poisson equation. Many works have been published that find travelling wave solutions to equations (1.3)-(1.4) with terms similar to other wave models, such as the Camassa-Holm equation [14], [15], Benjamin-Ono equation [16], [17], KdV equation, Rosenua equation [18], [19] and the RLW equation [20], [21]. The equations (1.3) and (1.4) can be written in the form

$$u_t + \alpha u u_x = \beta(1-\partial_x^2)^{-1}u_x, \tag{1.5}$$

where α, β are constants. In recent years, many authors have shown interest in studying the approximate solution of the Fornberg-Whitham equation (1.5). For instance, the homotopy analysis method (HAM) was applied to obtain an approximate analytical solution of the Fornberg-Whitham equation, and the result was compared with the exact solution by Abidi and Omrani [22]. With the method, numerical results could be obtained by using a few iterations. Comparison is made among the HAM results, the Adomian's decomposition method (ADM), and the homotopy perturbation method (HPM). Similarly, in [23], Biazar studied to propose an analytical approach based on the homotopy perturbation method (HPM) for solving the initial value problems associated with the Fornberg-Whitham type equations. However, convergence and stability analysis does not appear in this work. In [24], Sontakke and Shaikh studied a new iterative approach to get numerical solutions of time fractional Fornberg-Whitham and modified Fornberg-Whitham equations in wave breaking involving the Caputo derivative. The obtained solutions were compared with the exact solution. In [25], Gao et al. studied the $L^2(\mathbb{R})$ conservation law of solutions for the nonlinear Fornberg–Whitham equation. Making use of the Kruzkov's device of doubling the space variables, the stability of the solutions in $L^1(\mathbb{R})$ space was established under certain assumptions on the initial value.

In this research, we study the viscous Fornberg–Whitham equation

$$(1-\partial_x^2)(u_t + \alpha u u_x - \gamma u_{xx}) = \beta u_x, \quad (x,t) \in (a,b)\times(0,T), \tag{1.6}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in (a,b), \tag{1.7}$$

and boundary conditions

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t > 0, \tag{1.8}$$

$$u_x(a,t) = 0, \quad u_x(b,t) = 0, \quad t > 0, \tag{1.9}$$

$$u_{xx}(a,t) = 0, \quad u_{xx}(b,t) = 0, \quad t > 0, \tag{1.10}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma \geq 0$.

The nonlinear term uu_x poses a challenge for the design of the scheme. In an early attempt to approximate the term uu_x, Guo and Sanz-Serna [26] used the sum of two finite differences to approximate the term in the KdV equation. In [27], Wongsaijai and Poochinapan used the name Rosenua-KdV-RLW to refer to all three equations and applied a convex average value of two implicit finite differences to approximate the nonlinear term. This results in a collection of second-order θ-schemes proven to preserve invariant when $\theta = 1/3$ and modify their concept to fit the present problem. In recent works, Sun et al., applied second-order [28] and fourth-order [29] operators to the viscous Burgers' equation with zero boundary and get point-wise convergence analysis. We adopt their approach in the analysis.

Invariant-preserving is one of these models' most prevalent properties: certain quantities derived from the solution remain constant or do not change over time. Although in [30], Young provides other invariants for the model, we only focus on momentum and energy obtained from the solution of (1.6).

The rest of this paper is organized as follows. In Section 2, we go through some of the solution's analytic features that are critical to the stability and error evaluations. In Section 3, we describe the finite difference method (FDM) framework and propose a collection of θ-schemes to approximate the solution of the viscous Fornberg-Whitham equation, and we also demonstrate that the proposed schemes preserve invariant properties. The unique solvability is also shown in this section. The error analysis is presented in Section 4 to show that the approach has second-order accuracy in time and space. In Section 5, we validate the theoretical results by testing the proposed scheme on several examples of the inviscous and viscous Fornberg-Whitham equations. In Section 6, we conclude the paper.

2. Analytic Property

In this section, we show that the solution of (1.6) with conditions (1.7)-(1.10) preserves the momentum and energy.

Theorem 2.1 (Momentum preserving). *Let u be a solution of (1.6)-(1.10). Define $w = (1-\partial_x^2)^{-1}u$ and*

$$Q(t) = \int_a^b u(x,t)dx.$$

If $u_0, w \in L^1(a,b)$ and $w(a,t) = w(b,t) = 0$, then

$$Q(t) = Q(0),$$

for all $t > 0$.

Proof. From (1.6), we have $u_t = \beta(1-\partial_x^2)^{-1}u_x - \alpha u u_x + \gamma u_{xx}$. Therefore,

$$\begin{aligned} \frac{d}{dt}Q(t) &= \beta \int_a^b (1-\partial_x^2)^{-1}u_x \, dx - \alpha \int_a^b u u_x \, dx + \gamma \int_a^b u_{xx} \, dx \\ &= \beta \int_a^b w_x \, dx - \frac{\alpha}{2} \int_a^b (u^2)_x \, dx + \gamma \int_a^b (u_x)_x \, dx \\ &= \beta \left[w\right]_b^a - \frac{\alpha}{2} \left[u^2\right]_b^a + \gamma \left[u_x\right]_b^a \; = 0 \end{aligned}$$

because of the boundary conditions (1.8)-(1.10) and the hypothesis of the theorem.

Therefore, the function $Q(t)$ does not change over time. That is,

$$Q(t) = Q(0),$$

for all $t > 0$. ■

Theorem 2.2 (Energy preserving). *Let u be a solution of (1.6)-(1.10). Define*

$$E(t) = \int_a^b u^2(x,t)dx.$$

If $u_0, w \in L^2(a,b)$ *and* $w(a,t) = w(b,t) = w_x(a,t) = w_x(b,t) = 0$*, then*

$$E(t) \leq E(0),$$

for all $t > 0$.

Proof. Differentiating E with respect to t, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E(t) &= \beta \int_a^b u w_x \, dx - \alpha \int_a^b u^2 u_x \, dx + \gamma \int_a^b u u_{xx} \, dx \\
&= \beta \int_a^b (w - w_{xx}) w_x \, dx - \frac{\alpha}{3} \int_a^b (u^3)_x \, dx - \gamma \int_a^b u_x^2 \, dx \\
&= \frac{\beta}{2} \int_a^b w_x^2 \, dx - \frac{\beta}{2} \int_a^b (w_x^2)_x \, dx - \frac{\alpha}{3} \int_a^b (u^3)_x \, dx - \gamma \int_a^b u_x^2 \, dx \\
&= \frac{\beta}{2} \left[w^2\right]_b^a - \frac{\beta}{2} \left[w_x^2\right]_b^a - \frac{\alpha}{3} \left[u^3\right]_b^a - \gamma \int_a^b u_x^2 \, dx \\
&= -\gamma \int_a^b u_x^2 \, dx \ \leq 0.
\end{aligned}$$

Therefore, the function $E(t)$ does not increase over time. That is,

$$E(t) \leq E(0), \quad \text{for all } t > 0.$$

If $\gamma = 0$, then the proof above shows that $E'(0) = 0$. This gives $E(t) = E(0)$. ∎

3. Finite Difference Method

In this section, we first introduce basic settings for the FDM framework and propose a collection of three-level linearized schemes to approximate the solution of the viscous Fornberg-Whithem equation.

3.1. Discretization

We discretize the spatial domain $[a, b]$ into the partition $x_i = a + ih$, $i = 0, \dots, M$ where $h = (b - a)/M$ is the spatial step size. As for the temporal discretization, we define $t^n = n\tau$ for $n = 0, \dots, N$ where $\tau = T/N$ is the temporal step size. Let u_i^n be an approximation of $u(x_i, t^n)$ and

$$Z_h := \{\mathbf{u} = [u_i]_{i=-1}^{M+1} \mid u_{-1} = u_0 = u_1 = u_2 = u_{M-2} = u_{M-1} = u_M = u_{M+1} = 0\}.$$

For any $\mathbf{u}, \mathbf{v} \in Z_h$, we define an inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle_h = h \sum_{i=1}^{M-1} u_i v_i,$$

which allows us to define the discrete L_2 norm

$$\|\mathbf{u}\|_h^2 = \langle \mathbf{u}, \mathbf{u} \rangle_h \,. \tag{3.1}$$

We also define the discrete uniform norm

$$\|\mathbf{u}\|_{h,\infty} = \max_{1 \leq i \leq M-1} |u_i|, \tag{3.2}$$

for later use in the analysis.

We define finite difference methods for approximating some partial derivatives of u at (x_i, t^n) as follows:

$$\mathcal{D}_\tau^+ u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau}, \quad \mathcal{D}_\tau^0 u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$
$$u_i^{n+1/2} = \frac{u_i^{n+1} + u_i^n}{2}, \quad \overline{u}_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2},$$
$$D_h^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \quad D_h^0 u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}, \quad D_h^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h}.$$

Note that we can apply these operators to a vector in Z_h by acting on each element of the vector.

Using the boundary conditions in the definition of Z_h and summation by parts, one can prove the following results.

Lemma 3.1. *Let $\mathbf{u}, \mathbf{v} \in Z_h$. The following relations hold.*

$$\begin{aligned} \left\langle D_h^+ \mathbf{u}, \mathbf{v} \right\rangle_h &= -\left\langle \mathbf{u}, D_h^- \mathbf{v} \right\rangle_h \\ \left\langle D_h^0 \mathbf{u}, \mathbf{v} \right\rangle_h &= -\left\langle \mathbf{u}, D_h^0 \mathbf{v} \right\rangle_h \\ \left\langle D_h^+ D_h^- \mathbf{u}, \mathbf{v} \right\rangle_h &= -\left\langle D_h^+ \mathbf{u}, D_h^+ \mathbf{v} \right\rangle_h \\ \| D_h^0 \mathbf{u} \|_{h,\infty} &\leq \| D_h^+ \mathbf{u} \|_{h,\infty} \end{aligned}$$

Lemma 3.2. *Let $\mathbf{u} \in Z_h$, we have*

$$\begin{aligned} \left\langle D_h^+ D_h^- D_h^+ D_h^- \mathbf{u}, \mathbf{u} \right\rangle_h &= \| D_h^+ D_h^- \mathbf{u} \|_h^2 \\ \left\langle D_h^0 \mathbf{u}, \mathbf{u} \right\rangle_h &= 0. \end{aligned}$$

3.2. Formulation of the Scheme

First, we define

$$\Psi_\theta(\mathbf{u}, \mathbf{v})_i := 2\theta u_i D_h^0 v_i + (1-\theta) D_h^0 (uv)_i\,, \tag{3.3}$$

for any $\mathbf{u}, \mathbf{v} \in Z_h$. Using the finite difference framework above, we propose a family of explicit three-level finite difference schemes for solving the viscous Fornberg-Whithem equation (1.6) with conditions (1.7)-(1.10) as follows: for $\theta \in [0, 1]$, find $\mathbf{u}^{n+1}$ satisfying

$$(1 - D_h^+ D_h^-)\left(\mathcal{D}_\tau^+ u_i^0 + \frac{\alpha}{2}(\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i) - \gamma D_h^+ D_h^- u_i^{1/2}\right) = \beta D_h^0 u_i^{1/2} \tag{3.4}$$

$$u_i^0 = u_0(x_i), \tag{3.5}$$

for $i = 1, \ldots, M-1$, and

$$(1 - D_h^+ D_h^-)\left(\mathcal{D}_\tau^0 u_i^n + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i - \gamma D_h^+ D_h^- \overline{u}_i^n\right) = \beta D_h^0 \overline{u}_i^n, \tag{3.6}$$

for $i = 1, \ldots, M-1$ and $n = 1, \ldots, N-1$.

For the implementation of the three-level scheme, the first two initial steps of the solutions are required. Thus, we use (3.4)-(3.5) to compute $\mathbf{u}^1$. Then, we use (3.6) to compute $\mathbf{u}^{n+1}$, for $n = 1, 2, \ldots, N-1$.

Letting $(1 - D_h^+ D_h^-) w_i^n = \overline{u}_i^n$, with $i = 1, \ldots, M-1$ we can write a matrix-vector form

$$X_h \mathbf{w}^n = \overline{\mathbf{u}}^n, \quad \text{for} \quad i = 1, \ldots, M-1,$$

where X_h is a circulant tridiagonal matrix with $(-1/h^2, 2/h^2+1, -1/h^2)$ on the diagonal positions. It can be shown that X_h is strictly diagonally dominant, thus its inverse exists. Since X_h is symmetric, its inverse is also symmetric. This allows us to write the system (3.4), $i = 1, \ldots, M-1$, as a matrix-vector form:

$$\begin{aligned}
&\mathbf{u}^1 + \frac{\tau\theta\alpha}{4h} B^0\mathbf{u}^1 + \frac{\tau(1-\theta)\alpha}{8h} C^0\mathbf{u}^1 - \frac{1}{h^2} D^0\mathbf{u}^1 - \frac{\tau\theta\alpha}{4h^3} E^0\mathbf{u}^1 - \frac{\tau(1-\theta)\alpha}{8h^3} F^0\mathbf{u}^1 \\
&\quad - \frac{\tau\beta}{4h} G^0\mathbf{u}^1 - \frac{\gamma\tau}{2h^2} H^0\mathbf{u}^1 + \frac{\gamma\tau}{2h^4} J^0\mathbf{u}^1 \\
&= \mathbf{u}^0 - \frac{\tau\theta\alpha}{4h} B^0\mathbf{u}^0 - \frac{\tau(1-\theta)\alpha}{8h} C^0\mathbf{u}^0 - \frac{1}{h^2} D^0\mathbf{u}^0 + \frac{\tau\theta\alpha}{4h^3} E^0\mathbf{u}^0 + \frac{\tau(1-\theta)\alpha}{8h^3} F^0\mathbf{u}^0 \\
&\quad + \frac{\tau\beta}{4h} G^0\mathbf{u}^0 + \frac{\gamma\tau}{2h^2} H^0\mathbf{u}^0 - \frac{\gamma\tau}{2h^4} J^0\mathbf{u}^0. \qquad (3.7)
\end{aligned}$$

For $n = 1, 2, \ldots, N-1$, the equation (3.6), $i = 1, \ldots, M-1$, can be written as

$$\begin{aligned}
&\mathbf{u}^{n+1} + \frac{\tau\theta\alpha}{2h} B^n\mathbf{u}^{n+1} + \frac{\tau(1-\theta)\alpha}{4h} C^n\mathbf{u}^{n+1} - \frac{1}{h^2} D^n\mathbf{u}^{n+1} - \frac{\tau\theta\alpha}{2h^3} E^n\mathbf{u}^{n+1} \\
&\quad - \frac{\tau(1-\theta)\alpha}{4h^3} F^n\mathbf{u}^{n+1} - \frac{\tau\beta}{2h} G^n\mathbf{u}^{n+1} - \frac{\gamma\tau}{h^2} H^n\mathbf{u}^{n+1} + \frac{\gamma\tau}{h^4} J^n\mathbf{u}^{n+1} \\
&= \mathbf{u}^{n-1} - \frac{\tau\theta\alpha}{2h} B^n\mathbf{u}^{n-1} - \frac{\tau(1-\theta)\alpha}{4h} C^n\mathbf{u}^{n-1} - \frac{1}{h^2} D^n\mathbf{u}^{n-1} + \frac{\tau\theta\alpha}{2h^3} E^n\mathbf{u}^{n-1} \\
&\quad + \frac{\tau(1-\theta)\alpha}{4h^3} F^n\mathbf{u}^{n-1} + \frac{\tau\beta}{2h} G^n\mathbf{u}^{n-1} + \frac{\gamma\tau}{h^2} H^n\mathbf{u}^{n-1} - \frac{\gamma\tau}{h^4} J^n\mathbf{u}^{n-1}. \qquad (3.8)
\end{aligned}$$

Here, the coefficients B^n, C^n, D^n, G^n and H^n are circulant tridiagonal matrices whose nonzero entries on the i^{th} row are given by $(-u_{i+1}^n, 0, u_i^n), (-u_i^n, 0, u_{i+1}^n), (1, -2, 1), (-1, 0, 1)$ and $(1, -2, 1)$, respectively. The coefficients E^n, F^n and J^n are circular pentadiagonal matrices whose nonzero entries on the i^{th} row are given by $(-u_{i-1}^n, 2u_i^n, -u_{i+1}^n + u_{i-1}^n, -2u_i^n, u_{i+1}^n)$, $(-u_{i-2}^n, 2u_{i-1}^n, 0, -2u_{i+1}^n, u_{i+2}^n)$ and $(1, -4, 6, -4, 1)$, respectively.

The following result will be used in the stability analysis.

Lemma 3.3. *If $\overline{\mathbf{u}}, \mathbf{w} \in Z_h$, then $\langle X_h^{-1}(D_h^0\overline{\mathbf{u}}), \overline{\mathbf{u}}\rangle_h = 0$.*

Proof. Since X_h^{-1} is symmetric, we get

$$\begin{aligned}
\langle X_h^{-1}(D_h^0\overline{\mathbf{u}}), \overline{\mathbf{u}}\rangle_h &= \langle D_h^0\overline{\mathbf{u}}, X_h^{-1}\overline{\mathbf{u}}\rangle_h \\
&= \langle D_h^0(\mathbf{w} - D_h^+D_h^-\mathbf{w}), \mathbf{w}\rangle_h \\
&= \langle D_h^0\mathbf{w}, \mathbf{w}\rangle_h - \langle D_h^0D_h^+D_h^-\mathbf{w}, \mathbf{w}\rangle_h = 0
\end{aligned}$$

as needed. ■

3.3. Stability Analysis

In this section, we show that the numerical solution of (1.6) obtained from the proposed schemes (3.4)-(3.6) preserves the invariants in the discrete sense.

Theorem 3.4. *Define*

$$Q_h^n = h\sum_{i=1}^{M-1} u_i^{n+1/2} + \frac{\theta\tau\alpha}{2} h \sum_{i=1}^{M-1} u_i^n D_h^0 u_i^{n+1}. \qquad (3.9)$$

If $\mathbf{u}^n$ *is a solution of (3.4)–(3.6), then*

$$Q_h^n = Q_h^{n-1} = \ldots = Q_h^0 = h\sum_{i=1}^{M-1} u_i^0 + \frac{\theta\tau\alpha}{4} h \sum_{i=1}^{M-1} u_i^0 D_h^0 u_i^1, \tag{3.10}$$

for any $\theta \in [0,1]$.

Proof. We rewrite (3.6) to get

$$\begin{aligned} &\mathcal{D}_\tau^0 u_i^n + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i - D_h^+ D_h^- \mathcal{D}_\tau^0 u_i^n - \frac{\alpha}{2} D_h^+ D_h^- (\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i) \\ &= \beta D_h^0 \overline{u}_i^n + \gamma D_h^+ D_h^- \overline{u}_i^n - \gamma D_h^+ D_h^- D_h^+ D_h^- \overline{u}_i^n. \end{aligned} \tag{3.11}$$

Multiplying (3.11) by τh and summing on $i = 1, \ldots, M-1$ to arrive at

$$\begin{aligned} &\tau h \sum_{i=1}^{M-1} \mathcal{D}_\tau^0 u_i^n + \frac{\tau h\alpha}{2}\sum_{i=1}^{M-1} \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i - \tau h \sum_{i=1}^{M-1} D_h^+ D_h^- \mathcal{D}_\tau^0 u_i^n \\ &\quad - \frac{\tau h \alpha}{2} \sum_{i=1}^{M-1} D_h^+ D_h^- (\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i) \\ &= \tau h\beta \sum_{i=1}^{M-1} D_h^0 \overline{u}_i^n + \tau\gamma h\beta \sum_{i=1}^{M-1} D_h^+ D_h^- \overline{u}_i^n - \tau\gamma h\beta \sum_{i=1}^{M-1} D_h^+ D_h^- D_h^+ D_h^- \overline{u}_i^n. \end{aligned}$$

From (3.3) we obtain

$$\begin{aligned} &\tau h \sum_{i=1}^{M-1} \mathcal{D}_\tau^0 u_i^n + \theta\alpha\tau h \sum_{i=1}^{M-1} u_i^n D_h^0 \overline{u}_i^n + \frac{(1-\theta)\alpha\tau h}{2} \sum_{i=1}^{M-1} D_h^0 \left(u^n \overline{u}^n\right)_i \\ &\quad - \tau h \sum_{i=1}^{M-1} D_h^+ D_h^- \mathcal{D}_\tau^0 u_i^n - \theta\alpha\tau h \sum_{i=1}^{M-1} D_h^+ D_h^- (u_i^n D_h^0 \overline{u}_i^n) \\ &\quad + \frac{(1-\theta)\alpha\tau h}{2} \sum_{i=1}^{M-1} D_h^+ D_h^- (D_h^0 \left(u^n \overline{u}^n\right)_i) \\ &= \tau h\beta \sum_{i=1}^{M-1} D_h^0 \overline{u}_i^n + \tau\gamma h\beta \sum_{i=1}^{M-1} D_h^+ D_h^- \overline{u}_i^n - \tau\gamma h\beta \sum_{i=1}^{M-1} D_h^+ D_h^- D_h^+ D_h^- \overline{u}_i^n. \end{aligned}$$

Except for the first two, all terms are zero as a result of summation by parts and boundary condition on Z_h. We obtain

$$\frac{h}{2}\sum_{i=1}^{M-1} \left(u_i^{n+1} - u_i^{n-1}\right) + \theta\alpha\tau h \sum_{i=1}^{M-1} u_i^n D_h^0 \overline{u}_i^n = 0,$$

and thus,

$$\frac{h}{2}\sum_{i=1}^{M-1} \left(u_i^{n+1} - u_i^{n-1}\right) + \frac{\theta\alpha\tau}{4}\sum_{i=1}^{M-1} \left(u_i^n u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1} + u_i^n u_{i+1}^{n-1} - u_i^n u_{i-1}^{n-1}\right) = 0.$$

After rewriting, we get

$$\frac{h}{2}\sum_{i=1}^{M-1} \left(u_i^{n+1} - u_i^{n-1}\right) + \frac{\theta\alpha\tau}{4}\sum_{i=1}^{M-1} \left(u_i^n u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1} + u_{i-1}^n u_i^{n-1} - u_{i+1}^n u_i^{n-1}\right) = 0,$$

and hence,

$$\frac{1}{2}h\sum_{i=1}^{M-1}\left(u_i^{n+1}-u_i^{n-1}\right)+\frac{\theta\alpha\tau}{2}h\sum_{i=1}^{M-1}\left(u_i^n D_h^0 u_i^{n+1}-u_i^{n-1}D_h^0 u_i^n\right)=0.$$

This leads to the conclusion that $Q_h^n - Q_h^{n-1} = 0$ for $n = 1, \ldots, N-1$. Similarly, multiplying (3.4) by τh and summing over i to obtain

$$h\sum_{i=1}^{M-1}u_i^1 = h\sum_{i=1}^{M-1}u_i^0 - \frac{\theta\alpha\tau}{2}h\sum_{i=1}^{M-1}u_i^0 D_h^0 u_i^1.$$

Substitute this into Q_h^0 to find that

$$Q_h^0 = h\sum_{i=1}^{M-1}u_i^0 + \frac{\theta\alpha\tau}{4}h\sum_{i=1}^{M-1}u_i^0 D_h^0 u_i^1.$$

This completes the proof. ■

Theorem 3.5. *Let* $\mathbf{u}^n$ *be a solution of (3.4)-(3.6), define*

$$E_h^n = \frac{1}{2}\|\mathbf{u}^{n+1}\|_h^2 + \frac{1}{2}\|\mathbf{u}^n\|_h^2. \tag{3.12}$$

If $\theta = \frac{1}{3}$, *then*

$$E_h^n + 2\tau\gamma\sum_{k=1}^{n}\|D_h^+\overline{\mathbf{u}}^k\|_h^2 + \tau\gamma\|D_h^+\mathbf{u}^{1/2}\|_h^2 = \|\mathbf{u}^0\|_h^2, \tag{3.13}$$

for any $n > 0$.

Proof. We first consider,

$$\begin{aligned}
&h\sum_{i=1}^{M-1}\left[u_i^n\left(D_h^0\overline{u}_i^n\right)\overline{u}_i^n + D_h^0\left(u_i^n\overline{u}_i^n\right)\overline{u}_i^n\right]\\
&= h\sum_{i=1}^{M-1}\left[u_i^n\overline{u}_{i+1}^n\overline{u}_i^n - u_i^n\overline{u}_{i-1}^n\overline{u}_i^n + u_{i+1}^n\overline{u}_{i+1}^n\overline{u}_i^n - u_{i-1}^n\overline{u}_{i-1}^n\overline{u}_i^n\right]\\
&= h\sum_{i=1}^{M-1}\left[u_i^n\overline{u}_{i+1}^n\overline{u}_i^n - u_{i-1}^n\overline{u}_{i-1}^n\overline{u}_i^n\right] + h\sum_{i=1}^{M-1}\left[-u_i^n\overline{u}_{i-1}^n\overline{u}_i^n + u_{i+1}^n\overline{u}_{i+1}^n\overline{u}_i^n\right]\\
&= 0.
\end{aligned} \tag{3.14}$$

Taking the inner product of (3.6), where $i = 1, \ldots, M-1$, with $2\overline{\mathbf{u}}^n$, we rewrite it as follows:

$$\begin{aligned}
&\left\langle \mathcal{D}_\tau^0\mathbf{u}^n, 2\overline{\mathbf{u}}^n\right\rangle_h + \left\langle \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), 2\overline{\mathbf{u}}^n\right\rangle_h - \left\langle \gamma D_h^+ D_h^-\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\right\rangle_h\\
&= \left\langle \beta X_h^{-1}D_h^0\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\right\rangle_h.
\end{aligned} \tag{3.15}$$

Applying Lemma 3.1, we have

$$\begin{aligned}
&2\left\langle \mathcal{D}_\tau^0\mathbf{u}^n, \overline{\mathbf{u}}^n\right\rangle_h\\
&= 2\beta\left\langle X_h^{-1}D_h^0\overline{\mathbf{u}}^n, \overline{\mathbf{u}}^n\right\rangle_h - \alpha\left\langle \Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), \overline{\mathbf{u}}^n\right\rangle_h - 2\gamma\left\langle D_h^+\overline{\mathbf{u}}^n, D_h^+\overline{\mathbf{u}}^n\right\rangle_h.
\end{aligned} \tag{3.16}$$

Using Lemma 3.3, we get

$$2\left\langle \mathcal{D}_\tau^0 \mathbf{u}^n, \overline{\mathbf{u}}^n \right\rangle_h = -\alpha \left\langle \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n), \overline{\mathbf{u}}^n \right\rangle_h - 2\gamma \left\langle D_h^+ \overline{\mathbf{u}}^n, D_h^+ \overline{\mathbf{u}}^n \right\rangle_h. \tag{3.17}$$

Using Lemma 3.1 and equation (3.3), we obtain

$$\begin{aligned} 2h \sum_{i=1}^{M-1} \left(\mathcal{D}_\tau^0 u_i^n\right) \overline{u}_i^n = &-2\theta\alpha h \sum_{i=1}^{M-1} u_i^n \left(D_h^0 \overline{u}_i^n\right) \overline{u}_i^n - \alpha(1-\theta) h \sum_{i=1}^{M-1} D_h^0 \left(u^n \overline{u}^n\right)_i \overline{u}_i^n \\ &- 2\gamma h \sum_{i=1}^{M-1} \left(D_h^+ \overline{u}_i^n\right)^2. \end{aligned} \tag{3.18}$$

Using (3.14) and multiplying τ, we get

$$\begin{aligned} &2\tau h \sum_{i=1}^{M-1} \left(\mathcal{D}_\tau^0 u_i^n\right) \overline{u}_i^n \\ &= \tau\alpha(1-3\theta) h \sum_{i=1}^{M-1} u_i^n \left(D_h^0 \overline{u}_i^n\right) \overline{u}_i^n - 2\tau\gamma h \sum_{i=1}^{M-1} \left(D_h^+ \overline{u}_i^n\right)^2. \end{aligned} \tag{3.19}$$

Taking $\theta = \frac{1}{3}$ leads to

$$\frac{1}{2} h \sum_{i=1}^{M-1} \left[\left(u_i^{n+1}\right)^2 - \left(u_i^{n-1}\right)^2\right] + 2\tau\gamma h \sum_{i=1}^{M-1} \left(D_h^+ \overline{u}_i^n\right)^2 = 0. \tag{3.20}$$

This shows that

$$E_h^n - E_h^{n-1} + 2\tau\gamma h \sum_{i=1}^{M-1} \left(D_h^+ \overline{u}_i^n\right)^2 = 0, \tag{3.21}$$

for $n = 1, \ldots, N-1$.

Similarly, one can show from (3.4)-(3.5) that

$$\|\mathbf{u}^1\|_h^2 - \|\mathbf{u}^0\|_h^2 + 2\tau\gamma h \sum_{i=1}^{M-1} \left(D_h^+ u_i^{1/2}\right)^2 = 0. \tag{3.22}$$

Substitute this into E_h^0 to find that

$$E_h^0 = \|\mathbf{u}^0\|_h^2 - \tau\gamma \|D_h^+ \mathbf{u}^{1/2}\|^2,$$

as needed. ■

3.4. The Unique Solvability

We will show that the numerical solution exists and is unique. We begin by proving the boundedness theorem. The following result is needed.

Lemma 3.6 (Discrete Sobolev's inequality [31])**.** *If $\mathbf{u}^n \in Z_h$, then there exists a constant κ, depending only on L, such that*

$$\|\mathbf{u}^n\|_{h,\infty} \le \kappa \left(\|\mathbf{u}^n\|_h + \|D_h^+ \mathbf{u}^n\|\right).$$

Theorem 3.7. *Let $\mathbf{u}^n$ be a solution of the difference scheme (3.4)-(3.6). We have $\|D_h^+ \mathbf{u}^n\|_h$ and $\|\mathbf{u}^n\|_h$ are bounded. This implies $\|\mathbf{u}^n\|_{h,\infty}$ is also bounded.*

Proof. To prove the theorem, we proceed by the mathematical induction.

Assume $\|D_h^+\mathbf{u}^k\|_h$ and $\|\mathbf{u}^k\|_h$ are bounded for $k=1,2,\dots,n$. We will show that

$$\|D_h^+\mathbf{u}^{n+1}\|_h \quad \text{and} \quad \|\mathbf{u}^{n+1}\|_h$$

are bounded. Let C satisfy

$$\|D_h^+\mathbf{u}^k\|_h \le \frac{1}{2\kappa}C, \quad \|\mathbf{u}^k\|_h \le \frac{1}{2\kappa}C, \quad k=0,1,2,\dots,n. \tag{3.23}$$

We rewrite (3.6) to get

$$\begin{aligned}&\mathcal{D}_\tau^0 u_i^n + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i - D_h^+D_h^-\mathcal{D}_\tau^0 u_i^n - \frac{\alpha}{2}D_h^+D_h^-(\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i)\\ &= \beta D_h^0\overline{u}_i^n + \gamma D_h^+D_h^-\overline{u}_i^n - \gamma D_h^+D_h^-D_h^+D_h^-\overline{u}_i^n.\end{aligned} \tag{3.24}$$

Taking the inner product of (3.24), where $i=1,\dots,M-1$, with $2\overline{\mathbf{u}}^n$, we arrive at

$$\begin{aligned}&\langle \mathcal{D}_\tau^0\mathbf{u}^n, 2\overline{\mathbf{u}}^n\rangle_h - \langle D_h^+D_h^-\mathcal{D}_\tau^0\mathbf{u}^n, 2\overline{\mathbf{u}}^n\rangle_h\\ &= -\langle \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), 2\overline{\mathbf{u}}^n\rangle_h + \langle \frac{\alpha}{2}D_h^+D_h^-\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), 2\overline{\mathbf{u}}^n\rangle_h\\ &\quad + \langle \beta D_h^0\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h + \langle \gamma D_h^+D_h^-\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h - \langle \gamma D_h^+D_h^-D_h^+D_h^-\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h.\end{aligned} \tag{3.25}$$

The left-hand side of (3.25) can be simplified into

$$\langle \mathcal{D}_\tau^0\mathbf{u}^n, 2\overline{\mathbf{u}}^n\rangle_h = \frac{1}{2\tau}\left(\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^{n-1}\|_h^2\right), \tag{3.26}$$

and

$$-\langle D_h^+D_h^-\mathcal{D}_\tau^0\mathbf{u}^n, 2\overline{\mathbf{u}}^n\rangle_h = \frac{1}{2\tau}\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2 - \|D_h^+\mathbf{u}^{n-1}\|_h^2\right). \tag{3.27}$$

For the last three terms on the right-hand side of (3.25), using Lemmas 3.1-3.2 and Cauchy-Schwarz inequality, we have

$$\langle \beta D_h^0\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h = 0 \tag{3.28}$$

$$\langle \gamma D_h^+D_h^-\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h = -2\gamma\langle D_h^+\overline{\mathbf{u}}^n, D_h^+\overline{\mathbf{u}}^n\rangle_h = -2\gamma\|D_h^+\overline{\mathbf{u}}^n\|_h^2 \tag{3.29}$$

$$-\langle \gamma D_h^+D_h^-D_h^+D_h^-\overline{\mathbf{u}}^n, 2\overline{\mathbf{u}}^n\rangle_h = -2\gamma\|D_h^+D_h^-\overline{\mathbf{u}}^n\|_h^2. \tag{3.30}$$

As for the first terms on the right-hand side of (3.25), using Lemma 3.1, we have

$$\begin{aligned}&-\alpha\langle \Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n), \overline{\mathbf{u}}^n\rangle_h = -\alpha h\sum_{i=1}^{M-1}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i(\overline{u}_i^n)\\ &\quad = -\alpha h\sum_{i=1}^{M-1}\left(2\theta u_i^n(D_h^0\overline{u}_i^n) + (1-\theta)D_h^0(u^n\overline{u}^n)_i\right)\overline{u}_i^n\\ &\quad \le 2C\alpha\theta\|D_h^+\overline{\mathbf{u}}^n\|_h\|\overline{\mathbf{u}}^n\|_h + C\alpha(1-\theta)\|\overline{\mathbf{u}}^n\|_h\|D_h^+\overline{\mathbf{u}}^n\|_h\\ &\quad \le C\alpha(1+\theta)\|D_h^+\overline{\mathbf{u}}^n\|_h\|\overline{\mathbf{u}}^n\|_h\\ &\quad \le 2\gamma\|D_h^+\overline{\mathbf{u}}^n\|_h^2 + \frac{(C\alpha(1+\theta))^2}{8\gamma}\|\overline{\mathbf{u}}^n\|_h^2.\end{aligned} \tag{3.31}$$

Note that

$$D_h^0(uv)_i = \frac{1}{2}u_{i+1}D_h^+v_i + (D_h^0u_i)v_i + \frac{1}{2}u_{i-1}D_h^+v_{i-1}. \tag{3.32}$$

As for the second terms on the right-hand side of (3.25), using (3.32) and Lemma 3.1, we get

$$\begin{aligned}
&\alpha\langle \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n), (D_h^+ D_h^- \overline{\mathbf{u}}_i^n)\rangle_h \\
&= \alpha h \sum_{i=1}^{M-1} \left(2\theta u_i^n (D_h^0 \overline{u}_i^n)\right) D_h^+ D_h^- \overline{u}_i^n \\
&\quad + \alpha h \sum_{i=1}^{M-1} \left(\frac{(1-\theta)}{2} (u_{i+1}^n D_h^+ \overline{u}_i^n + 2(D_h^0 u_i^n)\overline{u}_i^n + u_{i-1}^n D_h^+ \overline{u}_{i-1}^n) \right) D_h^+ D_h^- \overline{u}_i^n \\
&\le C\alpha(1+\theta)\|D_h^+ \overline{\mathbf{u}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h + \frac{\alpha(1-\theta)}{2}\|\overline{\mathbf{u}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h \\
&\quad + \frac{\alpha(1-\theta)}{2}\|D_h^+ \overline{\mathbf{u}}^n\|_h \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h \\
&\le 2\gamma \|D_h^+ D_h^- \overline{\mathbf{u}}^n\|_h^2 + \frac{3(C\alpha(1+\theta))^2}{8\gamma}\|D_h^+ \overline{\mathbf{u}}^n\|_h^2 \\
&\quad + \frac{3(\alpha(1+\theta))^2}{32\gamma}\|\overline{\mathbf{u}}^n\|_h^2 + \frac{3(\alpha(1+\theta))^2}{32\gamma}\|D_h^+ \overline{\mathbf{u}}^n\|_h^2.
\end{aligned} \tag{3.33}$$

Substituting (3.26)-(3.33) into (3.25), we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^{n-1}\|_h^2\right) + \frac{1}{2\tau}\left(\|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right) \\
&\le \frac{(C\alpha(1+\theta))^2}{8\gamma}\|\overline{\mathbf{u}}^n\|_h^2 + \frac{3(C\alpha(1+\theta))^2}{8\gamma}\|D_h^+ \overline{\mathbf{u}}^n\|_h^2 + \frac{3(\alpha(1+\theta))^2}{32\gamma}\|\overline{\mathbf{u}}^n\|_h^2 \\
&\frac{3(\alpha(1+\theta))^2}{32\gamma}\|D_h^+ \overline{\mathbf{u}}^n\|_h^2.
\end{aligned} \tag{3.34}$$

Let

$$Q = \max\left\{ \frac{(C\alpha(1+\theta))^2}{8\gamma}, \frac{3(\alpha(1-\theta))^2}{32\gamma}, \frac{3(C\alpha(1+\theta))^2}{8\gamma}, \frac{3(C\alpha(1-\theta))^2}{32\gamma} \right\}.$$

Thus,

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^{n-1}\|_h^2\right) + \frac{1}{2\tau}\left(\|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right) \\
&\le Q\left(\|\overline{\mathbf{u}}^n\|_h^2 + \|D_h^+ \overline{\mathbf{u}}^n\|_h^2\right) \\
&\le \frac{Q}{2}\left(\|\mathbf{u}^{n+1}\|_h^2 + \|\mathbf{u}^{n-1}\|_h^2 + \|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right).
\end{aligned} \tag{3.35}$$

From (3.35), we rewrite to get

$$\begin{aligned}
&\left(\|\mathbf{u}^{n+1}\|_h^2 - \|\mathbf{u}^{n-1}\|_h^2 + \|D_h^+ \mathbf{u}^{n+1}\|_h^2 - \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right) \\
&\le \tau Q\left(\|\mathbf{u}^{n+1}\|_h^2 + \|\mathbf{u}^{n}\|_h^2 + \|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \|D_h^+ \mathbf{u}^{n}\|_h^2\right) \\
&\quad + \tau Q\left(\|\mathbf{u}^{n}\|_h^2 + \|\mathbf{u}^{n-1}\|_h^2 + \|D_h^+ \mathbf{u}^{n}\|_h^2 + \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right).
\end{aligned} \tag{3.36}$$

Let

$$B^n = \left(\|\mathbf{u}^n\|_h^2 + \|\mathbf{u}^{n-1}\|_h^2\right) + \left(\|D_h^+ \mathbf{u}^n\|_h^2 + \|D_h^+ \mathbf{u}^{n-1}\|_h^2\right).$$

Thus, equation (3.36) can be rewritten as follows:

$$\begin{aligned} B^{n+1} - B^n &\le \tau Q(B^{n+1} + B^n) \\ (1-\tau Q)B^{n+1} &\le (1+\tau Q)B^n. \end{aligned} \tag{3.37}$$

If τ is sufficiently small, i.e., $\tau Q \le \frac{1}{2}$, then

$$B^{n+1} \le (1+4\tau Q)\, B^n \le e^{(4QT)} B^1. \tag{3.38}$$

We have from (3.38) that

$$\|D_h^+ \mathbf{u}^{n+1}\|_h \le \sqrt{e^{4QT} B^1} \quad \text{and} \quad \|\mathbf{u}^{n+1}\|_h \le \sqrt{e^{4QT} B^1}.$$

Using Lemma 3.6, we get $\|\mathbf{u}^{n+1}\|_{h,\infty}$ bounded. ∎

Theorem 3.8. *The finite difference scheme (3.4)-(3.6) is uniquely solvable.*

Proof. We rewrite (3.4) to get

$$\begin{aligned} &\mathcal{D}_\tau^+ u_i^0 + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i - D_h^+ D_h^- \mathcal{D}_\tau^+ u_i^0 - \frac{\alpha}{2} D_h^+ D_h^- (\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i) \\ &= \beta D_h^0 u_i^{1/2} + \gamma D_h^+ D_h^- u_i^{1/2} - \gamma D_h^+ D_h^- D_h^+ D_h^- u_i^{1/2}. \end{aligned} \tag{3.39}$$

We first consider its homogeneous system. We have

$$\begin{aligned} \frac{1}{\tau} u_i^1 - \frac{1}{\tau} D_h^+ D_h^- u_i^1 = &-\frac{\alpha}{4}\Psi_\theta(\mathbf{u}_i^0, \mathbf{u}_i^1) + \frac{\alpha}{4} D_h^+ D_h^- \Psi_\theta(\mathbf{u}_i^0, \mathbf{u}_i^1) + \frac{\beta}{2} D_h^0 u_i^1 \\ &+ \frac{\gamma}{2} D_h^+ D_h^- u_i^1 - \frac{\gamma}{2} D_h^+ D_h^- D_h^+ D_h^- u_i^1, \end{aligned} \tag{3.40}$$

for $1 \le i \le M-1$, and $u_0^1 = u_M^1 = 0$.

Taking the inner product of (3.40), where $i = 1, \ldots, M-1$, with $\mathbf{u}^1$, we arrive at

$$\begin{aligned} &\frac{1}{\tau}\langle \mathbf{u}^1, \mathbf{u}^1\rangle_h - \frac{1}{\tau}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h \\ &= -\frac{\alpha}{4}\langle \Psi_\theta(\mathbf{u}^0, \mathbf{u}^1), \mathbf{u}^1\rangle_h + \frac{\alpha}{4}\langle D_h^+ D_h^- \Psi_\theta(\mathbf{u}^0, \mathbf{u}^1), \mathbf{u}^1\rangle_h + \frac{\beta}{2}\langle D_h^0 \mathbf{u}^1, \mathbf{u}^1\rangle_h \\ &\quad + \frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h - \frac{\gamma}{2}\langle D_h^+ D_h^- D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h. \end{aligned} \tag{3.41}$$

The left-hand side of (3.41) can be simplified into

$$\frac{1}{\tau}\|\mathbf{u}^1\|_h^2 + \frac{1}{\tau}\|D_h^+ \mathbf{u}^1\|_h^2. \tag{3.42}$$

For the last three terms on the right-hand side of (3.25), using Lemmas 3.1-3.2 and Cauchy-Schwarz inequality, we have

$$\frac{\beta}{2}\langle D_h^0 \mathbf{u}^1, \mathbf{u}^1\rangle_h = 0 \tag{3.43}$$

$$\frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h = -\frac{\gamma}{2}\langle D_h^+ \mathbf{u}^1, D_h^+ \mathbf{u}^1\rangle_h = -\frac{\gamma}{2}\|D_h^+ \mathbf{u}^1\|_h^2 \tag{3.44}$$

$$-\frac{\gamma}{2}\langle D_h^+ D_h^- D_h^+ D_h^- \mathbf{u}^1, \mathbf{u}^1\rangle_h, = -\frac{\gamma}{2}\|D_h^+ D_h^- \mathbf{u}^1\|_h^2. \tag{3.45}$$

We can use a similar idea from Theorem 3.7 for the first and second terms on the right-hand side of (3.41). We get

$$\begin{aligned} -\frac{\alpha}{4}\langle\Psi_\theta(\mathbf{u}^0,\mathbf{u}^1),\mathbf{u}^1\rangle_h &\le \frac{C\alpha(1+\theta)}{4}\|D_h^+\mathbf{u}^1\|_h\|\mathbf{u}^1\|_h \\ &\le \frac{\gamma}{2}\|D_h^+\mathbf{u}^1\|_h^2+\frac{(C\alpha(1+\theta))^2}{32\gamma}\|\mathbf{u}^1\|_h^2, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} &\frac{\alpha}{4}\langle D_h^+D_h^-\Psi_\theta(\mathbf{u}^0,\mathbf{u}^1),\mathbf{u}^1\rangle_h \\ &\le \frac{C\alpha(1+\theta)}{4}\|D_h^+\mathbf{u}^1\|_h\|D_h^+D_h^-\mathbf{u}^1\|_h+\frac{C\alpha(1-\theta)}{4}\|\mathbf{u}^1\|_h\|D_h^+D_h^-\mathbf{u}^1\|_h \\ &\le \frac{\gamma}{2}\|D_h^+D_h^-\mathbf{u}^1\|_h^2+\frac{(C\alpha(1+\theta))^2}{16\gamma}\|D_h^+\mathbf{u}^1\|_h^2+\frac{(C\alpha(1-\theta))^2}{16\gamma}\|\mathbf{u}^1\|_h^2. \end{aligned} \tag{3.47}$$

Substituting (3.42)-(3.47) into (3.41), we obtain

$$\begin{aligned} &\frac{1}{\tau}\|\mathbf{u}^1\|_h^2+\frac{1}{\tau}\|D_h^+\mathbf{u}^1\|_h^2 \\ &\le \frac{(C\alpha(1+\theta))^2}{16\gamma}\|D_h^+\mathbf{u}^1\|_h^2+\left(\frac{(C\alpha(1-\theta))^2}{16\gamma}+\frac{(C\alpha(1+\theta))^2}{32\gamma}\right)\|\mathbf{u}^1\|_h^2. \end{aligned} \tag{3.48}$$

Let

$$R=\max\left\{\frac{(C\alpha(1+\theta))^2}{16\gamma},\left(\frac{(C\alpha(1-\theta))^2}{16\gamma}+\frac{(C\alpha(1+\theta))^2}{32\gamma}\right)\right\}.$$

Thus,

$$\begin{aligned} \frac{1}{\tau}\|\mathbf{u}^1\|_h^2+\frac{1}{\tau}\|D_h^+\mathbf{u}^1\|_h^2 &\le R\left(\|D_h^+\mathbf{u}^1\|_h^2+\|\mathbf{u}^1\|_h^2\right) \\ \|\mathbf{u}^1\|_h^2+\|D_h^+\mathbf{u}^1\|_h^2 &\le 2\tau R\left(\|D_h^+\mathbf{u}^1\|_h^2+\|\mathbf{u}^1\|_h^2\right). \end{aligned} \tag{3.49}$$

If τ is sufficiently small, i.e., $1-2\tau R>0$, then

$$\|\mathbf{u}^1\|_h^2+\|D_h^+\mathbf{u}^1\|_h^2\le 0,$$

consequently,

$$\|\mathbf{u}^1\|_h=0 \quad \text{and} \quad \|D_h^+\mathbf{u}^1\|_h=0.$$

This implies that $\mathbf{u}^1$ uniquely exists. Therefor, the scheme (3.4) is uniquely solvable. Next, we will show that (3.6) is uniquely solvable. First, we rewrite (3.6) to get

$$\begin{aligned} &\mathcal{D}_\tau^0u_i^n+\frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i-D_h^+D_h^-\mathcal{D}_\tau^0u_i^n-\frac{\alpha}{2}D_h^+D_h^-(\Psi_\theta(\mathbf{u}^n,\overline{\mathbf{u}}^n)_i) \\ &=\beta D_h^0\overline{u}_i^n+\gamma D_h^+D_h^-\overline{u}_i^n-\gamma D_h^+D_h^-D_h^+D_h^-\overline{u}_i^n. \end{aligned} \tag{3.50}$$

Consider the homogeneous system of (3.50):

$$\begin{aligned} &\frac{1}{2\tau}u_i^{n+1}-\frac{1}{2\tau}D_h^+D_h^-u_i^{n+1} \\ &=-\frac{\alpha}{4}\Psi_\theta(\mathbf{u}_i^n,\mathbf{u}_i^{n+1})+\frac{\alpha}{4}D_h^+D_h^-\Psi_\theta(\mathbf{u}_i^n,\mathbf{u}_i^{n+1})+\frac{\beta}{2}D_h^0u_i^{n+1} \\ &\quad+\frac{\gamma}{2}D_h^+D_h^-u_i^{n+1}-\frac{\gamma}{2}D_h^+D_h^-D_h^+D_h^-u_i^{n+1}, \end{aligned} \tag{3.51}$$

for $1\le i\le M-1$, and $u_0^{n+1}=u_M^{n+1}=0$.

Taking the inner product of (3.51), where $i = 1, \dots, M-1$, with $\mathbf{u}^{n+1}$, we arrive at

$$\begin{aligned}
&\frac{1}{2\tau}\langle \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h - \frac{1}{2\tau}\langle D_h^+ D_h^- \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h \\
&= -\frac{\alpha}{4}\langle \Psi_\theta(\mathbf{u}^n, \mathbf{u}^{n+1}), \mathbf{u}^{n+1}\rangle_h + \frac{\alpha}{4}\langle D_h^+ D_h^- \Psi_\theta(\mathbf{u}^n, \mathbf{u}^{n+1}), \mathbf{u}^{n+1}\rangle_h \\
&\quad + \frac{\beta}{2}\langle D_h^0 \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h + \frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h \\
&\quad - \frac{\gamma}{2}\langle D_h^+ D_h^- D_h^+ D_h^- \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h. \qquad (3.52)
\end{aligned}$$

For the left-hand side of (3.52), using (3.1) and Lemma 3.1, we get

$$\frac{1}{2\tau}\|\mathbf{u}^{n+1}\|_h^2 + \frac{1}{2\tau}\|D_h^+ \mathbf{u}^{n+1}\|_h^2. \qquad (3.53)$$

For the last three terms on the right-hand side of (3.25), using Lemmas 3.1-3.2 and Cauchy-Schwarz inequality, we have

$$\frac{\beta}{2}\langle D_h^0 \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h = 0 \qquad (3.54)$$

$$\frac{\gamma}{2}\langle D_h^+ D_h^- \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h = -\frac{\gamma}{2}\langle D_h^+ \mathbf{u}^{n+1}, D_h^+ \mathbf{u}^{n+1}\rangle_h = -\frac{\gamma}{2}\|D_h^+ \mathbf{u}^{n+1}\|_h^2 \qquad (3.55)$$

$$-\frac{\gamma}{2}\langle D_h^+ D_h^- D_h^+ D_h^- \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle_h, = -\frac{\gamma}{2}\|D_h^+ D_h^- \mathbf{u}^{n+1}\|_h^2. \qquad (3.56)$$

We can apply similar idea from above to the first and second terms on the right-hand side of (3.52) to get

$$\begin{aligned}
&-\frac{\alpha}{4}\langle \Psi_\theta(\mathbf{u}^n, \mathbf{u}^{n+1}), \mathbf{u}^{n+1}\rangle_h \le \frac{C\alpha(1+\theta)}{4}\|D_h^+ \mathbf{u}^{n+1}\|_h \|\mathbf{u}^{n+1}\|_h \\
&\le \frac{\gamma}{2}\|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \frac{(C\alpha(1+\theta))^2}{32\gamma}\|\mathbf{u}^{n+1}\|_h^2. \qquad (3.57)
\end{aligned}$$

Using (3.32), we get

$$\begin{aligned}
&\frac{\alpha}{4}\langle D_h^+ D_h^- \Psi_\theta(\mathbf{u}^n, \mathbf{u}^{n+1}), \mathbf{u}^{n+1}\rangle_h \\
&\le \frac{C\alpha(1+\theta)}{4}\|D_h^+ \mathbf{u}^{n+1}\|_h \|D_h^+ D_h^- \mathbf{u}^{n+1}\|_h + \frac{C\alpha(1-\theta)}{4}\|\mathbf{u}^{n+1}\|_h \|D_h^+ D_h^- \mathbf{u}^{n+1}\|_h \\
&\le \frac{\gamma}{2}\|D_h^+ D_h^- \mathbf{u}^{n+1}\|_h^2 + \frac{(C\alpha(1+\theta))^2}{16\gamma}\|D_h^+ \mathbf{u}^{n+1}\|_h^2 + \frac{(C\alpha(1-\theta))^2}{16\gamma}\|\mathbf{u}^{n+1}\|_h^2. \qquad (3.58)
\end{aligned}$$

Substituting (3.53)-(3.58) into (3.52), we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\|\mathbf{u}^{n+1}\|_h^2 + \frac{1}{2\tau}\|D_h^+ \mathbf{u}^{n+1}\|_h^2 \\
&\le \frac{(C\alpha(1+\theta))^2}{16\gamma}\|D_h^+ \mathbf{u}^{n+1}\|_h^2 \\
&\quad + \left(\frac{(C\alpha(1-\theta))^2}{16\gamma} + \frac{(C\alpha(1+\theta))^2}{32\gamma}\right)\|\mathbf{u}^{n+1}\|_h^2. \qquad (3.59)
\end{aligned}$$

Let

$$P = \max\left\{\frac{(C\alpha(1+\theta))^2}{16\gamma}, \left(\frac{(C\alpha(1-\theta))^2}{16\gamma} + \frac{(C\alpha(1+\theta))^2}{32\gamma}\right)\right\}.$$

Thus,

$$\frac{1}{2\tau}\|\mathbf{u}^{n+1}\|_h^2+\frac{1}{2\tau}\|D_h^+\mathbf{u}^{n+1}\|_h^2 \le P\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2+\|\mathbf{u}^{n+1}\|_h^2\right)$$
$$\|\mathbf{u}^{n+1}\|_h^2+\|D_h^+\mathbf{u}^{n+1}\|_h^2 \le 2\tau P\left(\|D_h^+\mathbf{u}^{n+1}\|_h^2+\|\mathbf{u}^{n+1}\|_h^2\right). \tag{3.60}$$

If τ is sufficiently small, i.e., $1-2\tau P>0$, then

$$\|\mathbf{u}^{n+1}\|_h^2+\|D_h^+\mathbf{u}^{n+1}\|_h^2 \le 0.$$

Consequently,

$$\|\mathbf{u}^{n+1}\|_h=0 \quad \text{and} \quad \|D_h^+\mathbf{u}^{n+1}\|_h=0.$$

This implies that $\mathbf{u}^{n+1}$ uniquely exists. Therefor, the scheme (3.6) is uniquely solvable.

■

4. Convergence Analysis

In this section, we study the error analysis of the numerical solution obtained from the difference scheme (3.4)-(3.6).

Let v_i^n be the exact solutions at the point (x_i, t^n). Define $e_i^n = v_i^n - u_i^n$. From (3.4)-(3.5) the truncation error $\mathcal{T}^0$ satisfies

$$\begin{aligned}&\mathcal{D}_\tau^+ e_i^0+\frac{\alpha}{2}\left[\Psi_\theta(\mathbf{v}^0,\mathbf{v}^{1/2})_i-\Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2})_i\right]-D_h^+D_h^-\mathcal{D}_\tau^+e_i^0\\&-\frac{\alpha}{2}D_h^+D_h^-\left[\Psi_\theta(\mathbf{v}^0,\mathbf{v}^{1/2})_i-\Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2})_i\right]\\&=\beta D_h^0 e_i^{1/2}+\gamma D_h^+D_h^-e_i^{1/2}-\gamma D_h^+D_h^-D_h^+D_h^-e_i^{1/2}+\mathcal{T}_i^0,\end{aligned} \tag{4.1}$$

and by Taylor series expansion, there exists a positive constant c_0 such that

$$|\mathcal{T}_i^0| \le c_0(\tau+h^2). \tag{4.2}$$

On the other hand, from (3.6) the truncation error $\mathcal{T}^n$ satisfies

$$\begin{aligned}&\mathcal{D}_\tau^0 e_i^n+\frac{\alpha}{2}\left[\Psi_\theta(\mathbf{v}^n,\bar{\mathbf{v}}^n)_i-\Psi_\theta(\mathbf{u}^n,\bar{\mathbf{u}}^n)_i\right]-D_h^+D_h^-\mathcal{D}_\tau^0e_i^n\\&-\frac{\alpha}{2}D_h^+D_h^-\left[\Psi_\theta(\mathbf{v}^n,\bar{\mathbf{v}}^n)_i-\Psi_\theta(\mathbf{u}^n,\bar{\mathbf{u}}^n)_i\right]\\&=\beta D_h^0\bar{e}_i^n+\gamma D_h^+D_h^-\bar{e}_i^n-\gamma D_h^+D_h^-D_h^+D_h^-\bar{e}_i^n+\mathcal{T}_i^n.\end{aligned} \tag{4.3}$$

Using Taylor series expansion, there exists a positive constant c_1 such that

$$|\mathcal{T}_i^n| \le c_1(\tau^2+h^2). \tag{4.4}$$

Theorem 4.1. *Define*

$$c_2=\max_{x\in[a,b],\,t\in[0,T]}\{|v(x,t)|,|v_x(x,t)|\},$$

$$c_3=\max\left\{\left[\frac{(c_2\alpha\theta)^2}{8\gamma}+\frac{(c_2\alpha(1-\theta))^2}{32}+\frac{3(c_2\alpha(1-\theta))^2}{32\gamma}\right],\left[\frac{1}{2}+\frac{3(c_2\alpha\theta)^2}{8\gamma}+\frac{3(c_2\alpha(1-\theta))^2}{32\gamma}\right]\right\},$$

$$c_4=\max\left\{A_3+A_4+\left[\frac{c_2\alpha(1-\theta)}{2}+A_1\right],\frac{5(c_2\alpha(1-\theta))^2}{4\gamma}\right\},$$

$$A_1=c_2\alpha\theta+\frac{5(c_2\alpha\theta)^2}{\gamma}+\frac{5(c_2\alpha(1-\theta))^2}{4\gamma},$$

$$A_2=\frac{\alpha(1-\theta)}{2}+\beta+\frac{5(\alpha(1-\theta))^2}{16\gamma},$$

$$A_3 = c_2\alpha(1-\theta) + \frac{5(\alpha\theta)^2}{\gamma} + \frac{5(\alpha(1-\theta))^2}{4\gamma} + A_1 + A_2,$$

$$A_4 = 2c_2\alpha\theta + \frac{c_2\alpha(1-\theta)}{2} + \frac{(\alpha\theta)^2}{2\gamma} + \frac{5(c_2\alpha(1-\theta))^2}{4\gamma} + 1 + A_2,$$

$$c_5 = 2c_1^2, \quad c_6 = \max\left\{8c_0^2,\, 2c_5T\right\}, \quad c_7 = 2\kappa\sqrt{(2c_6)\exp(4c_4T)}.$$

If $(\tau^2+h^2) \leq \min\left\{\frac{1}{\kappa}c_0,\, \frac{1}{c_6}\right\}$ *and* $\tau \leq \min\left\{1,\, \frac{1}{2c_3},\, \frac{1}{2c_4}\right\}$*, then the solution* $\mathbf{u}^n$ *of the scheme (3.4)–(3.6) converges in the sense*

$$\|\mathbf{e}^n\|_{h,\infty} \leq C_7(\tau^2+h^2).$$

Proof. First, we estimate the error on the first time level. Since $\mathbf{e}^0 = 0$, we have

$$\begin{aligned}\Psi_\theta(\mathbf{v}^0,\mathbf{v}^{1/2}) - \Psi_\theta(\mathbf{u}^0,\mathbf{u}^{1/2}) &= \Psi_\theta(\mathbf{v}^0,\mathbf{e}^{1/2}) + \Psi_\theta(\mathbf{e}^0,\mathbf{v}^{1/2}) - \Psi_\theta(\mathbf{e}^0,\mathbf{e}^{1/2})\\ &= \Psi_\theta(\mathbf{v}^0,\mathbf{e}^{1/2}).\end{aligned}$$

Thus, the error equation (4.1) is reduced to

$$\begin{aligned}&\frac{e_i^1}{\tau} + \frac{\alpha}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1)_i - \frac{1}{\tau}D_h^+D_h^-e_i^1 - \frac{\alpha}{4}D_h^+D_h^-\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1)_i\\ &= \frac{\beta}{2}D_h^0e_i^1 + \frac{\gamma}{2}D_h^+D_h^-e_i^1 - \frac{\gamma}{2}D_h^+D_h^-D_h^+D_h^-e_i^1 + \mathcal{T}_i^0.\end{aligned} \tag{4.5}$$

Taking the inner product of (4.5), where $i = 1,\ldots,M-1$, with $\mathbf{e}^1$, we have

$$\begin{aligned}&\langle\frac{\mathbf{e}^1}{\tau},\mathbf{e}^1\rangle_h - \langle\frac{1}{\tau}D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h\\ &= -\langle\frac{\alpha}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),\mathbf{e}^1\rangle_h + \langle\frac{\alpha}{4}D_h^+D_h^-\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),\mathbf{e}^1\rangle_h + \langle\frac{\beta}{2}D_h^0\mathbf{e}^1,\mathbf{e}^1\rangle_h\\ &\quad + \langle\frac{\gamma}{2}D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h - \langle\frac{\gamma}{2}D_h^+D_h^-D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h + \langle\mathcal{T}^0,\mathbf{e}^1\rangle_h.\end{aligned} \tag{4.6}$$

We have the following estimations for each term. The left-hand side of (4.6) can be simplified into

$$\langle\frac{\mathbf{e}^1}{\tau},\mathbf{e}^1\rangle_h - \langle\frac{1}{\tau}D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h = \frac{1}{\tau}\|\mathbf{e}^1\|_h^2 + \frac{1}{\tau}\|D_h^+\mathbf{e}^1\|_h^2. \tag{4.7}$$

For the last four terms on the right-hand side of (4.6), using Lemmas 3.1-3.2 and Cauchy-Schwarz inequality, we get

$$\langle\frac{\beta}{2}D_h^0\mathbf{e}^1,\mathbf{e}^1\rangle_h = 0 \tag{4.8}$$

$$\langle\gamma D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h = -\frac{\gamma}{2}\langle D_h^+\mathbf{e}^1, D_h^+\mathbf{e}^1\rangle_h = -\frac{\gamma}{2}\|D_h^+\mathbf{e}^1\|_h^2 \tag{4.9}$$

$$-\langle\frac{\gamma}{2}D_h^+D_h^-D_h^+D_h^-\mathbf{e}^1,\mathbf{e}^1\rangle_h = -\frac{\gamma}{2}\|D_h^+D_h^-\mathbf{e}^1\|_h^2 \tag{4.10}$$

$$\langle\mathcal{T}^0,\mathbf{e}^1\rangle_h \leq \|\mathcal{T}^0\|_h\|\mathbf{e}^1\|_h. \tag{4.11}$$

For the first two terms on the right-hand side of (4.6), using (3.32) and Lemma 3.1, we have the following estimates:

$$\begin{aligned} -\langle \frac{\alpha}{4}\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),\mathbf{e}^1\rangle_h &= -\frac{\alpha\theta}{2}h\sum_{i=1}^{M-1} v_i^0(D_h^0 e_i^1)e_i^1 + \frac{\alpha(1-\theta)}{4}h\sum_{i=1}^{M-1} v_i^0 e_i^1(D_h^0 e_i^1) \\ &\leq \frac{c_2\alpha\theta}{2}\|D_h^+\mathbf{e}^1\|_h\|\mathbf{e}^1\|_h + \frac{c_2\alpha(1-\theta)}{4}\|\mathbf{e}^1\|_h\|D_h^+\mathbf{e}^1\|_h \\ &\leq \frac{(c_2\alpha\theta)^2}{8\gamma}\|\mathbf{e}^1\|_h^2 + \frac{\gamma}{2}\|D_h^+\mathbf{e}^1\|_h^2 + \frac{(c_2\alpha(1-\theta))^2}{32}\|\mathbf{e}^1\|_h^2 + \frac{1}{2}\|D_h^+\mathbf{e}^1\|_h^2, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \langle \frac{\alpha}{4}D_h^+D_h^-\Psi_\theta(\mathbf{v}^0,\mathbf{e}^1),\mathbf{e}^1\rangle_h &= \frac{\alpha\theta}{2}h\sum_{i=1}^{M-1} v_i^0(D_h^0 e_i^1)D_h^+D_h^- e_i^1 \\ &\quad + \frac{\alpha(1-\theta)}{4}h\sum_{i=1}^{M-1}\left(\frac{1}{2}v_{i+1}^0 D_h^+ e_i^1 + (D_h^0 v_i^0)e_i^1 + \frac{1}{2}v_{i-1}^0 D_h^+ e_{i-1}^1\right)D_h^+D_h^- e_i^1 \\ &\leq \frac{c_2\alpha\theta}{2}\|D_h^+\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h + \frac{c_2\alpha(1-\theta)}{4}\|D_h^+\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h \\ &\quad + \frac{c_2\alpha(1-\theta)}{4}\|\mathbf{e}^1\|_h\|D_h^+D_h^-\mathbf{e}^1\|_h \\ &\leq \frac{\gamma}{2}\|D_h^+D_h^-\mathbf{e}^1\|_h^2 + \frac{3(c_2\alpha\theta)^2}{8\gamma}\|D_h^+\mathbf{e}^1\|_h^2 + \frac{3(c_2\alpha(1-\theta))^2}{32\gamma}\|\mathbf{e}^1\|_h^2. \end{aligned} \tag{4.13}$$

Substituting (4.7)-(4.13) into (4.6), we get

$$\begin{aligned} \frac{1}{\tau}(\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2) &\leq c_3(\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2) + \|\mathcal{T}^0\|_h\|\mathbf{e}^1\|_h \\ (1-\tau c_3)\left(\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2\right) &\leq \tau\|\mathcal{T}^0\|_h\|\mathbf{e}^1\|_h. \end{aligned}$$

For $\tau c_3 \leq \frac{1}{2}$, we get

$$\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2 \leq 2\tau\|\mathcal{T}^0\|_h\|\mathbf{e}^1\|_h.$$

Noticing (4.2), if $\tau \leq 1$, we obtain

$$\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2 \leq 2\tau c_0(\tau + h^2)\|\mathbf{e}^1\|_h \leq 2c_0(\tau^2 + h^2)\|\mathbf{e}^1\|_h.$$

Squaring the above inequality, we get

$$(\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2)^2 \leq 4c_0^2(\tau^2+h^2)^2\|\mathbf{e}^1\|_h^2 \leq 4c_0^2(\tau^2+h^2)^2(\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2).$$

Hence,

$$\|\mathbf{e}^1\|_h^2 + \|D_h^+\mathbf{e}^1\|_h^2 \leq 4c_0^2(\tau^2+h^2)^2.$$

From this, we obtain

$$\|\mathbf{e}^1\|_h \leq 2c_0(\tau^2+h^2) \quad \text{and} \quad \|D_h^+\mathbf{e}^1\|_h \leq 2c_0(\tau^2+h^2). \tag{4.14}$$

From the hypothesis of the theorem, we have

$$\|\mathbf{e}^1\|_h \leq \frac{1}{2\kappa} \quad \text{and} \quad \|D_h^+\mathbf{e}^1\|_h \leq \frac{1}{2\kappa}.$$

This shows that

$$\|\mathbf{e}^1\|_{h,\infty} \leq 1.$$

For the time step $n, n > 1$, we use induction on n and proceed in a similar manner. Assume

$$\|\mathbf{e}^k\|_h \leq \frac{1}{2\kappa} \quad \text{and} \quad \|D_h^+\mathbf{e}^k\|_h \leq \frac{1}{2\kappa}. \tag{4.15}$$

That is $\|\mathbf{e}^k\|_{h,\infty} \leq 1$, for $k = 0, 1, \ldots, n$. We shall show that (4.15) holds for $k = n+1$. Taking the inner product of (4.3), where $i = 1, \ldots, M-1$, with $2\overline{\mathbf{e}}^n$, we arrive at

$$\begin{aligned}
&\langle \mathcal{D}_\tau^0 \mathbf{e}^n, 2\overline{\mathbf{e}}^n\rangle_h - \langle D_h^+ D_h^- \mathcal{D}_\tau^0 \mathbf{e}^n, 2\overline{\mathbf{e}}^n\rangle_h \\
&= -\langle \frac{\alpha}{2}\left[\Psi_\theta(\mathbf{v}^n, \overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)\right], 2\overline{\mathbf{e}}^n\rangle_h \\
&\quad + \langle \frac{\alpha}{2} D_h^+ D_h^- \left[\Psi_\theta(\mathbf{v}^n, \overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)\right], 2\overline{\mathbf{e}}^n\rangle_h \\
&\quad + \langle \beta D_h^0 \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h + \langle \gamma D_h^+ D_h^- \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h \\
&\quad - \langle \gamma D_h^+ D_h^- D_h^+ D_h^- \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h + \langle \mathcal{T}^n, 2\overline{\mathbf{e}}^n\rangle_h.
\end{aligned} \tag{4.16}$$

The left-hand side of (4.16) can be simplified into

$$\begin{aligned}
&\langle \mathcal{D}_\tau^0 \mathbf{e}^n, 2\overline{\mathbf{e}}^n\rangle_h - \langle D_h^+ D_h^- \mathcal{D}_\tau^0 \mathbf{e}^n, 2\overline{\mathbf{e}}^n\rangle_h \\
&= \frac{1}{2\tau}\left(\|\mathbf{e}^{n+1}\|_h^2 - \|\mathbf{e}^{n+1}\|_h^2\right) \\
&\quad + \frac{1}{2\tau}\left(\|D_h^+\mathbf{e}^{n+1}\|_h^2 - \|D_h^+\mathbf{e}^{n-1}\|_h^2\right).
\end{aligned} \tag{4.17}$$

For the last four terms on the right-hand side of (4.16), using Lemmas 3.1-3.2 and Cauchy-Schwarz inequality, we get

$$\langle \beta D_h^0 \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h = 0 \tag{4.18}$$

$$\langle \gamma D_h^+ D_h^- \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h = -2\gamma\langle D_h^+\overline{\mathbf{e}}^n, D_h^+\overline{\mathbf{e}}^n\rangle_h = -2\gamma\|D_h^+\overline{\mathbf{e}}^n\|_h^2 \tag{4.19}$$

$$-\langle \gamma D_h^+ D_h^- D_h^+ D_h^- \overline{\mathbf{e}}^n, 2\overline{\mathbf{e}}^n\rangle_h = -2\gamma\|D_h^+ D_h^-\overline{\mathbf{e}}^n\|_h^2 \tag{4.20}$$

$$\langle \mathcal{T}^n, 2\overline{\mathbf{e}}^n\rangle_h \leq 2\|\mathcal{T}^n\|_h\|\overline{\mathbf{e}}^n\|_h. \tag{4.21}$$

For the first term in (4.16), note that

$$\Psi_\theta(\mathbf{v}^n, \overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n) = \Psi_\theta(\mathbf{v}^n, \overline{\mathbf{e}}^n) + \Psi_\theta(\mathbf{e}^n, \overline{\mathbf{v}}^n) - \Psi_\theta(\mathbf{e}^n, \overline{\mathbf{e}}^n).$$

Using Lemma 3.1, we get

$$\begin{aligned}
&-\alpha\langle \Psi_\theta(\mathbf{v}^n, \overline{\mathbf{e}}^n), \overline{\mathbf{e}}^n\rangle_h \\
&= -2\alpha\theta h\sum_{i=1}^{M-1} v_i^n (D_h^0 \overline{e}_i^n)\overline{e}_i^n + \alpha(1-\theta)h\sum_{i=1}^{M-1} v_i^n \overline{e}_i^n (D_h^0 \overline{e}_i^n) \\
&\leq 2c_2\alpha\theta\|D_h^+\overline{\mathbf{e}}^n\|_h\|\overline{\mathbf{e}}^n\|_h + c_2\alpha(1-\theta)\|\overline{\mathbf{e}}^n\|_h\|D_h^+\overline{\mathbf{e}}^n\|_h \\
&\leq c_2\alpha\theta\left(\|D_h^+\overline{\mathbf{e}}^n\|_h^2 + \|\overline{\mathbf{e}}^n\|_h^2\right) + \frac{c_2\alpha(1-\theta)}{2}\left(\|\overline{\mathbf{e}}^n\|_h^2 + \|D_h^+\overline{\mathbf{e}}^n\|_h^2\right).
\end{aligned} \tag{4.22}$$

Similarly, we have

$$
\begin{aligned}
&-\alpha\langle\Psi_\theta(\mathbf{e}^n,\overline{\mathbf{v}}^n),\overline{\mathbf{e}}^n\rangle_h\\
&=-2\alpha\theta h\sum_{i=1}^{M-1}e_i^n(D_h^0\overline{v}_i^n)\overline{e}_i^n+\alpha(1-\theta)h\sum_{i=1}^{M-1}e_i^n\overline{v}_i^n(D_h^0\overline{e}_i^n)\\
&\le 2c_2\alpha\theta\|\mathbf{e}^n\|_h\|\overline{\mathbf{e}}^n\|_h+c_2\alpha(1-\theta)\|\mathbf{e}^n\|_h\|D_h^+\overline{\mathbf{e}}^n\|_h\\
&\le c_2\alpha\theta\left(\|\mathbf{e}^n\|_h^2+\|\overline{\mathbf{e}}^n\|_h^2\right)+\frac{c_2\alpha(1-\theta)}{2}\left(\|\mathbf{e}^n\|_h^2+\|D_h^+\overline{\mathbf{e}}^n\|_h^2\right).
\end{aligned}\tag{4.23}
$$

Using (4.15) and Lemma 3.1, we have

$$
\begin{aligned}
&\alpha\langle\Psi_\theta(\mathbf{e}^n,\overline{\mathbf{e}}^n),\overline{\mathbf{e}}^n\rangle_h\\
&=2\alpha\theta h\sum_{i=1}^{M-1}e_i^n(D_h^0\overline{e}_i^n)\overline{e}_i^n-\alpha(1-\theta)h\sum_{i=1}^{M-1}e_i^n\overline{e}_i^n(D_h^0\overline{e}_i^n)\\
&=2\alpha\theta h\sum_{i=1}^{M-1}\|\mathbf{e}^n\|_{h,\infty}|D_h^0\overline{e}_i^n||\overline{e}_i^n|-\alpha(1-\theta)h\sum_{i=1}^{M-1}\|\mathbf{e}^n\|_{h,\infty}|\overline{e}_i^n||D_h^0\overline{e}_i^n|\\
&\le 2\alpha\theta\|D_h^+\overline{\mathbf{e}}^n\|_h\|\overline{\mathbf{e}}^n\|_h+\alpha(1-\theta)\|\overline{\mathbf{e}}^n\|_h\|D_h^+\overline{\mathbf{e}}^n\|_h\\
&\le 2\gamma\|D_h^+\overline{\mathbf{e}}^n\|_h^2+\frac{(\alpha\theta)^2}{2\gamma}\|\overline{\mathbf{e}}^n\|_h^2+\frac{\alpha(1-\theta)}{2}\left(\|\overline{\mathbf{e}}^n\|_h^2+\|D_h^+\overline{\mathbf{e}}^n\|_h^2\right).
\end{aligned}\tag{4.24}
$$

As for the second terms on the right-hand side of (4.16), using (3.32) and Lemma 3.1, we get

$$
\begin{aligned}
&\alpha\langle\Psi_\theta(\mathbf{v}^n,\overline{\mathbf{e}}^n),(D_h^+D_h^-\overline{\mathbf{e}}^n)\rangle_h\\
&=2\alpha\theta h\sum_{i=1}^{M-1}v_i^n(D_h^0\overline{e}_i^n)D_h^+D_h^-\overline{e}^n\\
&\quad+\frac{\alpha(1-\theta)}{2}h\sum_{i=1}^{M-1}\left(v_{i+1}^nD_h^+\overline{e}_i^n+2(D_h^0v_i^n)\overline{e}_i^n+v_{i-1}^nD_h^+\overline{e}_{i-1}^n\right)(D_h^+D_h^-\overline{e}^n)\\
&\le 2c_2\alpha\theta\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\\
&\quad+c_2\alpha(1-\theta)\left(\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h+\|\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\right)\\
&\le\frac{3\gamma}{5}\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2+\left(\frac{5(c_2\alpha\theta)^2}{\gamma}+\frac{5(c_2\alpha(1-\theta))^2}{4\gamma}\right)\|D_h^+\overline{\mathbf{e}}^n\|_h^2\\
&\quad+\frac{5(c_2\alpha(1-\theta))^2}{4\gamma}\|\overline{\mathbf{e}}^n\|_h^2.
\end{aligned}\tag{4.25}
$$

Similarly, we have

$$\begin{aligned}
&\alpha\langle\Psi_\theta(\mathbf{e}^n,\overline{\mathbf{v}}^n),(D_h^+D_h^-\overline{\mathbf{e}}^n)\rangle_h \\
&\quad = 2\alpha\theta h\sum_{i=1}^{M-1} e_i^n(D_h^0\overline{v}_i^n)D_h^+D_h^-\overline{e}^n \\
&\qquad + \frac{\alpha(1-\theta)}{2}h\sum_{i=1}^{M-1}\left(e_{i+1}^n D_h^+\overline{v}_i^n + 2(D_h^0 e_i^n)\overline{v}_i^n + e_{i-1}^n D_h^+\overline{v}_{i-1}^n\right)(D_h^+D_h^-\overline{e}^n) \\
&\quad \le 2c_2\alpha\theta\|\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h \\
&\qquad + c_2\alpha(1-\theta)\left(\|\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h + \|D_h^+\mathbf{e}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h\right) \\
&\quad \le \frac{3\gamma}{5}\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2 + \left(\frac{5(c_2\alpha\theta)^2}{\gamma} + \frac{5(c_2\alpha(1-\theta))^2}{4\gamma}\right)\|\mathbf{e}^n\|_h^2 \\
&\qquad + \frac{5(c_2\alpha(1-\theta))^2}{4\gamma}\|D_h^+\mathbf{e}^n\|_h^2. \qquad (4.26)
\end{aligned}$$

Using (3.32), (4.15) and Lemma 3.1, we get

$$\begin{aligned}
&-\alpha\langle\Psi_\theta(\mathbf{e}^n,\overline{\mathbf{e}}^n),(D_h^+D_h^-\overline{\mathbf{e}}_i^n)\rangle_h \\
&= -2\alpha\theta h\sum_{i=1}^{M-1} e_i^n(D_h^0\overline{e}_i^n)D_h^+D_h^-\overline{e}_i^n \\
&\quad - \frac{\alpha(1-\theta)}{2}h\sum_{i=1}^{M-1}\left(e_{i+1}^n D_h^+\overline{e}_i^n + 2(D_h^0 e_i^n)\overline{e}_i^n + e_{i-1}^n D_h^+\overline{e}_{i-1}^n\right)(D_h^+D_h^-\overline{e}_i^n) \\
&\le 2\alpha\theta\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h + \alpha(1-\theta)\|D_h^+\overline{\mathbf{e}}^n\|_h\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h \\
&\quad + \alpha(1-\theta)\kappa\left(\|\overline{\mathbf{e}}^n\|_h + \|D_h^+\overline{\mathbf{e}}^n\|_h\right)\left(\frac{1}{2\kappa}\right)\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h \\
&\le \frac{4\gamma}{5}\|D_h^+D_h^-\overline{\mathbf{e}}^n\|_h^2 + \left(\frac{5(\alpha\theta)^2}{\gamma} + \frac{5(\alpha(1-\theta))^2}{4\gamma}\right)\|D_h^+\overline{\mathbf{e}}^n\|_h^2 \\
&\quad + \frac{5(\alpha(1-\theta))^2}{16\gamma}\|\overline{\mathbf{e}}^n\|_h^2 + \frac{5(\alpha(1-\theta))^2}{16\gamma}\|D_h^+\overline{\mathbf{e}}^n\|_h^2. \qquad (4.27)
\end{aligned}$$

Substituting (4.17)-(4.27) into (4.16) and multiplying 2 to both sides, we obtain

$$\begin{aligned}
&\frac{1}{\tau}\left(\|\mathbf{e}^{n+1}\|_h^2 - \|\mathbf{e}^{n-1}\|_h^2\right) + \frac{1}{\tau}\left(\|D_h^+\mathbf{e}^{n+1}\|_h^2 - \|D_h^+\mathbf{e}^{n-1}\|_h^2\right) \\
&\le c_4\left(\|\mathbf{e}^{n+1}\|_h^2 + 2\|\mathbf{e}^n\|_h^2 + \|\mathbf{e}^{n-1}\|_h^2\right) \\
&\quad + c_4\left(\|D_h^+\mathbf{e}^{n+1}\|_h^2 + 2\|D_h^+\mathbf{e}^n\|_h^2 + \|D_h^+\mathbf{e}^{n-1}\|_h^2\right) + c_5(\tau^2+h^2)^2. \qquad (4.28)
\end{aligned}$$

Define

$$S^n = (\|\mathbf{e}^n\|_h^2 + \|D_h^+\mathbf{e}^n\|_h^2) + (\|\mathbf{e}^{n+1}\|_h^2 + \|D_h^+\mathbf{e}^{n+1}\|_h^2).$$

It is possible to demonstrate that (4.28) leads to

$$\begin{aligned}
&\frac{1}{\tau}(S^n - S^{n-1}) \le c_4(S^n + S^{n-1}) + c_5(\tau^2+h^2)^2 \\
&\quad (1-c_4\tau)\,S^n \le (1+c_4\tau)\,S^{n-1} + c_5\tau(\tau^2+h^2)^2.
\end{aligned}$$

For $c_4\tau \leq \frac{1}{2}$, we arrive at

$$S^n \leq (1 + 4c_4\tau)\, S^{n-1} + 2c_5\tau(\tau^2 + h^2)^2. \tag{4.29}$$

The application of discrete Grönwall inequality yields

$$S^n \leq \exp\left(4c_4T\right)\left(S^0 + 2c_5T(\tau^2 + h^2)^2\right). \tag{4.30}$$

Take the value of S^0 from (4.14), we get

$$\begin{aligned} S^n &\leq \exp\left(4c_4T\right)\left(8c_0^2(\tau^2 + h^2)^2 + 2c_5T(\tau^2 + h^2)^2\right) \\ &\leq \frac{c_7^2}{(2\kappa)^2}(\tau^2 + h^2)^2. \end{aligned} \tag{4.31}$$

We have from (4.31) that

$$\|\mathbf{e}^{n+1}\|_h \leq \frac{c_7^2}{(2\kappa)^2}(\tau^2 + h^2) \quad \text{and} \quad \|D_h^+\mathbf{e}^{n+1}\|_h \leq \frac{c_7^2}{(2\kappa)^2}(\tau^2 + h^2).$$

Using Lemma 3.6, we get

$$\|\mathbf{e}^{n+1}\|_{h,\infty} \leq c_7(\tau^2 + h^2)$$

as needed. ∎

5. Numerical results

In this section, we test the proposed scheme on various examples. We begin by extending the scheme to more general settings.

5.1. Extension to the nonhomogeneous equation

For the nonhomogeneous Fornberg-Whitham equation

$$(1 - \partial_x^2)(u_t + \alpha u u_x - \gamma u_{xx}) - \beta u_x = f(x,t), \quad (x,t) \in (a,b) \times (0,T), \tag{5.1}$$

with conditions (1.7)-(1.10), we use the following difference scheme

$$\begin{aligned} &(1 - D_h^+D_h^-)\left(\mathcal{D}_\tau^+ u_i^0 + \frac{\alpha}{2}(\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i) - \gamma D_h^+D_h^- u_i^{1/2}\right) - \beta D_h^0 u_i^{1/2} \\ &= f(x_i, t^{1/2}), \end{aligned} \tag{5.2}$$

for $i = 1, \dots, M-1$, and

$$(1 - D_h^+D_h^-)\left(\mathcal{D}_\tau^0 u_i^n + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i - \gamma D_h^+D_h^- \overline{u}_i^n\right) - \beta D_h^0 \overline{u}_i^n = f(x_i, t^n), \tag{5.3}$$

for $i = 1, \dots, M-1$ and $n = 1, \dots, N-1$.

We can show that the difference scheme (5.2)–(5.3) converges uniformly and is accurate to the second order in both time and space by using a concept similar to Theorem 4.1.

5.2. Extension to the Periodic Problem

A slightly different setting is required when dealing with periodic boundary condition. The numerical solution $\mathbf{u}^n$ at the time t^n is taken from the solution space $\mathcal{Z}_h$ defined as

$$\mathcal{Z}_h := \{\mathbf{u} = [u_i]_{i\in\mathbb{Z}} \mid u_i = u_{M+i}\}.$$

For any $\mathbf{u}, \mathbf{v} \in \mathcal{Z}_h$, we define the inner product:

$$\langle \mathbf{u}, \mathbf{v}\rangle_h = h\sum_{i=1}^{M} u_i v_i.$$

The discrete L_2 norm and the discrete uniform norm are adjusted accordingly. We construct the difference scheme as follows:

$$(1 - D_h^+ D_h^-)\left(\mathcal{D}_\tau^+ u_i^0 + \frac{\alpha}{2}(\Psi_\theta(\mathbf{u}^0, \mathbf{u}^{1/2})_i) - \gamma D_h^+ D_h^- u_i^{1/2}\right) = \beta D_h^0 u_i^{1/2} \tag{5.4}$$

$$u_i^0 = u_0(x_i), \tag{5.5}$$

for $i = 1, \ldots, M$, and

$$(1 - D_h^+ D_h^-)\left(\mathcal{D}_\tau^0 u_i^n + \frac{\alpha}{2}\Psi_\theta(\mathbf{u}^n, \overline{\mathbf{u}}^n)_i - \gamma D_h^+ D_h^- \overline{u}_i^n\right) = \beta D_h^0 \overline{u}_i^n, \tag{5.6}$$

for $i = 1, \ldots, M$ and $n = 1, \ldots, N-1$.

With the setup above, using the same method as the zero boundary condition, we can show that the resultant scheme has a discrete invariant-preserving property when $\theta = 1/3$. We also proved that the numerical solution converges uniformly and is accurate to the second order in both time and space.

5.3. Determining the order of accuracy

If the exact solution is known, we compute r, the order of accuracy, using the formula

$$r = \log_2\left(\frac{\|\mathbf{e}_h\|}{\|\mathbf{e}_{h/2}\|}\right),$$

where $\mathbf{e}_h$ and $\mathbf{e}_{h/2}$ are the errors resulting from the divisions of the domain into subranges of sizes h and $h/2$ respectively.

When the exact solution is unknown, the order r is determined by comparing solutions in which h is successively decreased. Then, we get

$$r = \log_2\left(\frac{\|\mathbf{u}_h - \mathbf{u}_{h/2}\|}{\|\mathbf{u}_{h/2} - \mathbf{u}_{h/4}\|}\right),$$

where $\mathbf{u}_h$, $\mathbf{u}_{h/2}$ and $\mathbf{u}_{h/4}$ are the estimate solution resulting from the divisions of the domain into subranges of sizes h, $h/2$ and $h/4$ respectively.

5.4. Accuracy Test for the Viscous Problem

Example 1. For the case $\gamma \neq 0$, we used the homogeneous example with $\alpha = 1, \beta = -1, \gamma = 1$. It has zero boundary condition with no exact solution. The initial value is given by

$$u(x, 0) = \text{sech}(x).$$

The simulation was conducted on the domain $[-30, 30]$ with the final time $T = 1$ and $\tau = h$. The errors and orders of accuracy are shown in Table 1.

M	$\theta = 0$		$\theta = 1/3$		$\theta = 2/3$		$\theta = 1$	
	$\|u_h - u_{h/2}\|_\infty$	order	$\|u_h - u_{h/2}\|_\infty$	order	$\|u_h - u_{h/2}\|_\infty$	order	$\|u_h - u_{h/2}\|_\infty$	order
80	8.6481e-03	2.10	9.1294e-03	2.09	9.6509e-03	2.08	1.0152e-02	1.96
160	2.0239e-03	2.02	2.1451e-03	2.01	2.2779e-03	1.99	2.6077e-03	1.97
320	4.9808e-04	2.00	5.3239e-04	2.01	5.7329e-04	2.01	6.6398e-04	2.00
640	1.2424e-04		1.3191e-04		1.4221e-04		1.6598e-04	

TABLE 1. Orders showing optimal convergence rates for the homogeneous viscous problem (Example 1).

We see that the results from Table 1 demonstrate that numerical solution converges with second order accuracy with respect to the spatial variable.

5.5. Accuracy Test for the Inviscous Problem

For the viscous case ($\gamma = 0$), we use $\alpha = 1, \beta = -1$, and $\gamma = 1$ for the following examples.

Example 2. The simulation was conducted on the steady solution, which is periodic boundary on $[-p, p]$. We used the homogeneous example with initial data

$$u(x,0) = \frac{4}{3}\left(\frac{\cosh(1/2(x))}{\cosh(p/2)} - 1\right) + x_0.$$

Using domain $[-2, 2]$ with the final time $T = 1$ and $\tau = h$. The errors and orders of accuracy are given in Tables 2-3 and Figure 1.

M	$\theta = 0$				$\theta = 1/3$			
	$\|u - u_h\|_h$		$\|u - u_h\|_\infty$		$\|u - u_h\|_h$		$\|u - u_h\|_\infty$	
	error	order	error	order	error	order	error	order
40	9.7056e-04		7.6717e-04		6.1119e-04		5.3224e-04	
80	2.5136e-04	1.95	2.0749e-04	1.89	1.5889e-04	1.94	1.4505e-04	1.88
160	6.3953e-05	1.97	5.4015e-05	1.94	4.0501e-05	1.97	3.7898e-05	1.94
320	1.6129e-05	1.99	1.3784e-05	1.97	1.0223e-05	1.99	9.6887e-06	1.97
640	4.0498e-06	1.99	3.4820e-06	1.99	2.5681e-06	1.99	2.4496e-06	1.98

TABLE 2. Errors and orders showing optimal convergence rates for the homogeneous inviscous problem (Example 2) with $\theta = 0, 1/3$.

M	$\theta = 2/3$				$\theta = 1$			
	$\|u - u_h\|_h$		$\|u - u_h\|_\infty$		$\|u - u_h\|_h$		$\|u - u_h\|_\infty$	
	error	order	error	order	error	order	error	order
40	2.5847e-04		2.7513e-04		1.4228e-04		7.4647e-05	
80	6.8013e-05	1.93	7.6891e-05	1.84	3.5542e-05	2.00	1.8655e-05	2.00
160	1.7432e-05	1.96	2.0334e-05	1.92	8.8817e-06	2.00	4.6662e-06	2.00
320	4.4118e-06	1.98	5.2291e-06	1.96	2.2200e-06	2.00	1.1665e-06	2.00
640	1.1097e-06	1.99	1.3259e-06	1.98	5.5493e-07	2.00	2.9163e-07	2.00

TABLE 3. Errors and orders showing optimal convergence rates for the homogeneous inviscous problem (Example 2) with $\theta = 2/3, 1$.

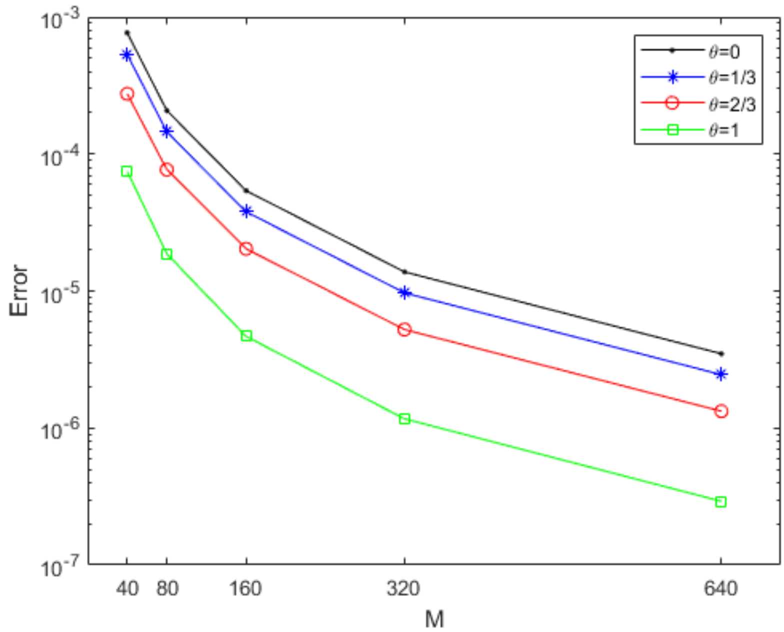

FIGURE 1. Comparison of the uniform errors at $t = 1$ when using values of $\theta = 0, 1/3, 2/3, 1$ for the homogeneous viscous problem (Example 2).

From results of Example 2, we see that the Tables 2-3 show that numerical solution converges to exact solution with second-order accuracy relative to spatial variables. Moreover, convergence can be inferred from the error curves in Figure 1.

Example 3. We used the periodic nonhomogeneous example with exact solution

$$u(x,t) = \sin(x - t).$$

The simulation was conducted on the periodic boundary $[0, 2\pi]$ with the final time $T = 1$ and $\tau = h$. The errors and orders of accuracy are given in Tables 4-5. We see that the convergence rates demonstrate that numerical solution converge to exact solution with second-order accuracy relative to spatial variables. The performances for each value of θ convergence can be inferred from the error curves are shown in Figure 2

M	$\theta = 0$				$\theta = 1/3$			
	$\|u - u_h\|_h$		$\|u - u_h\|_\infty$		$\|u - u_h\|_h$		$\|u - u_h\|_\infty$	
	error	order	error	order	error	order	error	order
80	2.6368e-03		4.5146e-03		2.1846e-03		3.7884e-03	
160	6.6280e-04	1.99	1.1381e-03	1.99	5.4850e-04	1.99	9.5397e-04	1.99
320	1.6606e-04	2.00	2.8520e-04	2.00	1.3739e-04	2.00	2.3879e-04	2.00
640	4.1555e-05	2.00	7.1363e-05	2.00	3.4378e-05	2.00	5.9731e-05	2.00
1280	1.0393e-05	2.00	1.7843e-05	2.00	8.5981e-06	2.00	1.4934e-05	2.00

TABLE 4. Errors and orders showing optimal convergence rates for the nonhomogeneous inviscous problem (Example 3) with $\theta = 0, 1/3$.

M	$\theta = 2/3$				$\theta = 1$			
	$\|u - u_h\|_h$		$\|u - u_h\|_\infty$		$\|u - u_h\|_h$		$\|u - u_h\|_\infty$	
	error	order	error	order	error	order	error	order
80	1.7366e-03		3.0582e-03		1.2976e-03		2.3252e-03	
160	4.3554e-04	2.00	7.6975e-04	1.99	3.2512e-04	2.00	5.8528e-04	1.99
320	1.0907e-04	2.00	1.9252e-04	2.00	8.1402e-05	2.00	1.4627e-04	2.00
640	2.7290e-05	2.00	4.8137e-05	2.00	2.0368e-05	2.00	3.6574e-05	2.00
1280	6.8256e-06	2.00	1.2035e-05	2.00	5.0943e-06	2.00	9.1435e-06	2.00

TABLE 5. Errors and orders showing optimal convergence rates for the nonhomogeneous inviscous problem (Example 3) with $\theta = 2/3, 1$.

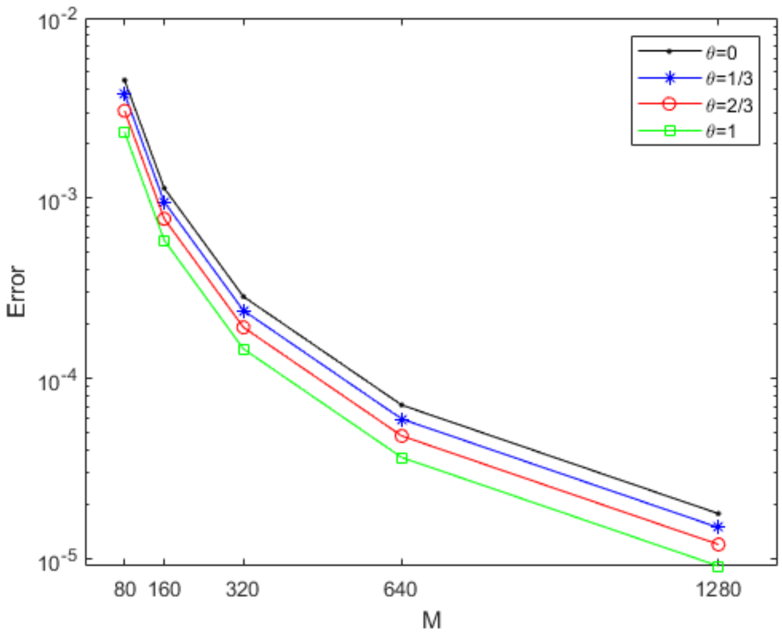

FIGURE 2. Comparison of the uniform errors at $t = 1$ when using values of $\theta = 0, 1/3, 2/3, 1$ for the nonhomogeneous problem (Example 3).

5.6. INVARIANT-PRESERVING TEST

To see how well the scheme performs on a long-time simulation. We tested the method on the exact solution

$$u(x,t) = \frac{4}{3}\left(e^{-\frac{1}{2}|x - x_0 T|} - 1\right) + x_0.$$

The simulation was done using $M = 320, N = 1250$ and $T = (0, 15, 30, 45, 60, 75, 90, 105)$ on the domain $[-30, 150]$ with $x_0 = 4/3$.

In Table 6, we see that the quantity $\|u^n\|_h$ is stable only when $\theta = 1/3$, but when we used $\theta = 0, 2/3, 1$, the numerical solution becomes unstable over time.

T	$\theta=0$		$\theta=1/3$		$\theta=2/3$		$\theta=1$	
	$\|u^n\|_h$	$\%\frac{\|u^n\|_h}{\|u^0\|_h}$	$\|u^n\|_h$	$\%\frac{\|u^n\|_h}{\|u^0\|_h}$	$\|u^n\|_h$	$\%\frac{\|u^n\|_h}{\|u^0\|_h}$	$\|u^n\|_h$	$\%\frac{\|u^n\|_h}{\|u^0\|_h}$
0	2.5032		2.5032		2.5032		2.5032	
15	2.5103	100.28	2.5032	100.00	2.4973	99.77	2.4938	99.62
30	2.5111	100.03	2.5032	100.00	2.4979	100.02	2.8907	115.92
45	2.5113	100.01	2.5032	100.00	2.5169	100.76	4.1403	143.23
60	2.5114	100.00	2.5032	100.00	2.7461	109.11	4.2895	103.60
75	2.5115	100.00	2.5032	100.00	3.9152	142.58	3.8790	90.43
90	2.5116	100.00	2.5032	100.00	3.9424	100.69	3.9468	101.75
105	2.5116	100.00	2.5032	100.00	3.0138	76.45	4.1360	104.79

TABLE 6. Comparison between the invariant-preserving scheme ($\theta = 1/3$) and the other schemes ($\theta = 0, 2/3, 1$).

6. Conclusions

In this paper, we propose a family of finite difference θ-schemes to solve the viscous Fornberg-Whithem equation which is a shallow water model describing waves breaking. The model is nonlinear, but our current approaches are linear and employ data from three-time steps. The method is explicit; this eliminates the requirement for iterative nonlinear solvers to solve unknowns. The model admits two invariants: momentum and energy. We showed that the resultant scheme has a discrete invariant-preserving property of $\theta = 1/3$. We also proved that the numerical solution converges uniformly and is accurate to the second order in both time and space. Finally, numerical examples are presented to show that the optimal order of convergence is achieved for any value of θ, but the invariant is only stable when $\theta = 1/3$. The numerical data produced through simulations agrees with the theoretical results in every aspects. Finally, we point out that the proposed method performs under specific conditions given in (1.7)-(1.10). In a more general settings, we expect that it will also give a satisfying result as we designed it so that the obtained numerical solution behaves similar to the exact solution in preserving some quantities.

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