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An extragradient approximation method for system of equilibrium problems and variational inequality problems[0]

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Abstract : The purpose of this paper is to investigate the problem of finding the common element of the set of common fixed points of an infinite family of nonexpansive mappings, the set of solutions of a system of equilibrium problems and the set of solutions of the variational inequality problem for a monotone and ζ-Lipschitz continuous mapping in Hilbert spaces. Then, we prove that the strong convergence of the proposed iterative algorithm to the unique solutions of variational inequality, which is the optimality condition for a minimization problem. Our results extend and improve the corresponding results of Colao, Marino and Xu [V. Colao, G. Marino and H.K. Xu b, An iterative method for finding common solutions of equilibrium and fixed point problems, J. Math. Anal. Appl. 344 (2008) 340-352] and Peng and Yao [J.W. Peng and J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings, Nonlinear Analysis. Doi.org/10.1016/j.na.2009.05.028] and many others.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In addition, Let $B : C \to H$ be a nonlinear mapping. The classical variational

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inequality problem is to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(C, B)$, that is,

$$VI(C, B) = \{ \ x \in C : \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C \ \}. \tag{1.2}$$

Let F be an bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(F)$, that is,

$$EP(F) = \{ \ x \in C : F(x, y) \geq 0, \quad \forall y \in C \ \}. \tag{1.4}$$

Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problems. Numerous problems in physics, optimization, saddle point problems, complementarity problems, mechanics and economics reduce to find a solution of (1.3). In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Let $\Im = \{F_k\}_{k \in \Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k \in \Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k \in \Lambda}$ such that

$$F_k(x, y) \geq 0, \quad \forall k \in \Lambda \quad \forall y \in C. \tag{1.5}$$

where Λ is an arbitrary index set. The set of solutions of (1.5) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \{ \ x \in C : F_k(x, y) \geq 0, \quad \forall k \in \Lambda \quad \forall y \in C \ \}. \tag{1.6}$$

If Λ is a singleton, then the problem (1.5) is reduced to the problem (1.3). The problem (1.5) is very general in the sense that it includes, as special case, some optimization , variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics and others (see, for instance, [1, 3, 4]).

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.
In order to prove our main results, we need the following lemmas.

Lemma 1.1. *For a given $z \in H$, $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \quad \forall v \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{1.7}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{1.8}$$

It is easy to see that (1.8) is equivalent to the following inequality:

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{1.9}$$

Using Lemma 1.1, one can see that the variational inequality (1.1) is equivalent to a fixed point problem.

It is easy to see that the following is true:

$$u \in VI(C, B) \Leftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{1.10}$$

The variational inequality has been extensively studied in the literature; see, for instance [5, 6, 8, 10, 22]. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall the following definitions:

(1) A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(2) B is called *β-strongly monotone* (see [2, 14]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C.$$

(3) B is called *ζ-Lipschitz continuous* if there exists a positive real number ζ such that

$$\|Bx - By\| \leq \zeta \|x - y\|, \quad \forall x, y \in C.$$

(4) B is called *β-inverse-strongly monotone* (see [2, 14]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2, \quad \forall x, y \in C.$$

Remark 1.2. *It is obvious that any β-inverse-strongly monotone mapping B is monotone and $\frac{1}{\beta}$-Lipschitz continuous.*

(5) A mapping T of C into itself is called *nonexpansive* (see [23]) if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T.

(6) Let $f : C \to C$ is said to be a *α-contraction* if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(7) An operator A is *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

(8) A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(9) A monotone mapping $T : H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone map of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [20].

In 1976, Korpelevich [13] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda B x_n), \\ x_{n+1} = P_C(x_n - \lambda B y_n), \end{cases} \tag{1.11}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{\zeta}), C$ is a closed convex subset of $\mathbb{R}^n$ and B is a monotone and ζ-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.11), converge to the same point $z \in VI(C, B)$. For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for an β-inverse-strongly monotone, Takahashi and Toyoda [24] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n), \ \forall n \geq 0, \end{cases} \tag{1.12}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.12) converges weakly to some $z \in F(S) \cap VI(C, B)$. Recently, Iiduka and Takahashi [12] proposed a new iterative scheme as following

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Bx_n), \quad \forall n \geq 0, \end{cases} \tag{1.13}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to some $z \in F(S) \cap VI(C, B)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see e.g., [11, 29, 30, 31] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.14}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Moreover, it is shown in [15] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A)Sx_n \tag{1.15}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [17] proposed the following iterative algorithm:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)Su_n. \end{cases} \tag{1.16}$$

They proved that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \quad \forall x \in F(S) \cap EP(F), \tag{1.17}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.18}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [16] introduced an iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (1.5), the set of solutions to the variational inequality for a monotone and Lipschitz continuous mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert spaces and proved a strong convergence theorem.

Definition 1.1. *[27]. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in [0,1]. For any $n \geq 1$, define a mapping W_n of C into itself as follows:*

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\ &\vdots \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\ U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\ W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{aligned} \tag{1.19}$$

Such a mappings W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$.

On the other hand, Colao et al. [7] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.3) and the set of common fixed points of a finite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{x_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1-\beta)I - \epsilon_n A\big)W_n u_n, \end{cases} \tag{1.20}$$

where $\{\epsilon_n\}$ be a sequences in $(0,1)$. It is proved [7] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.20) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and the unique solution of the variational inequality (1.17), i.e., $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap EP(F)}(I - (A - \gamma f))z$.

In 2009, Colao et al. [9] introduced and considered an implicit iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (1.5) and the set of common fixed points of an infinite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{z_n\}$ recursively by

$$z_n = \epsilon_n \gamma f(z_n) + (1 - \epsilon_n A) W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} z_n, \tag{1.21}$$

where $\{\epsilon_n\}$ be a sequences in $(0,1)$. It is proved [9] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.21) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big)$, where z is the unique solution of the variational inequality (1.17) and which is the optimality condition for the minimization problem (1.18).

In this paper, motivated by Colao et al. [7], Colao et al. [9] and Peng and Yao [16], we introduce a new iterative scheme in a Hilbert space H which is mixed the iterative schemes of (1.20) and (1.21). We prove that the sequence converges strongly to a common element of the set of solutions of the system equilibrium problems (1.5), the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of variational inequality (1.1) for be a monotone and ζ-Lipschitz continuous mapping in Hilbert spaces by using the extragradient approximation method. The results obtained in this paper improve and extend the recent ones announced by Colao, Marino and Xu [7], Colao, Acedo and Marino [9] and Peng and Yao [16] and many others.

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.1. [19] *Let $(C, \langle .,.\rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 2.2. [18]. *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for each $y \in H$ with $y \neq x$.

Lemma 2.3. [15]. *Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\Big\langle x-y, (A-\gamma f)x - (A-\gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x-y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.4. [15]. *Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.5. [1]. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in [4].

Lemma 2.6. [4]. *Assume that $F : C \times C \to R$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$J_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

(1) J_r^F is single- valued;

(2) J_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\left\|J_r^F x - J_r^F y\right\|^2 \leq \left\langle J_r^F x - J_r^F y, x - y\right\rangle;$$

(3) $F(J_r^F) = EP(F)$; and

(4) $EP(F)$ is closed and convex.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (1.19). Then we have the following crucial conclusions concerning W_n. You can find them in [28]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.7. [28]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.7, one can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \tag{2.1}$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$ Throughout this paper, we will assume that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, we have the following results.

Lemma 2.8. [28]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.9. [26]. *If $\{x_n\}$ is a bounded sequence in C, then $\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0$.*

Lemma 2.10. [21] . *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.11. [25]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - l_n)a_n + \sigma_n, \; n \ge 0,$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\sigma_n}{l_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12. . *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$,

(2) $\|x + y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$.

3 Main Results

In this section, we deal with the strong convergence of extragradient approximation method (3.1) for finding a common element of the set of solutions of the system equilibrium problems (1.5), the set of common fixed points of infinite family of nonexpansive mappings and the set of solutions of variational inequality (1.1) for be a monotone and ζ-Lipschitz continuous mapping in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be a monotone and ζ-Lipschitz continuous mapping of C into H such that*

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \epsilon_n \gamma f(W_n x_n) + \beta_n x_n + \big((1 - \beta_n)I - \epsilon_n A\big) W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{W_n\}$ is the sequence generated by (1.19) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\zeta})$ and $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3, \ldots, M\}$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.2}$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

Proof. Note that from the condition (C1), we may assume, without loss of generality, that $\epsilon_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.4, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\Big\{ \big|\langle Ax, x\rangle\big| : x \in H, \|x\| = 1 \Big\}.$$

Observe that

$$\begin{aligned} \Big\langle \big((1 - \beta_n)I - \epsilon_n A\big)x, x \Big\rangle &= 1 - \beta_n - \epsilon_n \langle Ax, x\rangle \\ &\geq 1 - \beta_n - \epsilon_n \|A\| \\ &\geq 0, \end{aligned}$$

this show that $(1-\beta_n)I-\epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n A\| &= \sup\Big\{\Big|\Big\langle\big((1-\beta_n)I-\epsilon_n A\big)x,x\Big\rangle\Big| : x\in H, \|x\|=1\Big\}\\ &= \sup\Big\{1-\beta_n-\epsilon_n\langle Ax,x\rangle : x\in H, \|x\|=1\Big\}\\ &\le 1-\beta_n-\epsilon_n\bar{\gamma}.\end{aligned}$$

Let $Q = P_\Theta$, where $\Theta := \cap_{n=1}^{\infty}F(T_n)\cap\big(\cap_{n=1}^{\infty}SEP(F_k)\big)\cap VI(C,B)\neq\emptyset$. Note that f is a contraction of H into itself with $\alpha\in(0,1)$. Then, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x)-Q(I-A+\gamma f)(y)\| &= \|P_\Theta(I-A+\gamma f)(x)-P_\Theta(I-A+\gamma f)(y)\|\\ &\le \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\ &\le \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|\\ &\le (1-\bar{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &= (1-\bar{\gamma}+\gamma\alpha)\|x-y\|\\ &= \big(1-(\bar{\gamma}-\gamma\alpha)\big)\|x-y\|,\quad \forall x,y\in H.\end{aligned}$$

Since $0<1-(\bar{\gamma}-\gamma\alpha)<1$, it follows that $Q(I-A+\gamma f)$ is a contraction of H into itself. Therefore by the Banach Contraction Mapping Principle, which implies that there exists a unique element $z\in H$ such that $z = Q(I-A+\gamma f)(z) = P_\Theta(I-A+\gamma f)(z)$.

We will divide the proof of Theorem 3.1 into seven steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, pick any $p\in\Theta$. Moreover, by taking $\Im_n^k = J_{r_{k,n}}^{F_k}J_{r_{k-1,n}}^{F_{k-1}}J_{r_{k-2,n}}^{F_{k-2}}\ldots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1}x_n$ for $k\in\{1,2,3,\ldots,M\}$ and $\Im_n^0 = I$ for all n. From the definition of $J_{r_{k,n}}^{F_k}$ is nonexpansive for each $k=1,2,3,\ldots,M$, then $\Im_n^k$ also and $p=\Im_n^k p$, we note that $u_n = \Im_n^M x_n$. If follows that

$$\|u_n-p\| = \|\Im_n^M x_n-\Im_n^M p\|\le\|x_n-p\|.$$

Put $v_n = P_C(u_n-\lambda_n By_n)$. Then, from (1.9) and the monotonicity of B, we have

$$\begin{aligned}\|v_n-p\|^2 &\le \|u_n-\lambda_n By_n-p\|^2-\|u_n-\lambda_n By_n-v_n\|^2\\ &= \|u_n-p\|^2-\|u_n-v_n\|^2+2\lambda_n\langle By_n,p-v_n\rangle\\ &= \|u_n-p\|^2-\|u_n-v_n\|^2\\ &\quad +2\lambda_n\big(\langle By_n-Bp,p-y_n\rangle+\langle Bp,p-y_n\rangle+\langle By_n,y_n-v_n\rangle\big)\\ &\le \|u_n-p\|^2-\|u_n-v_n\|^2+2\lambda_n\langle By_n,y_n-v_n\rangle\\ &= \|u_n-p\|^2-\|u_n-y_n\|^2-2\langle u_n-y_n,y_n-v_n\rangle-\|y_n-v_n\|^2\\ &\quad +2\lambda_n\langle By_n,y_n-v_n\rangle\\ &= \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2\\ &\quad +2\langle u_n-\lambda_n By_n-y_n,v_n-y_n\rangle.\end{aligned}$$

Moreover, since $y_n = P_C(u_n - \lambda_n B u_n)$ and (1.8), we have

$$\langle u_n - \lambda_n B u_n - y_n, v_n - y_n \rangle \leq 0. \tag{3.3}$$

Since A is ζ-Lipschitz continuous, from (3.3) we obtain that

$$\begin{aligned} &\langle u_n - \lambda_n B y_n - y_n, v_n - y_n \rangle \\ &\quad = \langle u_n - \lambda_n B u_n - y_n, v_n - y_n \rangle + \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n \rangle \\ &\quad \leq \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n \rangle \\ &\quad \leq \lambda_n \|B u_n - B y_n\| \|v_n - y_n\| \\ &\quad \leq \lambda_n \zeta \|u_n - y_n\| \|v_n - y_n\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|v_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \zeta \|u_n - y_n\| \|v_n - y_n\| \\ &\leq \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2 \zeta^2 \|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|u_n - p\|^2 + (\lambda_n^2 \zeta^2 - 1)\|u_n - y_n\|^2 \qquad (3.4) \\ &\leq \|u_n - p\|^2, \end{aligned}$$

and hence

$$\|v_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.5}$$

Thus, we can calculate

$$\begin{aligned} \|x_{n+1} - p\| &= \Big\| \epsilon_n \big(\gamma f(W_n x_n) - Ap\big) + \beta_n (x_n - p) + \big((1-\beta_n) I - \epsilon_n A\big)(W_n v_n - p) \Big\| \\ &\leq (1 - \beta_n - \epsilon_n \bar{\gamma}) \|v_n - p\| + \beta_n \|x_n - p\| + \epsilon_n \|\gamma f(W_n x_n) - Ap\| \\ &\leq (1 - \beta_n - \epsilon_n \bar{\gamma}) \|x_n - p\| + \beta_n \|x_n - p\| + \epsilon_n \|\gamma f(W_n x_n) - Ap\| \\ &= (1 - \epsilon_n \bar{\gamma}) \|x_n - p\| + \epsilon_n \gamma \|f(W_n x_n) - f(p)\| + \epsilon_n \|\gamma f(p) - Ap\| \\ &\leq (1 - \epsilon_n \bar{\gamma}) \|x_n - p\| + \epsilon_n \gamma \alpha \|x_n - p\| + \epsilon_n \|\gamma f(p) - Ap\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)\epsilon_n) \|x_n - p\| + (\bar{\gamma} - \gamma\alpha)\epsilon_n \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}. \qquad (3.6) \end{aligned}$$

By induction that

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha} \right\}, \quad n \in \mathbb{N}. \tag{3.7}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{Bu_n\}$, $\{Bv_n\}$, $\{W_n v_n\}$ and $\big\{f(W_n x_n)\big\}$.

Step 2. We claim that, if ω_n be a bounded sequence in C. Then

$$\lim_{n\to\infty} \|\Im_n^k \omega_n - \Im_{n+1}^k \omega_n\| = 0, \tag{3.8}$$

for every $k \in \{1, 2, 3, \ldots, M\}$. From Step 2 of the proof Theorem 3.1 in [7], we have that for $k \in \{1, 2, 3, \ldots, M\}$,

$$\lim_{n\to\infty} \|J_{r_{k,n+1}}^{F_k} \omega_n - J_{r_{k,n}}^{F_k} \omega_n\| = 0. \tag{3.9}$$

Note that for every $k \in \{1,2,3,\ldots,M\}$, we obtain

$$\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} = J_{r_{k,n}}^{F_k} \Im_n^{k-1}.$$

So, we have

$$\begin{aligned}
& \|\Im_n^k \omega_n - \Im_{n+1}^k \omega_n\| \qquad (3.10)\\
= & \|J_{r_{k,n}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1}\omega_n\| \\
\leq & \|J_{r_{k,n}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1}\omega_n\| + \|J_{r_{k,n+1}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1}\omega_n\| \\
\leq & \|J_{r_{k,n}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1}\omega_n\| + \|\Im_n^{k-1}\omega_n - \Im_{n+1}^{k-1}\omega_n\| \\
\leq & \|J_{r_{k,n}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1}\omega_n\| + \|J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2}\omega_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2}\omega_n\| \\
& + \|\Im_n^{k-2}\omega_n - \Im_{n+1}^{k-2}\omega_n\| \\
\leq & \|J_{r_{k,n}}^{F_k} \Im_n^{k-1}\omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1}\omega_n\| + \|J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2}\omega_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2}\omega_n\| \\
& + \ldots + \|J_{r_{2,n}}^{F_2} \Im_n^1\omega_n - J_{r_{2,n+1}}^{F_2} \Im_n^1\omega_n\| + \|J_{r_{1,n}}^{F_1}\omega_n - J_{r_{1,n+1}}^{F_1}\omega_n\|.
\end{aligned}$$

Now, apply (3.9) to conclude (3.8).

Step 3. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
On the other hand, from $u_n = \Im_n^M x_n$ and $u_{n+1} = \Im_{n+1}^M x_{n+1}$, by the triangular inequality, we also have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|\Im_{n+1}^M x_{n+1} - \Im_n^M x_n\| \\
&= \|\Im_{n+1}^M x_{n+1} - \Im_{n+1}^M x_n\| + \|\Im_{n+1}^M x_n - \Im_n^M x_n\| \\
&\leq \|x_{n+1} - x_n\| + \|\Im_{n+1}^M x_n - \Im_n^M x_n\|. \qquad (3.11)
\end{aligned}$$

Indeed, we observe that for any $x, y \in C$,

$$\begin{aligned}
\|(I-\lambda_n B)x - (I-\lambda_n B)y\|^2 &= \|(x-y) - \lambda_n(Bx - By)\|^2 \\
&= \|x-y\|^2 - 2\lambda_n\langle x-y, Bx-By\rangle + \lambda_n^2\|Bx-By\|^2 \\
&\leq \|x-y\|^2 + \lambda_n^2\zeta^2\|x-y\|^2 \\
&= (1+\lambda_n^2\zeta^2)\|x-y\|^2, \qquad (3.12)
\end{aligned}$$

which implies that

$$\|(I-\lambda_n A)x - (I-\lambda_n A)y\| \leq (1+\lambda_n\zeta)\|x-y\|. \qquad (3.13)$$

Note that

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}By_{n+1}) - P_C(u_n - \lambda_n By_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1}By_{n+1} - (u_n - \lambda_n By_n)\| \\
&= \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n) \\
&\quad + \lambda_{n+1}(Bu_{n+1} - By_{n+1} - Bu_n) + \lambda_n By_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1}Bu_n)\| \\
&\quad + \lambda_{n+1}(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|) + \lambda_n\|By_n\| \\
&\leq (1+\lambda_{n+1}\zeta)\|u_{n+1} - u_n\| + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\
&\quad + \lambda_n\|By_n\|. \qquad (3.14)
\end{aligned}$$

Substituting (3.11) into (3.14), we have

$$\begin{aligned}\|v_{n+1}-v_n\| &\leq (1+\lambda_{n+1}\zeta)\|u_{n+1}-u_n\| + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) + \lambda_n\|By_n\| \\ &\leq (1+\lambda_{n+1}\zeta)\|x_{n+1}-x_n\| + (1+\lambda_{n+1}\zeta)\|\Im^M_{n+1}x_n - \Im^M_n x_n\| \\ &\quad + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) + \lambda_n\|By_n\|. \qquad (3.15)\end{aligned}$$

Setting

$$z_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} = \frac{\epsilon_n\gamma f(W_n x_n) + \big((1-\beta_n)I - \epsilon_n A\big)W_n v_n}{1-\beta_n},$$

we have $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, $n \geq 1$. It follows that

$$\begin{aligned}z_{n+1}-z_n &= \frac{\epsilon_{n+1}\gamma f(W_{n+1}x_{n+1}) + \big((1-\beta_{n+1})I - \epsilon_{n+1}A\big)W_{n+1}v_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\epsilon_n\gamma f(W_n x_n) + \big((1-\beta_n)I - \epsilon_n A\big)W_n v_n}{1-\beta_n} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(W_{n+1}x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(W_n x_n) + W_{n+1}v_{n+1} - W_n v_n \\ &\quad + \frac{\epsilon_n}{1-\beta_n}AW_n v_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}AW_{n+1}v_{n+1} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\gamma f(W_{n+1}x_{n+1}) - AW_{n+1}v_{n+1}\big) + \frac{\epsilon_n}{1-\beta_n}\big(AW_n v_n - \gamma f(W_n x_n)\big) \\ &\quad + W_{n+1}v_{n+1} - W_{n+1}v_n + W_{n+1}v_n - W_n v_n. \qquad (3.16)\end{aligned}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\ &\quad + \frac{\epsilon_n}{1-\beta_n}\big(\|AW_n v_n\| + \|\gamma f(W_n x_n)\|\big) \\ &\quad + \|W_{n+1}v_{n+1} - W_{n+1}v_n\| \\ &\quad + \|W_{n+1}v_n - W_n v_n\| - \|x_{n+1}-x_n\| \\ &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\ &\quad + \frac{\epsilon_n}{1-\beta_n}\big(\|AW_n v_n\| + \|\gamma f(W_n x_n)\|\big) + \|v_{n+1}-v_n\| \\ &\quad + \|W_{n+1}v_n - W_n v_n\| - \|x_{n+1}-x_n\| \\ &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\ &\quad + \frac{\epsilon_n}{1-\beta_n}\big(\|AW_n v_n\| + \|\gamma f(W_n x_n)\|\big) + \lambda_{n+1}\zeta\|x_{n+1}-x_n\| \\ &\quad + (1+\lambda_{n+1}\zeta)\|\Im^M_{n+1}x_n - \Im^M_n x_n\| \\ &\quad + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\ &\quad + \lambda_n\|By_n\| + \|W_{n+1}v_n - W_n v_n\|. \qquad (3.17)\end{aligned}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}v_n - W_nv_n\| &= \|\mu_1T_1U_{n+1,2}v_n - \mu_1T_1U_{n,2}v_n\| \\
&\leq \mu_1\|U_{n+1,2}v_n - U_{n,2}v_n\| \\
&= \mu_1\|\mu_2T_2U_{n+1,3}v_n - \mu_2T_2U_{n,3}v_n\| \\
&\leq \mu_1\mu_2\|U_{n+1,3}v_n - U_{n,3}v_n\| \\
&\vdots \\
&\leq \mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}v_n - U_{n,n+1}v_n\| \\
&\leq M_1\prod_{i=1}^{n}\mu_i, \qquad (3.18)
\end{aligned}$$

where $M_1 \geq 0$ is a constant such that $\|U_{n+1,n+1}v_n - U_{n,n+1}v_n\| \leq M_1, \quad \forall n \geq 0$. Combining (3.17) and (3.18), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq\; & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(W_{n+1}x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\
& + \frac{\epsilon_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(W_nx_n)\|\big) + \lambda_{n+1}\zeta\|x_{n+1} - x_n\| \\
& + (1+\lambda_{n+1}\zeta)\|\Im_{n+1}^{M}x_n - \Im_n^{M}x_n\| \\
& + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\
& + \lambda_n\|By_n\| + M_1\prod_{i=1}^{n}\mu_i.
\end{aligned}$$

which implies that (noting that (C1), (C2), (C4) and $0 < \mu_i \leq b < 1, \quad \forall i \geq 1$)

$$\limsup_{n\to\infty}\big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Hence, by Lemma 2.10, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \qquad (3.19)$$

Applying (3.8), (3.19) and (C4) to (3.11) and (3.14), we obtain that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = \lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \qquad (3.20)$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - W_nv_n\| = 0$.
Since $x_{n+1} = \epsilon_n\gamma f(W_nx_n) + \beta_nx_n + ((1-\beta_n)I - \epsilon_nA)W_nv_n$, we have

$$\begin{aligned}
\|x_n - W_n v_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n v_n\| \\
&= \|x_n - x_{n+1}\| + \Big\|\epsilon_n \gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)W_n v_n - W_n v_n\Big\| \\
&= \|x_n - x_{n+1}\| + \Big\|\epsilon_n\big(\gamma f(W_n x_n) - AW_n v_n\big) + \beta_n(x_n - W_n v_n)\Big\| \\
&\leq \|x_n - x_{n+1}\| + \epsilon_n\big(\|\gamma f(W_n x_n)\| + \|AW_n v_n\|\big) + \beta_n\|x_n - W_n v_n\|,
\end{aligned}$$

that is

$$\|x_n - W_n v_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\epsilon_n}{1-\beta_n}\big(\|\gamma f(W_n x_n)\| + \|AW_n v_n\|\big).$$

By (C1), (C2) and (3.19) it follows that

$$\lim_{n\to\infty}\|W_n v_n - x_n\| = 0. \tag{3.21}$$

Step 5. We claim that the following statements hold:

1. $\lim_{n\to\infty}\|u_n - y_n\| = 0$;
2. $\lim_{n\to\infty}\|u_n - v_n\| = 0$;
3. $\lim_{n\to\infty}\|W_n y_n - y_n\| = 0$.

For any $p \in \Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{n=1}^{\infty} SEP(F_k)\big) \cap VI(C,B)$ and (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \Big\|\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - p) + \beta_n(x_n - p) + \epsilon_n\big(\gamma f(W_n x_n) - Ap\big)\Big\|^2 \\
&= \|\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - p) + \beta_n(x_n - p)\|^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\Big\langle\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - p), \gamma f(W_n x_n) - Ap\Big\rangle \\
&\leq \big[(1-\beta_n-\epsilon_n\bar\gamma)\|W_n v_n - p\| + \beta_n\|x_n - p\|\big]^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Ap\|^2 \\
&\quad + 2\beta_n\epsilon_n\langle x_n - p, \gamma f(W_n x_n) - Ap\rangle \\
&\quad + 2\epsilon_n\Big\langle\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - p), \gamma f(W_n x_n) - Ap\Big\rangle \\
&\leq \big[(1-\beta_n-\epsilon_n\bar\gamma)\|v_n - p\| + \beta_n\|x_n - p\|\big]^2 + c_n \\
&\leq (1-\beta_n-\epsilon_n\bar\gamma)^2\|v_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + 2(1-\beta_n-\epsilon_n\bar\gamma)\beta_n\|v_n - p\|\|x_n - p\| + c_n \\
&\leq (1-\beta_n-\epsilon_n\bar\gamma)^2\|v_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\epsilon_n\bar\gamma)\beta_n\big(\|v_n - p\|^2 + \|x_n - p\|^2\big) + c_n \\
&= \big[(1-\epsilon_n\bar\gamma)^2 - 2(1-\epsilon_n\bar\gamma)\beta_n + \beta_n^2\big]\|v_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + \big((1-\epsilon_n\bar\gamma)\beta_n - \beta_n^2\big)\big(\|v_n - p\|^2 + \|x_n - p\|^2\big) + c_n \\
&= (1-\epsilon_n\bar\gamma)^2\|v_n - p\|^2 - (1-\epsilon_n\bar\gamma)\beta_n\|v_n - p\|^2 + (1-\epsilon_n\bar\gamma)\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)\|v_n - p\|^2 + (1-\epsilon_n\bar\gamma)\beta_n\|x_n - p\|^2 + c_n,
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned} c_n &= \epsilon_n^2\|\gamma f(W_nx_n)-Ap\|^2+2\beta_n\epsilon_n\langle x_n-p,\gamma f(W_nx_n)-Ap\rangle \\ &\quad +2\epsilon_n\Big\langle\big((1-\beta_n)I-\epsilon_nA\big)(W_nv_n-p),\gamma f(W_nx_n)-Ap\Big\rangle. \end{aligned}$$

It follows from condition (C1) that

$$\lim_{n\to\infty} c_n=0. \tag{3.23}$$

Substituting (3.4) into (3.22), and using (C4), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\le (1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)\|v_n-p\|^2+(1-\epsilon_n\bar\gamma)\beta_n\|x_n-p\|^2+c_n \\ &\le (1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)\Big\{\|u_n-p\|^2+(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2\Big\} \\ &\quad +(1-\epsilon_n\bar\gamma)\beta_n\|x_n-p\|^2+c_n \\ &\le (1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)\Big\{\|x_n-p\|^2+(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2\Big\} \\ &\quad +(1-\epsilon_n\bar\gamma)\beta_n\|x_n-p\|^2+c_n \\ &= (1-\epsilon_n\bar\gamma)^2\|x_n-p\|^2+(1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n \\ &\le \|x_n-p\|^2+(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n. \end{aligned}$$

It follows that

$$\begin{aligned} (1-\lambda_n^2\zeta^2)\|u_n-y_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n \\ &= (\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|)+c_n \\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n. \end{aligned}$$

Since $\lim_{n\to\infty} c_n=0$ and from (3.19), we obtain

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.24}$$

By the same argument as in (3.4), we obtain

$$\begin{aligned} \|v_n-p\|^2 &\le \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2+2\lambda_n\zeta\|u_n-y_n\|\|v_n-y_n\| \\ &\le \|u_n-p\|^2-\|u_n-y_n\|^2-\|y_n-v_n\|^2+\|u_n-y_n\|^2+\lambda_n^2\zeta^2\|v_n-y_n\|^2 \\ &= \|u_n-p\|^2+(\lambda_n^2\zeta^2-1)\|y_n-v_n\|^2 \\ &\le \|x_n-p\|^2+(\lambda_n^2\zeta^2-1)\|y_n-v_n\|^2. \end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.22), and using (C4), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\le (1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)\Big\{\|x_n-p\|^2+(\lambda_n^2\zeta^2-1)\|y_n-v_n\|^2\Big\} \\ &\quad +(1-\epsilon_n\bar\gamma)\beta_n\|x_n-p\|^2+c_n \\ &= (1-\epsilon_n\bar\gamma)^2\|x_n-p\|^2+(1-\epsilon_n\bar\gamma)(1-\beta_n-\epsilon_n\bar\gamma)(\lambda_n^2\zeta^2-1)\|y_n-v_n\|^2+c_n \\ &\le \|x_n-p\|^2+(\lambda_n^2\zeta^2-1)\|y_n-v_n\|^2+c_n. \end{aligned}$$

It follows that

$$\begin{aligned}(1-\lambda_n^2\zeta^2)\|y_n-v_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\ &\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.\end{aligned}$$

Since $\lim_{n\to\infty} c_n = 0$ and from (3.19), we obtain

$$\lim_{n\to\infty}\|y_n-v_n\|=0. \tag{3.26}$$

On the other hand, we observe that

$$\|u_n-v_n\| \leq \|u_n-y_n\|+\|y_n-v_n\|.$$

Applying (3.24)and (3.26), we have

$$\lim_{n\to\infty}\|u_n-v_n\|=0. \tag{3.27}$$

For any $p \in \Theta$, note that $J^{F_k}_{r_{k,n}}$ is firmly nonexpansive (Lemma 2.6) for $k \in \{1, 2, 3, \dots, M\}$, then we have

$$\begin{aligned}\|\Im_n^k x_n-p\|^2 &= \|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n-J^{F_k}_{r_{k,n}}p\|^2\\ &\leq \left\langle J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n-J^{F_k}_{r_{k,n}}p,\Im_n^{k-1}x_n-p\right\rangle\\ &= \left\langle \Im_n^k x_n-p,\Im_n^{k-1}x_n-p\right\rangle\\ &= \frac{1}{2}\Big(\|\Im_n^k x_n-p\|^2+\|\Im_n^{k-1}x_n-p\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\Big)\end{aligned}$$

and hence

$$\|\Im_n^k x_n-p\|^2 \leq \|\Im_n^{k-1}x_n-p\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2, \quad k=1,2,3,\dots,M$$

which implies that for each $k \in \{1, 2, 3, \dots, M\}$,

$$\begin{aligned}\|\Im_n^k x_n-p\|^2 &\leq \|\Im_n^0 x_n-p\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\ &\quad -\|\Im_n^{k-1}x_n-\Im_n^{k-2}x_n\|^2-\ldots-\|\Im_n^2 x_n-\Im_n^1 x_n\|^2-\|\Im_n^1 x_n-\Im_n^0 x_n\|^2\\ &\leq \|x_n-p\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2.\end{aligned}$$

Together with (3.22) gives

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\beta_n-\epsilon_n\bar{\gamma})\|v_n-p\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\Big\{\|u_n-p\|^2+(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2\Big\}\\
&+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|u_n-p\|^2+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-p\|^2+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
\le\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\Big\{\|x_n-p\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\Big\}\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+(1-\epsilon_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
=\ & (1-\epsilon_n\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n\\
=\ & \big[1-2\epsilon_n\bar{\gamma}+(\epsilon_n\bar{\gamma})^2\big]\|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n\\
\le\ & \|x_n-p\|^2+(\epsilon_n\bar{\gamma})^2\|x_n-p\|^2-(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
\le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\epsilon_n\bar{\gamma})^2\|x_n-p\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n\\
\le\ & \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+(\epsilon_n\bar{\gamma})^2\|x_n-p\|^2\\
&+(1-\epsilon_n\bar{\gamma})(1-\epsilon_n\bar{\gamma}-\beta_n)(\lambda_n^2\zeta^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

Using $\epsilon_n\to 0$, $c_n\to 0$ as $n\to\infty$, (3.19) and (3.24), we obtain

$$\lim_{n\to\infty}\|\Im_n^k x_n-\Im_n^{k-1}x_n\|=0. \tag{3.28}$$

Observe that

$$\begin{aligned}
\|W_ny_n-y_n\| \le\ & \|W_ny_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|\\
\le\ & \|y_n-v_n\|+\|W_nv_n-x_n\|+\|x_n-\Im_n^k x_n\|+\|u_n-y_n\|\\
\le\ & \|y_n-v_n\|+\|W_nv_n-x_n\|+\|\Im_n^0 x_n-\Im_n^1 x_n\|+\|\Im_n^1 x_n-\Im_n^2 x_n\|\\
&+\ldots+\|\Im_n^{M-1}x_n-\Im_n^M x_n\|+\|u_n-y_n\|
\end{aligned}$$

Applying (3.21), (3.24), (3.26) and (3.28) to the last inequality, we obtain

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{3.29}$$

Let W be the mapping defined by (2.1). Since $\{y_n\}$ is bounded, Applying Lemma 2.9 and (3.29), we have

$$\|Wy_n - y_n\| \leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.30}$$

Step 6. We claim that $\limsup_{n\to\infty}\big\langle (A-\gamma f)z, z - x_n\big\rangle \leq 0$, which z is the unique solution of the variational inequality $\big\langle (A-\gamma f)z, x - z\big\rangle \geq 0,\ \ \forall x \in \Theta$.

Since $z = P_\Theta(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.2). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\big\langle (A-\gamma f)z, z - x_{n_i}\big\rangle = \limsup_{n\to\infty}\big\langle (A-\gamma f)z, z - x_n\big\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $\{x_{n_i}\} \rightharpoonup w$. Since $\lim_{n\to\infty}\|\Im_n^k x_n - \Im_n^{k-1}x_n\| = 0$ for $k = 1,2,3,\ldots,M$, we have $\Im_{n_i}^k x_{n_i} \rightharpoonup w$ for $k = 1,2,3,\ldots,M$.

From $\|u_n - y_n\| \to 0$ and $\|u_n - v_n\| \to 0$, we obtain $y_{n_i} \rightharpoonup w$ and $v_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

Next, we show that $w \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B)$.

First, we show that $w \in \cap_{k=1}^{M} SEP(F_k)$. Since $u_n = \Im_n^k x_n$ for $k = 1,2,3,\ldots,M$, we also have

$$F_k(\Im_n^k x_n, y) + \frac{1}{r_n}\langle y - \Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \geq 0, \quad \forall y \in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - \Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \geq -F_k(\Im_n^k x_n, y) \geq F_k(y, \Im_n^k x_n)$$

and hence

$$\Big\langle y - \Im_{n_i}^k x_{n_i}, \frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\Big\rangle \geq F_k(y, \Im_{n_i}^k x_{n_i}).$$

Since $\frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}} \to 0$ and $\Im_{n_i}^k x_{n_i} \rightharpoonup w$, it follows by (A4) that

$$F_k(y, w) \leq 0 \quad \forall y \in C,$$

for each $k = 1,2,3,\ldots,M$.

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F_k(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 \;=\; F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, w) \leq tF_k(y_t, y)$$

and hence $F_k(y_t, y) \geq 0$. From (A3), we have $F_k(w, y) \geq 0$ for all $y \in C$ and hence $w \in EP(F_k)$ for $k = 1, 2, 3, \ldots, M$, that is, $w \in \cap_{k=1}^{M} SEP(F_k)$.

Next, we show that $w \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 2.8, we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $w \notin F(W)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Ww$, it follows by the Opial's condition (Lemma 2.2) that

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty} \big\{ \|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\| \big\} \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - w\|
\end{aligned}$$

which derives a contradiction. Thus, we have $w \in F(W) = \cap_{n=1}^{\infty} F(T_n)$.

Finally, we show that $w \in VI(C, B)$. Define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone. Let $(v, w_1) \in G(T)$. Since $w_1 - Bv \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, w_1 - Bv\rangle \geq 0$. On the other hand, $v_n = P_C(u_n - \lambda_n B y_n)$, we have

$$\langle v - v_n, v_n - (u_n - \lambda_n B y_n)\rangle \geq 0,$$

and hence

$$\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + By_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, w\rangle &\geq \langle v - v_{n_i}, Bv\rangle \\
&\geq \langle v - v_{n_i}, Bv\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + By_{n_i} \right\rangle \\
&= \left\langle v - v_{n_i}, Bv - By_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - v_{n_i}, Bv - Bv_{n_i}\rangle + \langle v - v_{n_i}, Bv_{n_i} - By_{n_i}\rangle \\
&\quad - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - v_{n_i}, Bv_{n_i} - By_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \|v_n - y_n\| = 0$, $u_{n_i} \rightharpoonup w$ and B is Lipschitz continuous, we obtain that $\lim_{n\to\infty} \|Bv_n - By_n\| = 0$ and $v_{n_i} \rightharpoonup p$. From $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty} \|v_n - u_n\| = 0$, we obtain

$$\langle v - w, w_1\rangle \geq 0.$$

Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, B)$. Hence $w \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B)$.

Since $z = P_\Theta(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle (A-\gamma f)z, z - x_n\big\rangle &= \limsup_{n\to\infty}\big\langle (A-\gamma f)z, z - x_n\big\rangle \\ &= \lim_{i\to\infty}\big\langle (A-\gamma f)z, z - x_{n_i}\big\rangle \\ &= \big\langle (A-\gamma f)z, z - w\big\rangle \le 0. \end{aligned} \tag{3.31}$$

It follows from the last inequality and (3.21) that

$$\limsup_{n\to\infty}\big\langle \gamma f(z) - Az, W_n v_n - z\big\rangle \le 0. \tag{3.32}$$

Step 7. Finally, we claim that $\{x_n\}$ converges strongly to $z = P_\Theta(I - A + \gamma f)(z)$.
Indeed, from (3.1) , we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \qquad (3.33)\\
= {} & \big\|\epsilon_n\gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)W_n v_n - z\big\|^2 \\
= {} & \|\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - z) + \beta_n(x_n - z) + \epsilon_n\big(\gamma f(W_n x_n) - Az\big)\|^2 \\
= {} & \|\big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - z) + \beta_n(x_n - z)\|^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Az\|^2 \\
& + 2\beta_n\epsilon_n\big\langle x_n - z, \gamma f(W_n x_n) - Az\big\rangle \\
& + 2\epsilon_n\Big\langle \big((1-\beta_n)I - \epsilon_n A\big)(W_n v_n - z), \gamma f(W_n x_n) - Az\Big\rangle \\
\le {} & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|W_n v_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Az\|^2 \\
& + 2\beta_n\epsilon_n\gamma\big\langle x_n - z, f(W_n x_n) - f(z)\big\rangle + 2\beta_n\epsilon_n\big\langle x_n - z, \gamma f(z) - Az\big\rangle \\
& + 2(1-\beta_n)\gamma\epsilon_n\big\langle W_n v_n - z, f(W_n x_n) - f(z)\big\rangle + 2(1-\beta_n)\epsilon_n\big\langle W_n v_n - z, \gamma f(z) - Az\big\rangle \\
& - 2\epsilon_n^2\big\langle A(W_n v_n - z), \gamma f(z) - Az\big\rangle \\
\le {} & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|W_n v_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Az\|^2 \\
& + 2\beta_n\epsilon_n\gamma\|x_n - z\|\|f(W_n x_n) - f(z)\| + 2\beta_n\epsilon_n\big\langle x_n - z, \gamma f(z) - Az\big\rangle \\
& + 2(1-\beta_n)\gamma\epsilon_n\|W_n v_n - z\|\|f(W_n x_n) - f(z)\| + 2(1-\beta_n)\epsilon_n\big\langle W_n v_n - z, \gamma f(z) - Az\big\rangle \\
& - 2\epsilon_n^2\big\langle A(W_n v_n - z), \gamma f(z) - Az\big\rangle \\
\le {} & \Big[(1-\beta_n-\epsilon_n\bar{\gamma})\|x_n - z\| + \beta_n\|x_n - z\|\Big]^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Az\|^2 \\
& + 2\beta_n\epsilon_n\gamma\alpha\|x_n - z\|^2 + 2\beta_n\epsilon_n\big\langle x_n - z, \gamma f(z) - Az\big\rangle \\
& + 2(1-\beta_n)\gamma\epsilon_n\alpha\|x_n - z\|^2 + 2(1-\beta_n)\epsilon_n\big\langle W_n v_n - z, \gamma f(z) - Az\big\rangle \\
& - 2\epsilon_n^2\big\langle A(W_n v_n - z), \gamma f(z) - Az\big\rangle \\
= {} & \Big[(1-\epsilon_n\bar{\gamma})^2 + 2\beta_n\epsilon_n\gamma\alpha + 2(1-\beta_n)\gamma\epsilon_n\alpha\Big]\|x_n - z\|^2 + \epsilon_n^2\|\gamma f(W_n x_n) - Az\|^2 \\
& + 2\beta_n\epsilon_n\big\langle x_n - z, \gamma f(z) - Az\big\rangle + 2(1-\beta_n)\epsilon_n\big\langle W_n v_n - z, \gamma f(z) - Az\big\rangle \\
& - 2\epsilon_n^2\big\langle A(W_n v_n - z), \gamma f(z) - Az\big\rangle
\end{aligned}$$

(3.34)

$$
\begin{aligned}
&\leq \big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\bar{\gamma}^2\epsilon_n^2\|x_n-z\|^2+\epsilon_n^2\|\gamma f(W_nx_n)-Az\|^2 \\
&\quad +2\beta_n\epsilon_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle+2(1-\beta_n)\epsilon_n\big\langle W_nv_n-z,\gamma f(z)-Az\big\rangle \\
&\quad +2\epsilon_n^2\|A(W_nv_n-z)\|\|\gamma f(z)-Az\| \\
&= \big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\epsilon_n\bigg\{\epsilon_n\Big[\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(W_nx_n)-Az\|^2 \\
&\quad +2\|A(W_nv_n-z)\|\|\gamma f(z)-Az\|\Big]+2\beta_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle \\
&\quad +2(1-\beta_n)\big\langle W_nv_n-z,\gamma f(z)-Az\big\rangle\bigg\}
\end{aligned}
$$

Since $\{x_n\}$, $\{f(W_nx_n)\}$ and $\{W_nv_n\}$ are bounded, we can take a constant $K>0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(W_nx_n)-Az\|^2+2\|A(W_nv_n-z)\|\|\gamma f(z)-Az\|\leq K,$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-z\|^2\leq\big[1-2(\bar{\gamma}-\alpha\gamma)\epsilon_n\big]\|x_n-z\|^2+\epsilon_n\sigma_n, \tag{3.35}$$

where

$$\sigma_n=2\beta_n\big\langle x_n-z,\gamma f(z)-Az\big\rangle+2(1-\beta_n)\big\langle W_nv_n-z,\gamma f(z)-Az\big\rangle+\epsilon_nK.$$

Using (C1), (3.31) and (3.32), we get $\limsup_{n\to\infty}\sigma_n\leq 0$. Applying Lemma 2.11 to (3.35), we conclude that $x_n\to z$ in norm. This completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k\in\{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be a monotone and ζ-Lipschitz continuous mapping of C into H such that*

$$\Theta:=\big(\cap_{k=1}^{M}SEP(F_k)\big)\cap VI(C,B)\neq\emptyset.$$

Let f be a contraction of H into itself with $\alpha\in(0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$
\begin{cases}
x_1=x\in C \text{ chosen arbitrary},\\
u_n=J^{F_M}_{r_{M,n}}J^{F_{M-1}}_{r_{M-1,n}}J^{F_{M-2}}_{r_{M-2,n}}\cdots J^{F_2}_{r_{2,n}}J^{F_1}_{r_{1,n}}x_n,\\
y_n=P_C(u_n-\lambda_nBu_n),\\
x_{n+1}=\epsilon_n\gamma f(x_n)+\beta_nx_n+\big((1-\beta_n)I-\epsilon_nA\big)P_C(u_n-\lambda_nBy_n),\quad\forall n\geq 1,
\end{cases}
$$

where $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\lambda_n\}\subset[a,b]\subset(0,\frac{1}{\zeta})$ and $\{r_{k,n}\}, k\in\{1,2,3,\ldots,M\}$ are real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\epsilon_n=0$ *and* $\sum_{n=1}^{\infty}\epsilon_n=\infty$,

(C2) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,

(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

Proof. Put $T_n = I$ for all $n \in \mathbb{N}$ and for all $x \in E$. Then $W_n = I$ for all $x \in E$. The conclusion follows from Theorem 3.1. This completes the proof. □

If $A = I, \gamma \equiv 1$ and $\gamma_n = 1 - \epsilon_n - \beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*, let* $F_k, k \in \{1, 2, 3, \ldots, M\}$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4), let* $\{T_n\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* B *be a monotone and* ζ*-Lipschitz continuous mapping of* C *into* H *such that*

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C, B) \neq \emptyset.$$

Let f *be a contraction of* H *into itself with* $\alpha \in (0, 1)$*. Let* $\{x_n\}$*,* $\{y_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C \ \text{chosen arbitrary}, \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \epsilon_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_C(u_n - \lambda_n B y_n), \ \ \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ *is the sequence generated by (1.19)and* $\{\epsilon_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are sequences in* $(0, 1)$*,* $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\zeta})$ *and* $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ *are a real sequence in* $(0, \infty)$ *satisfy the following conditions:*

(C1) $\epsilon_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3, \ldots, M\}$,

(C5) $\lim_{n\to\infty} \lambda_n = 0$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Theta$ *which is the unique solution of the variational inequality*

$$\Big\langle z - f(z), x - z \Big\rangle \geq 0, \ \ \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta f(z)$.

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