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# Γ-Paired Dominating Graphs of Some Paths and Some Cycles

**Pannawat Eakawinrujee and Nantapath Trakultraipruk***

*Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 12120, Thailand*
*e-mail : p.eakawinrujee@gmail.com (P. Eakawinrujee); n.trakultraipruk@yahoo.com (N. Trakultraipruk)*

**Abstract** A set $D$ of vertices in a graph $G$ is a paired dominating set of $G$ if every vertex of $G$ is adjacent to some vertex in $D$ and the subgraph of $G$ induced by $D$ contains a perfect matching. The maximum cardinality of a minimal paired dominating set of $G$ is called the upper paired domination number, denoted by $\Gamma_{pr}(G)$. A minimal paired dominating set with cardinality $\Gamma_{pr}(G)$ is a $\Gamma_{pr}(G)$-set. The $\Gamma$-paired dominating graph $PD_\Gamma(G)$ of $G$ is the graph whose vertices are $\Gamma_{pr}(G)$-sets and any two $\Gamma_{pr}(G)$-sets are adjacent in $PD_\Gamma(G)$ if they differ by exactly one vertex. In this paper, we first correct some results on the $\Gamma$-paired dominating graphs of some paths that appeared in [P. Eakawinrujee, N. Trakultraipruk, $\Gamma$-paired dominating graphs of some paths, MATEC Web Conf. 189 (2018) 03029] and then we determine the $\Gamma$-paired dominating graphs of some cycles.

**MSC:** 05C69
**Keywords:** paired dominating graph; upper paired dominating set; upper paired domination number; gamma graph

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## 1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $D \subseteq V(G)$ is a *dominating set* of $G$ if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in $D$. The *domination number* $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. A $\gamma(G)$*-set* is a dominating set of $G$ with cardinality $\gamma(G)$. For detailed surveys on domination parameters, see [11, 12].

The *gamma graph* $\gamma \cdot G$ of $G$ is the graph with $V(\gamma \cdot G)$ as the set of all $\gamma(G)$-sets, and two vertices $D_1$ and $D_2$ of $\gamma \cdot G$ are adjacent if they satisfy the condition as follows:

$$D_2 = (D_1 \setminus \{u\}) \cup \{v\} \text{ for some } u \in D_1 \text{ and } v \notin D_1. \tag{1.1}$$

The gamma graph $\gamma \cdot G$ was introduced by Subramanian and Sridharan [18]. For additional results on $\gamma \cdot G$, see [2, 13, 16, 17]

*Corresponding author.

The *gamma graph* $G(\gamma)$ of $G$ is the graph with $V(G(\gamma)) = V(\gamma\cdot G)$, and two vertices of $G(\gamma)$ are adjacent if they satisfy the condition (1.1) and $uv \in E(G)$. The gamma graph $G(\gamma)$ was defined by Fricke *et al.* [9], and was further studied in [3].

The concept of a gamma graph $\gamma\cdot G$ using other types of domination has been studied by several authors. For example, the $\gamma$-*total dominating graph*, the $\gamma$-*independent dominating graph*, the $\gamma$-*induced-paired dominating graph*, and the $\gamma$-*paired dominating graph* were defined by Wongsriya and Trakultraipruk [19], Samanmoo *et al.* [14], Sanguanpong and Trakultraipruk [15], and Eakawinrujee and Trakultraipruk [6, 7], respectively.

A *paired dominating set* of $G$, introduced by Haynes and Slater [10], is a dominating set whose induced subgraph has a perfect matching. A paired dominating set $D$ of $G$ is *minimal* if no proper subset of $D$ is a paired dominating set of $G$. The *upper paired domination number* $\Gamma_{pr}(G)$ of $G$ is the maximum cardinality of a minimal paired dominating set of $G$. A minimal paired dominating set $D$ is called a $\Gamma_{pr}(G)$-*set* if $|D| = \Gamma_{pr}(G)$. The upper paired domination was introduced by Dorbec *et al.* [5], and was further studied in [1, 4, 20].

In [8], we defined the $\Gamma$-*paired dominating graph* $PD_\Gamma(G)$ of $G$ to be the graph with $V(PD_\Gamma(G))$ as the set of all $\Gamma_{pr}(G)$-sets, and two vertices $D_1$ and $D_2$ of $PD_\Gamma(G)$ are adjacent if they satisfy the condition (1.1). We presented the $\Gamma$-paired dominating graph of a path $P_n$, where $n \in \{2, 3, \ldots, 13\}$. Unfortunately, there are some mistakes in these results when $n \in \{8, 10, 11, 12, 13\}$. In this paper, we correct these mistakes and then determine the $\Gamma$-paired dominating graphs of some cycles.

## 2. Preliminary Results

In this section, we provide some definitions and known results. We denote a *path*, a *cycle*, and a *complete graph* with $n$ vertices by $P_n$, $C_n$, and $K_n$, respectively.

The *paired domination number* $\gamma_{pr}(G)$ of $G$ is the minimum cardinality of a paired dominating set of $G$. Haynes and Slater [10] provided the paired domination numbers of paths and cycles.

**Lemma 2.1** ([10]). *For any integer $n \geq 3$, $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$.*

Dorbec *et al.* [5] established the upper paired domination numbers of paths.

**Lemma 2.2** ([5]). *For any integer $n \geq 2$, $\Gamma_{pr}(P_n) = 8\lfloor \frac{n+1}{11} \rfloor + 2\lfloor \frac{(n+1) \pmod{11}}{3} \rfloor$.*

Ulatowski [20] gave the upper paired domination numbers of cycles together with two additional results.

**Lemma 2.3** ([20]). *For any integer $n \geq 3$, $\Gamma_{pr}(C_n) = 8\lfloor \frac{n}{11} \rfloor + 2\lfloor \frac{n \pmod{11}}{3} \rfloor$.*

**Proposition 2.4** ([20]). *If $n \in \{2, 3, 4, 5, 6, 7, 9\}$, then $\Gamma_{pr}(P_n) = \gamma_{pr}(P_n)$.*

**Proposition 2.5** ([20]). *If $n \in \{3, 4, 5, 6, 7, 8, 9, 10, 13\}$, then $\Gamma_{pr}(C_n) = \gamma_{pr}(C_n)$.*

The *Cartesian product* $G\square H$ of graphs $G$ and $H$ is the graph with $V(G\square H) = V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

Let $P_p : 1, 2, 3, \ldots, p$ and $P_q : 1, 2, 3, \ldots, q$ be two paths, where $p$ and $q$ are positive integers. A *stepgrid* $SG_{p,q}$, defined by Fricke *et al.* [9], is the subgraph of $P_p\square P_q$ induced by $\{(x, y) : 1 \leq x \leq p, 1 \leq y \leq q, x - y \leq 1\}$. For example, the stepgrids $SG_{1,1}$, $SG_{2,2}$, and $SG_{3,3}$ are shown in Figure 1.

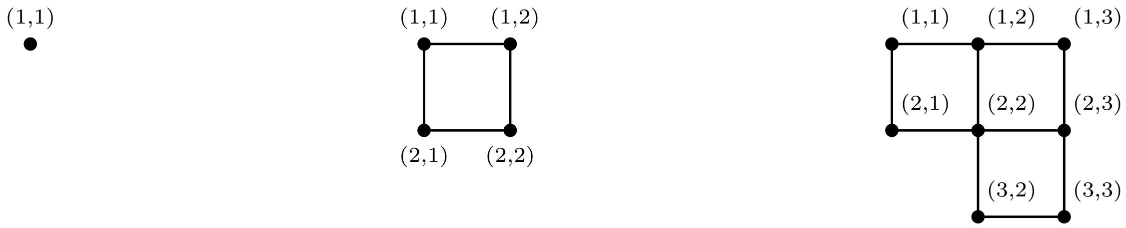


FIGURE 1. The stepgrids $SG_{1,1}$ (left), $SG_{2,2}$ (center), and $SG_{3,3}$ (right)

Let $P_p : 1, 2, 3, \ldots, p$, $P_q : 1, 2, 3, \ldots, q$, and $P_r : 1, 2, 3, \ldots, r$ be three paths, where $p, q$, and $r$ are positive integers. In [7], we defined a *stepgrid* $SG_{p,q,r}$ to be the graph with

$$V(SG_{p,q,r}) = \{(x, y, z) \in V(P_p \square P_q \square P_r) : x - y \leq 0, x - z \leq 1, y - z \geq 0\}$$

and

$$E(SG_{p,q,r}) = E(P_p \square P_q \square P_r) \cup \{(x, x, x)(x + 1, x + 1, x) : 1 \leq x \leq p - 1\}.$$

For example, the stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$ are shown in Figure 2.

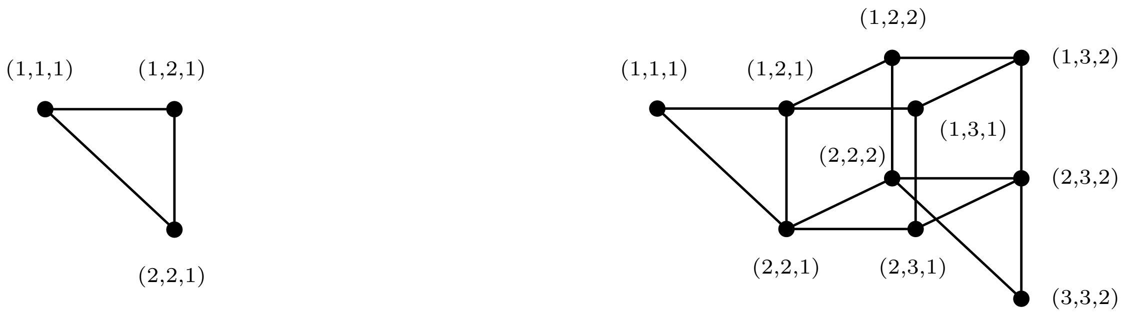


FIGURE 2. The stepgrids $SG_{2,2,1}$(left) and $SG_{3,3,2}$ (right)

The $\gamma$*-paired dominating graph* $PD_\gamma(G)$ of $G$, defined in [7], is the graph whose vertices are minimum paired dominating sets, and two vertices $D_1$ and $D_2$ of $PD_\gamma(G)$ are adjacent if they satisfy the condition (1.1).

We determined the $\gamma$-paired dominating graphs of paths. These results are revealed in the following theorem.

**Theorem 2.6** ([7]). *For any integer $n \geq 2$,*

$$PD_\gamma(P_n) \cong \begin{cases} K_1 & \text{if } n \equiv 0 \pmod{4}; \\ P_{\frac{n+5}{4}} & \text{if } n \equiv 3 \pmod{4}; \\ SG_{\frac{n+2}{4}, \frac{n+2}{4}} & \text{if } n \equiv 2 \pmod{4}; \\ SG_{\frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}} & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

Let $G_1 : 1, 2, 3, \ldots, 2k - 1$ and $G_2 : 1, 2, 3, \ldots, 3k - 1$ be two paths, where $k$ is a positive integer. In [6], we defined a *loopgrid* $LG_k$ of size $k$ to be the graph with

$$V(LG_k) = \{(x, y) \in V(G_1 \Box G_2) : 0 \leq y - x \leq k\}$$

and

$$E(LG_k) = E(G_1 \Box G_2) \cup \{(1, y)(2k-1, y+2k-1) : 1 \leq y \leq k\}.$$

Figure 3 illustrates the loopgrid of size 3.

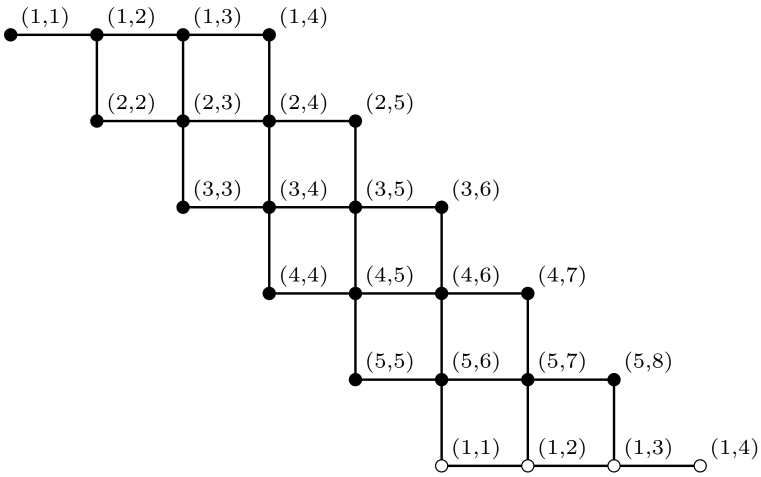


FIGURE 3. The loopgrid of size 3

Let $G_1 : 1, 2, 3, \ldots, 2k$, $G_2 : 1, 2, 3, \ldots, 2k$, and $G_3 : 1, 2, 3, \ldots, 2k+1$ be the paths, where $k$ is a positive integer. We defined a *loopbox* $LB_k$ of size $k$ in [6] to be the graph with

$V(LB_k) = \{(x, y, z) \in V(G_1 \Box G_2 \Box G_3) : 0 \leq y - x \leq k, -1 \leq y - z \leq k-1, 0 \leq z - x \leq k\}$

and

$$\begin{aligned} E(LB_k) = {} & E(G_1 \Box G_2 \Box G_3) \cup \{(x, x+k-1, x)(x, x+k, x+1) : 1 \leq x \leq k\} \\ & \cup \{(x, x, x+1)(x+1, x+1, x+1) : 1 \leq x \leq 2k-1\} \\ & \cup \{(x, x+k, x+k)(x+1, x+k, x+k+1) : 1 \leq x \leq k\} \\ & \cup \{(1, y, z)(z+k, 2k, y+k+1) : 1 \leq y, z \leq k, -1 \leq y - z \leq k-1\} \\ & \cup \{(1, 1, 1)(k+1, 2k, k+1)\} \cup \{(1, k, k+1)(2k, 2k, 2k+1)\}. \end{aligned}$$

The loopboxes of size 1, 2, and 3 are shown in Figure 4.

We determined the $\gamma$-paired dominating graphs of cycles, which are shown in the following theorem.

**Theorem 2.7** ([6]). *For any integer $n \geq 3$,*

$$PD_\gamma(C_n) \cong \begin{cases} C_3 \Box C_3 & \text{if } n = 6; \\ 4K_1 & \text{if } n \equiv 0 \pmod 4 \text{ and } n \neq 4; \\ C_n & \text{if } n \equiv 3 \pmod 4 \text{ or } n = 4; \\ LG_{\frac{n+2}{4}} & \text{if } n \equiv 2 \pmod 4 \text{ and } n \neq 6; \\ LB_{\frac{n-1}{4}} & \text{if } n \equiv 1 \pmod 4). \end{cases}$$

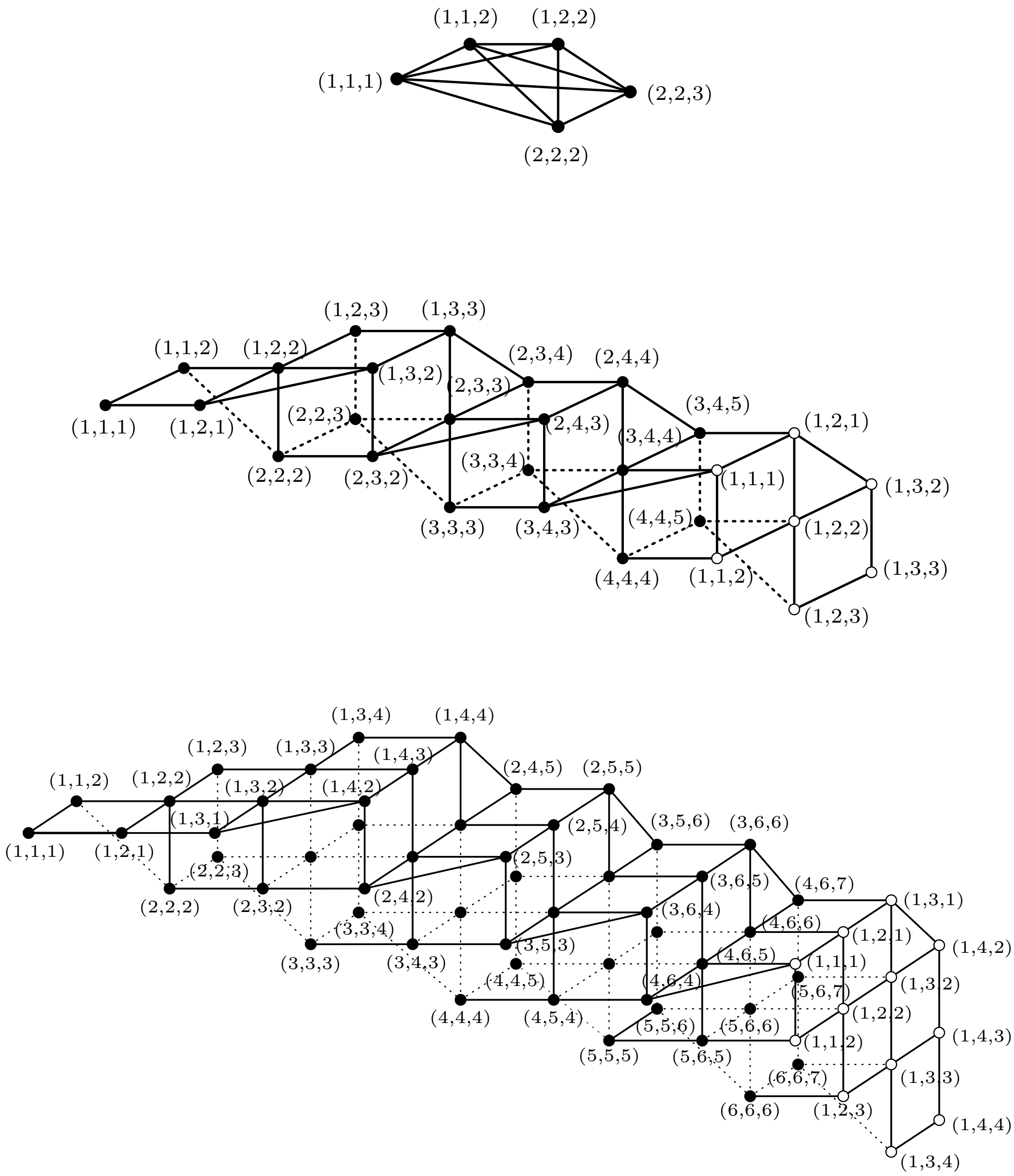


FIGURE 4. The loopboxes of size 1 (top), of size 2 (middle), and of size 3 (bottom)

## 3. Γ-Paired Dominating Graphs of Some Paths

In this section, we recall the Γ-paired dominating graphs of some paths that appear in [8]. For $n \in \{2, 3, 4, 5, 6, 7, 9\}$, we get the Γ-paired dominating graph of $P_n$ as follows.

**Theorem 3.1** ([8]). *The* Γ*-paired dominating graph of* $P_n$ *is*

$$PD_\Gamma(P_n) \cong \begin{cases} K_1 & \text{if } n = 4; \\ P_{\frac{n+5}{4}} & \text{if } n \in \{3, 7\}; \\ SG_{\frac{n+2}{4}, \frac{n+2}{4}} & \text{if } n \in \{2, 6\}; \\ SG_{\frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}} & \text{if } n \in \{5, 9\}. \end{cases}$$

Next, we correct the results on the $\Gamma$-paired dominating graph of $P_n$ for $n \in \{8, 10, 11, 12, 13\}$, which are appeared in [8]. By Theorem 2.2, we get that $\Gamma_{pr}(P_8) = 6$, $\Gamma_{pr}(P_{10}) = \Gamma_{pr}(P_{11}) = \Gamma_{pr}(P_{12}) = 8$, and $\Gamma_{pr}(P_{13}) = 10$. Again, we let $P_n : 1, 2, 3, \ldots, n$ be the path with $n$ vertices.

**Theorem 3.2.** *The $\Gamma$-paired dominating graph of $P_8$ is the graph shown in Figure 5, where* $D_1 = \{1, 2, 3, 4, 6, 7\}$, $D_2 = \{1, 2, 3, 4, 7, 8\}$, $D_3 = \{1, 2, 4, 5, 6, 7\}$, $D_4 = \{1, 2, 4, 5, 7, 8\}$, $D_5 = \{1, 2, 5, 6, 7, 8\}$, $D_6 = \{2, 3, 4, 5, 7, 8\}$, $D_7 = \{2, 3, 5, 6, 7, 8\}$.

**Theorem 3.3.** *The $\Gamma$-paired dominating graph of $P_{10}$ is the graph shown in Figure 5, where* $D_1 = \{1, 2, 3, 4, 6, 7, 8, 9\}$, $D_2 = \{1, 2, 3, 4, 6, 7, 9, 10\}$, $D_3 = \{1, 2, 3, 4, 7, 8, 9, 10\}$, $D_4 = \{1, 2, 4, 5, 6, 7, 9, 10\}$, $D_5 = \{1, 2, 4, 5, 7, 8, 9, 10\}$, $D_6 = \{2, 3, 4, 5, 7, 8, 9, 10\}$.


Figure 5. The $\Gamma$-paired dominating graphs of $P_8$ (left) and $P_{10}$ (right)

**Theorem 3.4.** *The $\Gamma$-paired dominating graph of $P_{11}$ is the graph shown in Figure 6, where* $D_1 = \{1, 2, 3, 4, 6, 7, 9, 10\}$, $D_2 = \{1, 2, 3, 4, 6, 7, 10, 11\}$, $D_3 = \{1, 2, 3, 4, 7, 8, 9, 10\}$, $D_4 = \{1, 2, 3, 4, 7, 8, 10, 11\}$, $D_5 = \{1, 2, 4, 5, 6, 7, 9, 10\}$, $D_6 = \{1, 2, 4, 5, 6, 7, 10, 11\}$, $D_7 = \{1, 2, 4, 5, 7, 8, 9, 10\}$, $D_8 = \{1, 2, 4, 5, 7, 8, 10, 11\}$, $D_9 = \{1, 2, 4, 5, 8, 9, 10, 11\}$, $D_{10} = \{1, 2, 5, 6, 7, 8, 10, 11\}$, $D_{11} = \{1, 2, 5, 6, 8, 9, 10, 11\}$, $D_{12} = \{2, 3, 4, 5, 7, 8, 9, 10\}$, $D_{13} = \{2, 3, 4, 5, 7, 8, 10, 11\}$, $D_{14} = \{2, 3, 4, 5, 8, 9, 10, 11\}$, $D_{15} = \{2, 3, 5, 6, 7, 8, 10, 11\}$, $D_{16} = \{2, 3, 5, 6, 8, 9, 10, 11\}$.

**Theorem 3.5.** *The $\Gamma$-paired dominating graph of $P_{12}$ is the graph shown in Figure 7, where* $D_1 = \{1, 2, 3, 4, 6, 7, 10, 11\}$, $D_2 = \{1, 2, 3, 4, 7, 8, 10, 11\}$, $D_3 = \{1, 2, 3, 4, 7, 8, 11, 12\}$, $D_4 = \{1, 2, 4, 5, 6, 7, 10, 11\}$, $D_5 = \{1, 2, 4, 5, 7, 8, 10, 11\}$, $D_6 = \{1, 2, 4, 5, 7, 8, 11, 12\}$, $D_7 = \{1, 2, 4, 5, 8, 9, 10, 11\}$, $D_8 = \{1, 2, 4, 5, 8, 9, 11, 12\}$, $D_9 = \{1, 2, 5, 6, 7, 8, 10, 11\}$, $D_{10} = \{1, 2, 5, 6, 7, 8, 11, 12\}$, $D_{11} = \{1, 2, 5, 6, 8, 9, 10, 11\}$, $D_{12} = \{1, 2, 5, 6, 8, 9, 11, 12\}$, $D_{13} = \{1, 2, 5, 6, 9, 10, 11, 12\}$, $D_{14} = \{2, 3, 4, 5, 7, 8, 10, 11\}$, $D_{15} = \{2, 3, 4, 5, 7, 8, 11, 12\}$, $D_{16} = \{2, 3, 4, 5, 8, 9, 10, 11\}$, $D_{17} = \{2, 3, 4, 5, 8, 9, 11, 12\}$, $D_{18} = \{2, 3, 5, 6, 7, 8, 10, 11\}$, $D_{19} = \{2, 3, 5, 6, 7, 8, 11, 12\}$, $D_{20} = \{2, 3, 5, 6, 8, 9, 10, 11\}$, $D_{21} = \{2, 3, 5, 6, 8, 9, 11, 12\}$, $D_{22} = \{2, 3, 5, 6, 9, 10, 11, 12\}$, $D_{23} = \{2, 3, 6, 7, 8, 9, 11, 12\}$, $D_{24} = \{2, 3, 6, 7, 9, 10, 11, 12\}$.

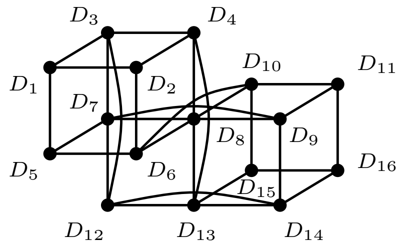


FIGURE 6. The Γ-paired dominating graph of $P_{11}$

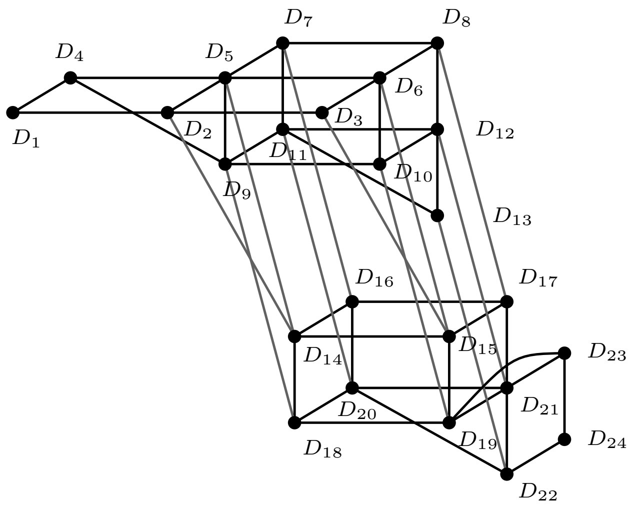


FIGURE 7. The Γ-paired dominating graph of $P_{12}$

**Theorem 3.6.** *The* $\Gamma$*-paired dominating graph of* $P_{13}$ *is the graph shown in Figure 8, where* $D_1 = \{1,2,3,4,6,7,8,9,11,12\}$, $D_2 = \{1,2,3,4,6,7,8,9,12,13\}$, $D_3 = \{1,2,3,4,6,7,9,10,11,12\}$, $D_4 = \{1,2,3,4,6,7,9,10,12,13\}$, $D_5 = \{1,2,3,4,6,7,10,11,12,13\}$, $D_6 = \{1,2,3,4,7,8,9,10,12,13\}$, $D_7 = \{1,2,3,4,7,8,10,11,12,13\}$, $D_8 = \{1,2,4,5,6,7,9,10,11,12\}$, $D_9 = \{1,2,4,5,6,7,9,10,12,13\}$, $D_{10} = \{1,2,4,5,6,7,10,11,12,13\}$, $D_{11} = \{1,2,4,5,7,8,9,10,12,13\}$, $D_{12} = \{1,2,4,5,7,8,10,11,12,13\}$, $D_{13} = \{1,2,5,6,7,8,10,11,12,13\}$, $D_{14} = \{2,3,4,5,7,8,9,10,12,13\}$, $D_{15} = \{2,3,4,5,7,8,10,11,12,13\}$, $D_{16} = \{2,3,5,6,7,8,10,11,12,13\}$.

## 4. Γ-Paired Dominating Graphs of Some Cycles

This secction shows the Γ-paired dominating graph of $C_n$ for $n \in \{3,4,\dots,13\}$. We let $C_n : 1,2,\dots,n,1$ be the cycle with $n$ vertices.

Note that if $G$ is a graph with $\Gamma_{pr}(G) = \gamma_{pr}(G)$, then $PD_\Gamma(G) \cong PD_\gamma(G)$. By Proposition 2.5 and Theorem 2.7, we get the Γ-paired dominating graph of $C_n$ for $n \in \{3,4,5,6,7,8,9,10,13\}$ as follows.

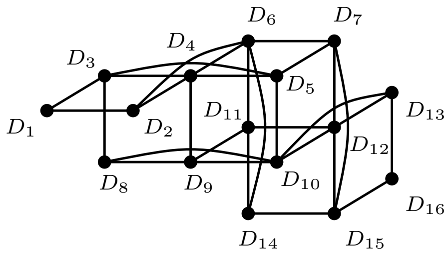


FIGURE 8. The $\Gamma$-paired dominating graph of $P_{13}$

**Theorem 4.1.** *The* $\Gamma$*-paired dominating graph of* $C_n$ *is*

$$PD_\Gamma(C_n) \cong \begin{cases} C_3\square C_3 & \text{if } n=6;\\ 4K_1 & \text{if } n=8;\\ C_n & \text{if } n\in\{3,4,7\};\\ LG_{\frac{n+2}{4}} & \text{if } n=10;\\ LB_{\frac{n-1}{4}} & \text{if } n\in\{5,9,13\}. \end{cases}$$

By Theorem 2.3, we have $\Gamma_{pr}(C_{11}) = \Gamma_{pr}(C_{12}) = 8$. Then we get the $\Gamma$-paired dominating graphs of $C_{11}$ and $C_{12}$ as follows.

**Theorem 4.2.** *The* $\Gamma$*-paired dominating graph of* $C_{11}$ *is the graph shown in Figure 9, where* $D_1 = \{1,2,3,4,6,7,8,9\}$, $D_2 = \{1,2,3,4,6,7,9,10\}$, $D_3 = \{1,2,3,4,7,8,9,10\}$, $D_4 = \{1,2,4,5,6,7,9,10\}$, $D_5 = \{1,2,4,5,6,7,10,11\}$, $D_6 = \{1,2,4,5,7,8,9,10\}$, $D_7 = \{1,2,4,5,7,8,10,11\}$, $D_8 = \{1,2,5,6,7,8,10,11\}$, $D_9 = \{2,3,4,5,7,8,9,10\}$, $D_{10} = \{2,3,4,5,7,8,10,11\}$, $D_{11} = \{2,3,4,5,8,9,10,11\}$, $D_{12} = \{2,3,5,6,7,8,10,11\}$, $D_{13} = \{2,3,5,6,7,8,11,1\}$, $D_{14} = \{2,3,5,6,8,9,10,11\}$, $D_{15} = \{2,3,5,6,8,9,11,1\}$, $D_{16} = \{2,3,6,7,8,9,11,1\}$, $D_{17} = \{3,4,5,6,8,9,10,11\}$, $D_{18} = \{3,4,5,6,8,9,11,1\}$, $D_{19} = \{3,4,5,6,9,10,11,1\}$, $D_{20} = \{3,4,6,7,8,9,11,1\}$, $D_{21} = \{3,4,6,7,9,10,11,1\}$, $D_{22} = \{4,5,6,7,9,10,11,1\}$.

**Theorem 4.3.** *The* $\Gamma$*-paired dominating graph of* $C_{12}$ *is the graph shown in Figure 10, where* $D_1 = \{1,2,3,4,6,7,9,10\}$, $D_2 = \{1,2,3,4,6,7,10,11\}$, $D_3 = \{1,2,3,4,7,8,9,10\}$, $D_4 = \{1,2,3,4,7,8,10,11\}$, $D_5 = \{1,2,4,5,6,7,9,10\}$, $D_6 = \{1,2,4,5,6,7,10,11\}$, $D_7 = \{1,2,4,5,7,8,9,10\}$, $D_8 = \{1,2,4,5,7,8,10,11\}$, $D_9 = \{1,2,4,5,7,8,11,12\}$, $D_{10} = \{1,2,4,5,8,9,10,11\}$, $D_{11} = \{1,2,4,5,8,9,11,12\}$, $D_{12} = \{1,2,5,6,7,8,10,11\}$, $D_{13} = \{1,2,5,6,7,8,11,12\}$, $D_{14} = \{1,2,5,6,8,9,10,11\}$, $D_{15} = \{1,2,5,6,8,9,11,12\}$, $D_{16} = \{2,3,4,5,7,8,10,11\}$, $D_{17} = \{2,3,4,5,7,8,11,12\}$, $D_{18} = \{2,3,4,5,8,9,10,11\}$, $D_{19} = \{2,3,4,5,8,9,11,12\}$, $D_{20} = \{2,3,5,6,7,8,10,11\}$, $D_{21} = \{2,3,5,6,7,8,11,12\}$, $D_{22} = \{2,3,5,6,8,9,10,11\}$, $D_{23} = \{2,3,5,6,8,9,11,12\}$, $D_{24} = \{2,3,5,6,8,9,12,1\}$, $D_{25} = \{2,3,5,6,9,10,11,12\}$, $D_{26} = \{2,3,5,6,9,10,12,1\}$, $D_{27} = \{2,3,6,7,8,9,11,12\}$, $D_{28} = \{2,3,6,7,8,9,12,1\}$, $D_{29} = \{2,3,6,7,9,10,11,12\}$, $D_{30} = \{2,3,6,7,9,10,12,1\}$, $D_{31} = \{3,4,5,6,8,9,11,12\}$, $D_{32} = \{3,4,5,6,8,9,12,1\}$, $D_{33} = \{3,4,5,6,9,10,11,12\}$, $D_{34} = \{3,4,5,6,9,10,12,1\}$, $D_{35} = \{3,4,6,7,8,9,11,12\}$, $D_{36} = \{3,4,6,7,8,9,12,1\}$, $D_{37} = \{3,4,6,7,9,10,11,12\}$, $D_{38} = \{3,4,6,7,9,10,12,1\}$, $D_{39} = \{3,4,6,7,10,11,12,1\}$, $D_{40} = \{3,4,7,8,9,10,12,1\}$, $D_{41} = \{3,4,7,8,10,11,12,1\}$, $D_{42} = \{4,5,6,7,9,10,12,1\}$, $D_{43} = \{4,5,6,7,10,11,12,1\}$, $D_{44} = \{4,5,7,8,9,10,12,1\}$, $D_{45} = \{4,5,7,8,10,11,12,1\}$.

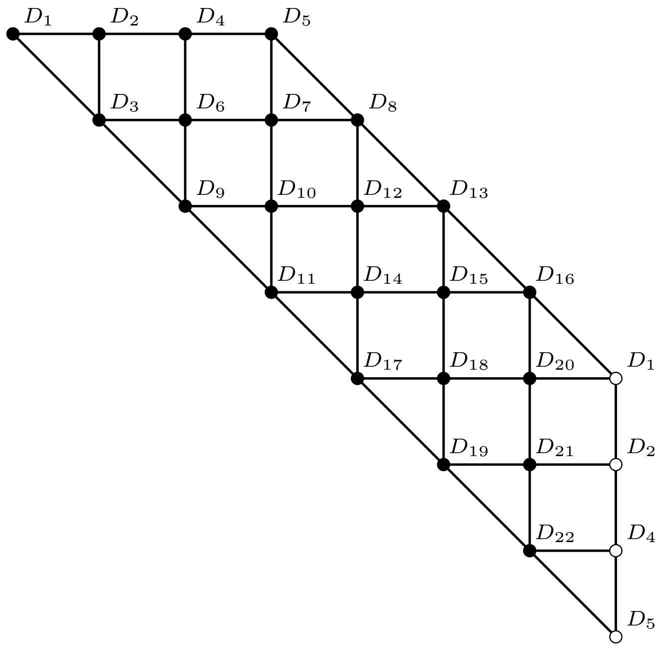


FIGURE 9. The Γ-paired dominating graph of $C_{11}$

## 5. Open Problem

In this paper, we have obtained the Γ-paired dominating graphs of $P_n$ and $C_n$ for $n \leq 13$. It seems that finding the Γ-paired dominating graphs of $P_n$ and $C_n$ is much more complicated when $n \geq 14$; this leads an open problem.

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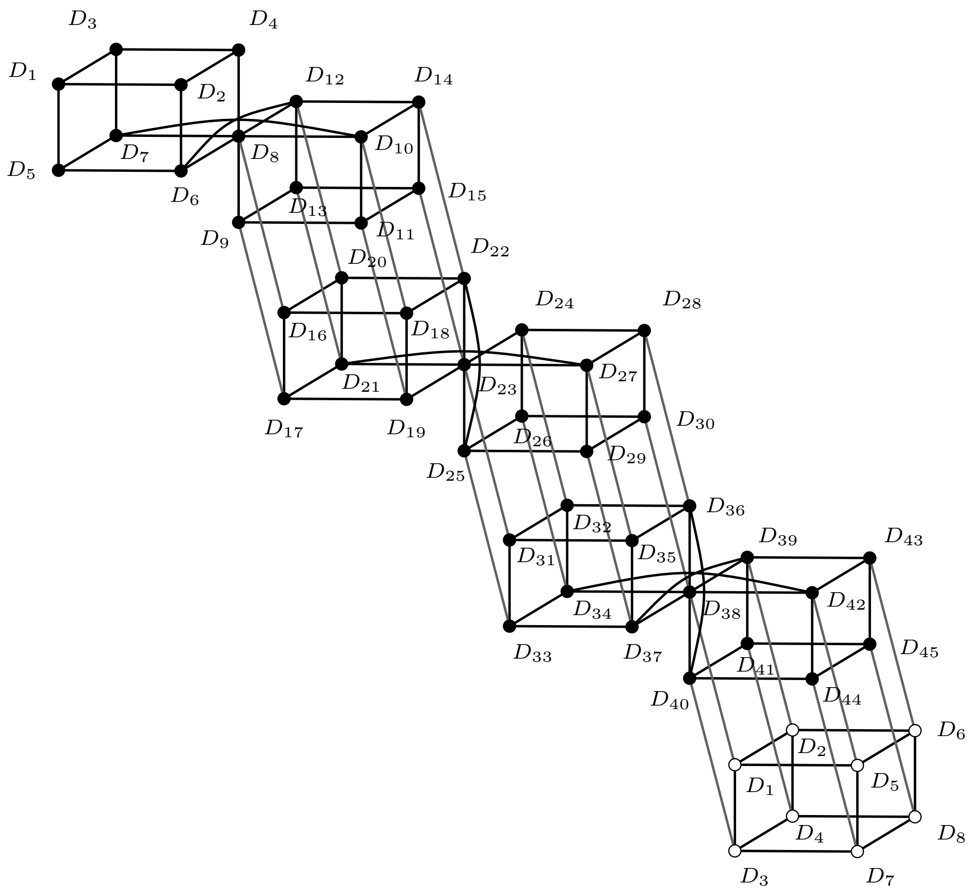


FIGURE 10. The $\Gamma$-paired dominating graph of $C_{12}$

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