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Strong Convergence of a New Hybrid Iteration for Three Asymptotically Quasi $\mathcal{G}$-ϕ-Nonexpansive Mappings in Banach Spaces with Directed Graphs

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Abstract The purpose of the paper is to combine the shrinking projection method with SP-iteration to introduce a new hybrid iterative scheme for approximating common fixed points of three asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mappings. A strong convergence result for the proposed iteration processes is proved under some suitable conditions in uniformly smooth and uniformly convex Banach spaces with directed graphs without any requirement of the semicompact property of such mappings. Finally, we apply our proposed method to find a solution of the system of the nonlinear integral equations.

MSC: 47H09; 47H10

Keywords: asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping; directed graph; SP-iteration, uniformly smooth and uniformly convex Banach space

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1. Introduction

The notion of a nonexpansive mapping was introduced by Browder [1]. This notion was generalized to an asymptotically nonexpansive mapping by Goebel and Kirk [2]. These mappings have played very important role in fixed point theory and its applications. Inspired by Matsushita's idea [3], Qin et al. [4] introduced quasi-ϕ-nonexpansive mapping in smooth Banach spaces and then introduced asymptotically quasi-ϕ-nonexpansive mapping in [5]. In another way, by Jachymski's idea [6], Aleomraninejad et al. [7] generalized a nonexpansive mapping to a $\mathcal{G}$-nonexpansive mapping. Also, Sangago et al. [8] generalized it to a $\mathcal{G}$-asymptotically nonexpansive mapping. After that, many weak convergence

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results for such mappings were presented [9–14]. The natural question that arises in the case of an infinite dimensional Banach space is how to construct an algorithm that gives a strong convergence theorem. This question was made clearly by many authors, see in [9, 15, 16]. One of the algorithms is hybrid iteration processes which was introduced by Hammad et al. [17] for two $\mathcal{G}$-nonexpansive mappings in Hilbert spaces with directed graphs without adding the condition to $\mathcal{G}$-nonexpansive mappings. Recently, Hieu and Tu [18] generalized the main results in [17] to two $\mathcal{G}$-asymptotically nonexpansive mappings in Hilbert spaces with directed graphs. The authors proposed four new hybrid iteration processes for two $\mathcal{G}$-asymptotically nonexpansive mappings and prove some strong convergence results of such iterations to common fixed points of two $\mathcal{G}$-asymptotically nonexpansive mappings in Hilbert spaces with graphs.

In 2020, Dung et al. [19] studied a quasi $\mathcal{G}$-ϕ-nonexpansive mapping and an asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping, and also introduced algorithms for strong convergence results in uniformly smooth and uniformly convex Banach spaces with directed graphs. Later on, in the same space of Dung et al. [19], Hieu and Huy [20] proposed a modified inertial hybrid iteration for two asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive. Some strong convergence theorems for such iteration were proved under suitable conditions in Banach spaces with directed graphs. These results are extensions of the main theorems in [17, 18] from Hilbert spaces with directed graphs to uniformly smooth and uniformly convex Banach spaces with directed graphs.

In 2019, Sridarat et al. [9] proposed the modified SP-iteration for approximating a common fixed points of three $\mathcal{G}$-nonexpansive mappings in uniformly convex Banach spaces with directed graphs. The modified SP-iteration is presented in [9] as follows.

$$\begin{cases} t_1 \in \mathcal{K} \\ h_n = (1-\gamma_n)t_n + \gamma_n \mathcal{S}_3 t_n \\ k_n = (1-\beta_n)h_n + \beta_n \mathcal{S}_2 h_n \\ t_{n+1} = (1-\alpha_n)k_n + a_n \mathcal{S}_1 k_n, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \longrightarrow \mathcal{K}$ are three $\mathcal{G}$-nonexpansive mappings. Furthermore, some authors also generalized the SP-iteration to some new iterations for generalized nonexpansive mappings and variational inequality problems in some various spaces [21–24].

Recently, Wattanataweekul [14] extended the results of the method (1.1) in [9] to three $\mathcal{G}$-asymptotically nonexpansive mappings. Weak and strong convergence theorems of the proposed iteration were proved in uniformly convex Banach spaces with directed graphs. The following iteration is presented in [14].

$$\begin{cases} t_1 \in \mathcal{K} \\ h_n = (1-\gamma_n)t_n + \gamma_n \mathcal{S}_3^n t_n \\ k_n = (1-\beta_n)h_n + \beta_n \mathcal{S}_2^n h_n \\ t_{n+1} = (1-\alpha_n)k_n + a_n \mathcal{S}_1^n k_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \longrightarrow \mathcal{K}$ are three $\mathcal{G}$-asymptotically nonexpansive mappings. Note that the author used the semicompact property of such mappings to prove the strong convergence of (1.2) to common fixed points of three $\mathcal{G}$-asymptotically nonexpansive mappings.

Motivated by the iteration processes discussed above, we propose a new hybrid iteration method for approximating common fixed points of three asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mappings. This method utilizes the shrinking projection technique combined with a modified SP-iteration. Subsequently, we establish several strong convergence results for these iteration processes in uniformly smooth and uniformly convex Banach spaces with directed graphs, without necessitating the semicompact property of such mappings. As a practical application, we demonstrate how our proposed method can be applied to solve a system of nonlinear integral equations.

2. Preliminaries

First, let $\mathcal{W}$ be a smooth Banach space, and *Lyapunov functional* $\phi : \mathcal{W} \times \mathcal{W} \longrightarrow [0, \infty)$ be defined by

$$\phi(h, k) = \|h\|^2 - 2\langle h, \mathcal{J}k\rangle + \|k\|^2, \forall h, k \in \mathcal{W}. \tag{2.1}$$

Remark 2.1. [25–29]

Let $\mathcal{W}$ be a smooth Banach space. Then we have the following statements.

(1) If $\mathcal{W}$ is a Hilbert space, then (2.1) becomes $\phi(h, k) = \|h - k\|^2$ for all $h, k \in \mathcal{W}$.

(2) For all $h, k, l \in \mathcal{W}$ and $t \in [0, 1]$, we have

  (a) $(\|h\| - \|k\|)^2 \leq \phi(h, k) \leq (\|h\| + \|k\|)^2$. This fact ensures that $\phi(h, k) \geq 0$ and hence ϕ is well-defined.

  (b) $\phi(h, k) = \phi(h, l) + \phi(z, l) + 2\langle h - l, \mathcal{J}l - \mathcal{J}k\rangle$.

  (c) $\phi\big(h, \mathcal{J}^{-1}\big((1 - t)\mathcal{J}k + t\mathcal{J}l\big)\big) \leq (1 - t)\phi(h, k) + t\phi(h, l)$.

  (d) $\phi(h, k) = \langle h, \mathcal{J}h - \mathcal{J}k\rangle + \langle k - h, \mathcal{J}k\rangle \leq \|h\|.\|\mathcal{J}h - \mathcal{J}k\| + \|k - h\|.\|k\|$.

(3) If $\mathcal{W}$ is a smooth, strictly convex and reflexive Banach space, then for $h, k \in \mathcal{W}$, $\phi(h, k) = 0$ if and only if $h = k$.

Lemma 2.2. [30] *Assume that $\mathcal{W}$ is a uniformly smooth and uniformly convex Banach space. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of $\mathcal{W}$ such that $\lim_{n\to\infty} \phi(\alpha_n, \beta_n) = 0$ and either $\{\alpha_n\}$ or $\{\beta_n\}$ is bounded. Then $\lim_{n\to\infty} \|\alpha_n - \beta_n\| = 0$.*

Note that the metric projection in a Hilbert space was extended to the generalized projection mapping in a Banach space. Next, we collect some necessary lemmas of the generalized projection mapping for proving our main results.

Lemma 2.3. [27] *Assume that $\mathcal{W}$ is a smooth, strictly convex, reflexive Banach space, and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{W}$ and $a \in \mathcal{W}$. Then there exists a unique element $h_0 \in \mathcal{K}$ such that $\phi(h_0, h) = \inf\{\phi(k, h) : k \in \mathcal{K}\}$. The mapping $\Pi_{\mathcal{K}} : \mathcal{W} \longrightarrow \mathcal{K}$ defined by $\Pi_{\mathcal{K}}h = h_0$ is called the* generalized projection.

Lemma 2.4. [27] *Assume that $\mathcal{W}$ is a smooth Banach space, $\mathcal{K}$ is a nonempty closed convex subset of $\mathcal{W}$, $h \in \mathcal{W}$, and $h_0 \in \mathcal{K}$. Then $h_0 = \Pi_{\mathcal{K}}h$ if and only if $\langle h_0 - k, Jh - Jh_0\rangle \geq 0$ for all $k \in \mathcal{K}$.*

Lemma 2.5. [27] *Assume that $\mathcal{W}$ is a smooth, strictly convex, reflexive Banach space, and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{W}$ and $h \in \mathcal{W}$. Then for all $k \in \mathcal{K}$, we have*

$$\phi(k, \Pi_{\mathcal{K}}h) + \phi(\Pi_{\mathcal{K}}h, h) \leq \phi(k, h).$$

The following result was presented in [31].

Lemma 2.6. [31] *Assume that $\mathcal{W}$ is a uniformly convex Banach space and $\varepsilon > 0$. Then there exists a continuous, convex and strictly increasing function $\varphi : [0, 2\varepsilon] \longrightarrow [0, \infty)$ such that $\varphi(0) = 0$ and*

$$\|th + (1-t)k\|^2 \leq \lambda\|h\|^2 + (1-t)\|k\|^2 - t(1-t)\varphi(\|h-k\|)$$

for all $t \in [0, 1]$ and $h, k \in B_\varepsilon = \{h \in \mathcal{W} : \|h\| \leq \varepsilon\}$.

Next, let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})$ be a directed graph where $V(\mathcal{G})$ is a set of vertices of graph $\mathcal{G}$ and $E(\mathcal{G})$ is a set of edges of graph $\mathcal{G}$. We assume that $E(\mathcal{G})$ contains no parallel edges, $(h, h) \in E(\mathcal{G})$ for all $h \in V(\mathcal{G})$, and $E(\mathcal{G}^{-1}) = \big\{(h, k) \in \mathcal{W} \times \mathcal{W} : (k, h) \in E(\mathcal{G})\big\}$, where $\mathcal{G}^{-1}$ is the conversion of a graph $\mathcal{G}$.

Definition 2.7. A metric space $\mathcal{W}$ is said to be endowed with a transitive directed graph $\mathcal{G}$ if $G = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed graph such that the following hold:

(1) $\mathcal{G}$ is transitive, that is, for any $u, v, z \in V(\mathcal{G})$,

$$(u, v), (v, z) \in E(\mathcal{G}) \quad \Rightarrow \quad (u, z) \in E(\mathcal{G});$$

(2) the set of vertices $V(\mathcal{G})$ coincides with $\mathcal{W}$.

Definition 2.8. [32] Assume that $\mathcal{W}$ is a normed space, $\mathcal{K}$ is a nonempty subset of $\mathcal{W}$, and $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed graph such that $V(\mathcal{G}) = \mathcal{K}$. Then $\mathcal{K}$ is called to have *property* $(\mathcal{G})$ if for any sequence $\{h_n\}$ in $\mathcal{K}$ such that $(h_n, h_{n+1}) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$ and $\{h_n\}$ weakly converging to $h \in \mathcal{K}$, there exists a subsequence $\{h_{k(n)}\}$ of $\{h_n\}$ such that $(h_{k(n)}, h) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$.

In 2020, the following coordinate-convexity of $E(\mathcal{G})$ was first introduced by the authors in [33].

Definition 2.9. [33] Assume that $\mathcal{W}$ is a vector space and $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed graph such that $E(\mathcal{G}) \subset \mathcal{W} \times \mathcal{W}$. Then $E(\mathcal{G})$ is called *coordinate-convex* if for all $(u, h), (u, k), (h, p), (k, p) \in E(\mathcal{G})$ and for all $t \in [0, 1]$, we have

$$t(u, h) + (1-t)(u, k) \in E(\mathcal{G}) \text{ and } t(h, p) + (1-t)(k, p) \in E(\mathcal{G}).$$

It is remarkable that if $E(\mathcal{G})$ is convex, then $E(\mathcal{G})$ is coordinate-convex.

Next, we recall the definitions of generalized nonexpansive mappings in normed spaces with graphs.

Definition 2.10. Assume that $\mathcal{W}$ is a normed space, $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed graph such that $V(\mathcal{G}) \subset \mathcal{W}$ and $\mathcal{S} : V(\mathcal{G}) \longrightarrow V(\mathcal{G})$ is a mapping. Then

(1) $\mathcal{S}$ is called $\mathcal{G}$-*nonexpansive* [34] if
   (a) $\mathcal{S}$ is *edge-preserving*, that is, if for all $h, k \in V(\mathcal{G})$ with $(h, k) \in E(\mathcal{G})$, then $(\mathcal{S}h, \mathcal{S}k) \in E(\mathcal{G})$.
   (b) $\|\mathcal{S}h - \mathcal{S}k\| \leq \|h - k\|$ for all $h, k \in V(\mathcal{G})$ with $(h, k) \in E(\mathcal{G})$;

(2) $\mathcal{S}$ is called *asymptotically* $\mathcal{G}$-*nonexpansive* [8] if $\mathcal{S}$ is edge-preserving and there exists a sequence $\{\kappa_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(\kappa_n - 1) < \infty$ such that

$$\|\mathcal{S}^n h - \mathcal{S}^n k\| \leq \kappa_n \|h - k\|$$

for all $h, k \in V(\mathcal{G})$ with $(h, k) \in E(\mathcal{G})$, where $\{\kappa_n\}$ is said to be an *asymptotic coefficient sequence.*

Now, $F(\mathcal{S}) = \{h \in \mathcal{K} : \mathcal{S}h = h\}$ is denoted for the set of fixed points of the mapping $\mathcal{S} : \mathcal{K} \longrightarrow \mathcal{K}$. We next recall some the following notions in smooth Banach spaces.

Definition 2.11. Assume that $\mathcal{W}$ is a smooth Banach space, $\mathcal{K}$ is a nonempty subset of $\mathcal{W}$, $\mathcal{S} : \mathcal{K} \longrightarrow \mathcal{K}$ is a mapping and $\phi : \mathcal{W} \times \mathcal{W} \longrightarrow [0, \infty)$ is a Lyapunov functional. Then

(1) $\mathcal{S}$ is called *quasi ϕ-nonexpansive* [4] if $F(\mathcal{S}) \neq \emptyset$ and $\phi(u, \mathcal{S}h) \leq \phi(u, h)$ for all $u \in F(\mathcal{S})$ and $h \in \mathcal{K}$.

(2) $\mathcal{S}$ is called *asymptotically quasi ϕ-nonexpansive* [5] if $F(\mathcal{S}) \neq \emptyset$ and there exists a real sequence $\{\kappa_n\}$ with $\kappa_n \geq 1$ and $\lim_{n\to\infty} \kappa_n = 1$ such that $\phi(u, \mathcal{S}^n h) \leq \kappa_n \phi(u, h)$ for all $n \geq 1$, $u \in F(\mathcal{S})$ and $h \in \mathcal{K}$.

(3) $\mathcal{S}$ is called *uniformly asymptotically regular* [5] on $\mathcal{K}$ if for any bounded subset $\mathcal{U}$ of $\mathcal{K}$, we have $\lim_{n\to\infty} \sup_{h\in\mathcal{U}} \|\mathcal{S}^{n+1}h - \mathcal{S}^n h\| = 0$.

For quasi $\mathcal{G}$-ϕ-nonexpansive mapping and asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping, the authors in [19] introduced in smooth Banach spaces with directed graphs as follows.

Definition 2.12. Let $\mathcal{W}$ be a smooth Banach space, $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a directed graph such that $V(\mathcal{G}) \subset \mathcal{W}$. Assume that $\mathcal{S} : V(\mathcal{G}) \longrightarrow V(\mathcal{G})$ is a mapping and $\phi : \mathcal{W} \times \mathcal{W} \longrightarrow [0, \infty)$ is a Lyapunov functional. Then

(1) $\mathcal{S}$ is called *quasi $\mathcal{G}$-ϕ-nonexpansive* if $F(\mathcal{S}) \neq \emptyset$ and
   (a) $(u, \mathcal{S}h) \in E(\mathcal{G})$ for all $u \in F(\mathcal{S})$ and $h \in V(\mathcal{G})$ with $(u, h) \in E(\mathcal{G})$.
   (b) $\phi(u, \mathcal{S}u) \leq \phi(u, u)$ for all $u \in F(\mathcal{S})$ and $h \in V(\mathcal{G})$ with $(u, h) \in E(\mathcal{G})$.

(2) $\mathcal{S}$ is called *asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive* if
   (a) $(u, \mathcal{S}h) \in E(\mathcal{G})$ for all $u \in F(\mathcal{S})$ and $h \in V(\mathcal{G})$ with $(u, h) \in E(\mathcal{G})$.
   (b) there exists a sequence $\{\kappa_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \kappa_n = 1$ such that

$$\phi(u, \mathcal{S}^n h) \leq \kappa_n \phi(u, h)$$

   whenever $(u, h) \in E(\mathcal{G})$ for all $u \in F(\mathcal{S})$ and $h \in V(\mathcal{G})$, where $\{\kappa_n\}$ is called an *asymptotic coefficient sequence.*

It's clearly that in Hilbert spaces, a $\mathcal{G}$-ϕ-nonexpansive mapping and asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping become a quasi $\mathcal{G}$-nonexpansive mapping and asymptotically quasi $\mathcal{G}$-nonexpansive mapping [19], respectively.

The following fixed point theory of an asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping in uniformly smooth and uniformly convex Banach spaces that we need in our proofs.

Proposition 2.13. [20] *Assume that $\mathcal{W}$ is a uniformly smooth and uniformly convex Banach space, $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed graph such that $V(\mathcal{G}) = \mathcal{W}$, $\mathcal{S} : \mathcal{W} \longrightarrow \mathcal{W}$ is an asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping and $F(\mathcal{S}) \times F(\mathcal{S}) \subset E(\mathcal{G})$. Then we have the following statements.*

(1) If $\mathcal{W}$ has property $(\mathcal{G})$, then $F(\mathcal{S})$ is closed.

(2) If $E(\mathcal{G})$ is coordinate-convex and $\mathcal{S}$ is closed, then $F(\mathcal{S})$ is convex.

3. Main Results

First, we consider the following assumptions.

(H1) $\mathcal{W}$ is a uniformly smooth and uniformly convex Banach space, and $\mathcal{K}$ is a nonempty, closed and convex subset of $\mathcal{W}$, and $\mathcal{W}$ satisfies the property $(\mathcal{G})$.

(H2) $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a directed and transitive graph, $V(\mathcal{G}) = \mathcal{K}$ and $E(\mathcal{G})$ is coordinate-convex.

(H3) $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : V(\mathcal{G}) \longrightarrow V(\mathcal{G})$ are three closed, uniformly asymptotically regular and asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mappings with the asymptotic coefficient sequences $\{\kappa_n^{(i)}\} \subset [1, \infty)$ satisfying $\lim\limits_{n\to\infty} \kappa_n^{(i)} = 1$ for each $i = 1, 2, 3$ such that

(1) $F(\mathcal{S}_i) \times F(\mathcal{S}_i) \subset E(\mathcal{G})$ for all $i = 1, 2, 3$.

(2) the set $\mathcal{F} = \bigcap\limits_{i=1}^{3} F(\mathcal{S}_i)$ is nonempty and bounded in $V(\mathcal{G})$, that is, there exists a positive number κ such that $\mathcal{F} \subset B_\kappa = \{h \in V(\mathcal{G}) : \|h\| \leq \kappa\}$.

Put $\kappa_n = \max\{\kappa_n^{(i)} : i = 1, 2, 3\}$. Then $\lim\limits_{n\to\infty} \kappa_n = 1$. Furthermore, by Definition 2.12, for each $i = 1, 2, 3$, we get that $\phi(u, \mathcal{S}_i^n h) \leq \kappa_n \phi(u, h)$, for all $(u, h) \in E(\mathcal{G})$, $u \in \mathcal{F} := \bigcap\limits_{i=1}^{3} F(\mathcal{S}_i)$ and $h \in V(\mathcal{G})$.

Motivated by the iteration processes (1.1) and (1.2), we introduce the sequence $\{h_n\}$ which is a hybrid iterative scheme for three asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mappings. The sequence $\{h_n\}$ is generated by

$$\begin{cases} h_1 \in \mathcal{K}, \mathcal{K}_1 = \mathcal{K} \\ w_n = \mathcal{J}^{-1}\big((1-\gamma_n)\mathcal{J}h_n + \gamma_n \mathcal{J}\mathcal{S}_3^n h_n\big) \\ t_n = \mathcal{J}^{-1}\big((1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\big) \\ k_n = \mathcal{J}^{-1}\big((1-\alpha_n)\mathcal{J}t_n + a_n \mathcal{J}\mathcal{S}_1^n t_n\big) \\ \mathcal{K}_{n+1} = \big\{w \in \mathcal{K}_n : \phi(w, k_n) \leq \phi(w, h_n) + \varepsilon_n\big\} \\ h_{n+1} = \Pi_{\mathcal{K}_{n+1}}(h_1), n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$, $\Pi_{\mathcal{K}_{n+1}}(h_1)$ is the generalized projection mapping h_1 onto $\mathcal{K}_{n+1}$, ϕ is a Lyapunov functional, $\mathcal{J}$ is a normalized duality mapping, $\varepsilon_n = (\kappa_n - 1)\big(a_n + \beta_n + \gamma_n + (a_n\beta_n + \beta_n\gamma_n + \gamma_n a_n)(\kappa_n - 1) + a_n\beta_n\gamma_n(\kappa_n - 1)^2\big)(\kappa + \|h_n\|)^2$.

Proposition 3.1. *Assume that the assumptions* $(H1)$-$(H3)$ *are satisfied. Then the generalized projection* $\Pi_{\mathcal{F}}(h_1)$ *is well-defined.*

Proof. By Proposition 2.13, we find that $F(\mathcal{S}_i)$ is convex and closed for each $i = 1, 2, 3$. Therefore, $\mathcal{F} = \bigcap\limits_{i=1}^{3} F(\mathcal{S}_i)$ is convex and closed. Next, by the assumption, we conclude that $\mathcal{F}$ is nonempty. By the above, we find that $\Pi_{\mathcal{F}}(h_1)$ is well-defined. ∎

Proposition 3.2. *Assume that the assumptions* $(H1)$-$(H3)$ *are satisfied. Let* $\{h_n\}$ *be generated by* (3.1) *such that* $(u, k_n), (u, t_n), (u, w_n) \in E(\mathcal{G})$ *for all* $u \in \mathcal{F}$. *Then*

(1) $\mathcal{F} \subset \mathcal{K}_n$ *for all* $n \in \mathbb{N}$.

(2) the generalization projection $\Pi_{\mathcal{K}_{n+1}}(h_1)$ *is well-defined for all* $n \in \mathbb{N}$.

Proof. (1). Obviously, $\mathcal{F} \subset \mathcal{K} = \mathcal{K}_1$. Suppose that $\mathcal{F} \subset \mathcal{K}_m$ for some $m \geq 1$. For $u \in \mathcal{F}$, we have $u \in \mathcal{F} \subset \mathcal{K}_m$ and $\mathcal{S}_1^m u = \mathcal{S}_2^m u = \mathcal{S}_3^m u = u$. By using Remark 2.1.(2c), $(u, h_m), (u, t_m), (u, w_m) \in E(\mathcal{G})$, and the fact that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are asymptotically

$\mathcal{G}$-ϕ-nonexpansive mappings, we get

$$\begin{aligned} \phi(u,k_m) &= \phi\big(u,\mathcal{J}^{-1}\big((1-\alpha_m)\mathcal{J}t_m + a_m\mathcal{J}\mathcal{S}_1^m u_m\big)\big) \\ &\leq (1-\alpha_m)\phi(u,t_m) + a_m\phi(u,\mathcal{S}_1^m t_m) \\ &\leq (1-\alpha_m)\phi(u,t_m) + a_m\kappa_m\phi(u,t_m) \\ &\leq (1+\alpha_m(\kappa_m-1))\phi(u,t_m), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \phi(u,t_m) &= \phi\big(u,\mathcal{J}^{-1}\big((1-\beta_m)\mathcal{J}w_m + \beta_m\mathcal{J}\mathcal{S}_2^m w_m\big)\big) \\ &\leq (1-\beta_m)\phi(u,w_m) + \beta_m\phi(u,\mathcal{S}_2^m w_m) \\ &\leq (1-\beta_m)\phi(u,w_m) + \beta_m\kappa_m\phi(u,w_m) \\ &= \big(1+\beta_m(\kappa_m-1)\big)\phi(u,w_m) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \phi(u,w_m) &= \phi\big(u,\mathcal{J}^{-1}\big((1-\gamma_m)\mathcal{J}h_m + \gamma_m\mathcal{J}\mathcal{S}_3^m h_m\big)\big) \\ &\leq (1-\gamma_m)\phi(u,h_m) + \gamma_m\phi(u,\mathcal{S}_3^m h_m) \\ &\leq (1-\gamma_m)\phi(u,h_m) + \gamma_m\kappa_m\phi(u,h_m) \\ &= \big(1+\gamma_m(\kappa_m-1)\big)\phi(u,h_m). \end{aligned} \tag{3.4}$$

It follows from (3.2), (3.3), (3.4) that

$$\begin{aligned} \phi(u,k_m) &\leq (1+\alpha_m(\kappa_m-1))\phi(u,t_m) \\ &\leq (1+\alpha_m(\kappa_m-1))\big(1+\alpha_m(\kappa_m-1)\big)\phi(u,w_m) \\ &\leq (1+\alpha_m(\kappa_m-1))\big(1+\alpha_m(\kappa_m-1)\big)\big(1+\gamma_m(\kappa_m-1)\big)\phi(u,h_m) \\ &= \phi(u,h_m) + (\kappa_m-1)\big(a_m+\beta_m+\gamma_m+(a_m\beta_m+\beta_m\gamma_m+\gamma_m a_m)(\kappa_m-1) \\ &\qquad +a_m\beta_m\gamma_m(\kappa_m-1)^2\big)\phi(u,h_m) \\ &\leq \phi(u,h_m) + (\kappa_m-1)\big(a_m+\beta_m+\gamma_m+(a_m\beta_m+\beta_m\gamma_m+\gamma_m a_m)(\kappa_m-1) \\ &\qquad +a_m\beta_m\gamma_m(\kappa_m-1)^2\big)(\kappa+\|h_m\|)^2 \\ &= \phi(u,h_m) + \varepsilon_m. \end{aligned} \tag{3.5}$$

This leads to $u \in \mathcal{K}_{m+1}$ and hence $\mathcal{F} \subset \mathcal{K}_{m+1}$.Therefore, $\mathcal{F} \subset \mathcal{K}_n$ for all $n \in \mathbb{N}$.

(2). We first show that $\mathcal{K}_n$ is convex and closed for $n \in \mathbb{N}$ by mathematical induction. Obviously, $\mathcal{K}_1 = \mathcal{K}$ is convex and closed. Suppose that $\mathcal{K}_m$ is closed and convex. By the definition of $\mathcal{K}_{m+1}$, we have

$$\begin{aligned} \mathcal{K}_{m+1} &= \big\{w \in \mathcal{K}_n : \phi(w,k_m) \leq \phi(w,h_n) + \varepsilon_n\big\} \\ &= \big\{w \in \mathcal{K}_n : 2\langle w, \mathcal{J}h_m - \mathcal{J}k_m\rangle \leq \|h_m\|^2 - \|k_m\|^2 + \varepsilon_m\big\}. \end{aligned} \tag{3.6}$$

By directly checking, we find that $\mathcal{K}_{m+1}$ is convex. Furthermore, it follows from (3.6) and the continuity of $J(.)$ that $\mathcal{K}_{m+1}$ is closed. This proves that $\mathcal{K}_{m+1}$ is convex and closed. Therefore, $\mathcal{K}_n$ is convex and closed for all $cn \in \mathbb{N}$.

Since $\mathcal{F} \neq \emptyset$ and $\mathcal{F} \subset \mathcal{K}_{n+1}$, we have $\mathcal{K}_{n+1} \neq \emptyset$ for all $\in \mathbb{N}$. By the above, we find that $\mathcal{K}_{n+1}$ is nonempty, closed and convex for all $n \in \mathbb{N}$. This proves that $\Pi_{\mathcal{K}_{n+1}}(h_1)$ is well-defined for all $n \in \mathbb{N}$. ■

Proposition 3.3. *Assume that the assumptions (H1)-(H3) are satisfied. Let $\{h_n\}$ be generated by (3.1) such that $(u,h_n),(u,t_n),(u,w_n) \in E(\mathcal{G})$ for all $u \in \mathcal{F}$. Then*

(1) the $\lim_{n\to\infty}\phi(h_n,h_1)$ *exists.*

(2) there exists $w^*\in\mathcal{K}$ *such that* $\lim_{n\to\infty}h_n=w^*$.

Proof. (1). Since $h_n=\Pi_{\mathcal{K}_n}(h_1)$, from Lemma 2.3, we have

$$\phi(h_n,h_1)\le\phi(y,h_1)\quad\text{for all}\quad y\in\mathcal{K}_n. \tag{3.7}$$

Note that $h_{n+1}=\Pi_{\mathcal{K}_{n+1}}(h_1)\in\mathcal{K}_{n+1}\subset\mathcal{K}_n$. By taking $y=h_{n+1}$ in (3.7), we get

$$\phi(h_n,h_1)\le\phi(h_{n+1},h_1). \tag{3.8}$$

By Proposition 3.1, we find that there exists a unique $w=\Pi_{\mathcal{F}}(h_1)$. It means $w\in\mathcal{F}\subset\mathcal{K}_n$ by Proposition 3.2. By taking $y=w$ in (3.7), we have

$$\phi(h_n,h_1)\le\phi(w,h_1). \tag{3.9}$$

It follows from (3.8) and (3.9) that the sequence $\{\phi(h_n,h_1)\}$ is nondecreasing and bounded. Therefore, $\lim_{n\to\infty}\phi(h_n,h_1)$ exists.

(2). It follows from Lemma 2.5 and $h_n=\Pi_{\mathcal{K}_n}(h_1)$, we get

$$\phi(y,h_n)+\phi(h_n,h_1)\le\phi(y,h_1)\quad\text{for all}\quad y\in\mathcal{K}_n. \tag{3.10}$$

Let $m>n$. We find that $h_m=\Pi_{\mathcal{K}_m}(h_1)\in\mathcal{K}_m\subset\mathcal{K}_n$. By taking $y=h_m$ in (3.10), we have $\phi(h_m,h_n)+\phi(h_n,h_1)\le\phi(h_m,h_1)$. This implies that

$$\phi(h_m,h_n)\le\phi(h_m,h_1)-\phi(h_n,h_1). \tag{3.11}$$

It follows from (3.11) and the existence of the $\lim_{n\to\infty}\phi(h_n,h_1)$ that $\lim_{m,n\to\infty}\phi(h_m,h_n)=0$. By Lemma 2.2, we get $\lim_{m,n\to\infty}\|h_m-h_n\|=0$. Therefore, $\{h_n\}$ is a Cauchy sequence in $\mathcal{K}$. Then there exists $w^*\in\mathcal{K}$ such that $\lim_{n\to\infty}h_n=w^*$. ∎

Theorem 3.4. *Assume that the assumptions* $(H1)$*-*$(H3)$ *are satisfied. Let* $\{h_n\}$ *be generated by* (3.1) *such that* $(u,h_n),(u,t_n),(u,w_n)\in E(\mathcal{G})$ *for all* $u\in\mathcal{F}$, $\liminf_{n\to\infty}a_n(1-\alpha_n)>0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ *and* $\liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0$. *Then the sequence* $\{h_n\}$ *strongly converges to* $\Pi_{\mathcal{F}}(h_1)$.

Proof. The proof is divided into the following two steps.

Step 1. We prove that $w^*\in\mathcal{F}$ *with* $w^*=\lim_{n\to\infty}h_n$. Indeed, from Proposition 3.3, we find that $\{h_n\}$ is a Cauchy sequence. Hence, there exists $w^*\in\mathcal{K}$ such that $\lim_{n\to\infty}h_n=w^*$ and

$$\lim_{n\to\infty}\|h_{n+1}-h_n\|=0. \tag{3.12}$$

Since $h_{n+1}\in\mathcal{K}_{n+1}\subset\mathcal{K}_n$, from the definition of $\mathcal{K}_{n+1}$, we obtain

$$\phi(h_{n+1},k_n)\le\phi(h_{n+1},h_n)+\gamma_n. \tag{3.13}$$

Since J is uniformly continuous on bounded sets, by (3.12), we obtain

$$\lim_{n\to\infty}\|\mathcal{J}h_{n+1}-\mathcal{J}h_n\|=0. \tag{3.14}$$

Moreover, by Remark 2.1.(2d), we have

$$\phi(h_{n+1},h_n)\le\|h_{n+1}\|.\|\mathcal{J}h_{n+1}-\mathcal{J}h_n\|+\|h_n-h_{n+1}\|.\|h_n\|. \tag{3.15}$$

By combining (3.12), (3.14) and (3.15), we find that

$$\lim_{n\to\infty} \phi(h_{n+1}, h_n) = 0. \tag{3.16}$$

Next, we conclude from the boundedness of the sequence $\{h_n\}$ that there exists $M > 0$ such that

$$\begin{aligned} 0 \leq \varepsilon_n &= (\kappa_n - 1)\big(a_n + \beta_n + \gamma_n + (a_n\beta_n + \beta_n\gamma_n + \gamma_n a_n)(\kappa_n - 1) \\ &\qquad + a_n\beta_n\gamma_n(\kappa_n - 1)^2\big)(\kappa + \|h_n\|)^2 \leq M(\kappa_n - 1). \end{aligned} \tag{3.17}$$

It follows from (3.17) and using $\lim_{n\to\infty} \kappa_n = 1$, we obtain that $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, it follows from (3.13), (3.16) and $\lim_{n\to\infty} \varepsilon_n = 0$ that

$$\lim_{n\to\infty} \phi(h_{n+1}, k_n) = 0. \tag{3.18}$$

Therefore, by Lemma 2.2 and using (3.18), we find that

$$\lim_{n\to\infty} \|h_{n+1} - k_n\| = 0. \tag{3.19}$$

By (3.12) and (3.19), we get

$$\lim_{n\to\infty} \|h_n - k_n\| = 0. \tag{3.20}$$

Since J is uniformly continuous on bounded sets, from (3.20), we get

$$\lim_{n\to\infty} \|\mathcal{J}h_n - \mathcal{J}k_n\| = 0. \tag{3.21}$$

Let $p \in \mathcal{F}$. By using Remark 2.1.(2a), $(u, h_n) \in E(\mathcal{G})$ and the fact that $\mathcal{S}_3$ is an asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping, we have

$$(\|u\| - \|\mathcal{S}_3^n h_n\|)^2 \leq \phi(u, \mathcal{S}_3^n h_n) \leq \kappa_n \phi(u, h_n) \leq \kappa_n(\|u\| + \|h_n\|)^2 \leq \kappa_n(\kappa + \|h_n\|)^2. \tag{3.22}$$

By (3.22) and the boundedness of $\{h_n\}$ and $\{\kappa_n\}$, thus $\{\mathcal{S}_3^n h_n\}$ is also bounded. Put

$$r = \max\{\sup_{n\in\mathbb{N}} \|h_n\|, \sup_{n\in\mathbb{N}} \|\mathcal{S}_3^n h_n\|\} = \max\{\sup_{n\in\mathbb{N}} \|\mathcal{J}h_n\|, \sup_{n\in\mathbb{N}} \|J\mathcal{S}_3^n h_n\|\}.$$

Then $\mathcal{J}h_n \in B_r$ and $\mathcal{J}\mathcal{S}_3^n h_n \in B_r$ for all $n \in \mathbb{N}$. By using $\mathcal{W}$ is uniformly smooth and Lemma 2.6, there exists a strictly increasing, continuous and convex function $\varphi_r : [0, 2r] \longrightarrow [0, \infty)$ such that $\varphi_r(0) = 0$ and

$$\begin{aligned} & \|(1 - \gamma_n)\mathcal{J}h_n + \gamma_n \mathcal{J}\mathcal{S}_3^n h_n\|^2 \\ \leq\ & (1 - \gamma_n)\|\mathcal{J}h_n\|^2 + \gamma_n\|\mathcal{J}\mathcal{S}_3^n h_n\|^2 - \gamma_n(1 - \gamma_n)\varphi_r(\|\mathcal{J}h_n - \mathcal{J}\mathcal{S}_3^n h_n\|) \\ =\ & (1 - \gamma_n)\|h_n\|^2 + \gamma_n\|\mathcal{S}_3^n h_n\|^2 - \gamma_n(1 - \gamma_n)\varphi_r(\|\mathcal{J}h_n - \mathcal{J}\mathcal{S}_3^n h_n\|). \end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(u,w_n) &= \phi\big(u,\mathcal{J}^{-1}\big((1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\big)\big)\\
&= \|u\|^2-2\langle p,(1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\rangle+\|\mathcal{J}^{-1}\big((1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\big)\|^2\\
&= \|u\|^2-2\langle p,(1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\rangle+\|(1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\|^2\\
&\leq \|u\|^2-2\langle p,(1-\gamma_n)\mathcal{J}h_n+\gamma_n\mathcal{J}\mathcal{S}_3^n h_n\rangle+(1-\gamma_n)\|h_n\|^2+\gamma_n\|\mathcal{S}_3^n h_n\|^2\\
&\quad -\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)\\
&= (1-\gamma_n)\big(\|u\|^2-2\langle u,\mathcal{J}h_n\rangle+\|h_n\|^2\big)+\gamma_n\big(\|u\|^2-2\langle u,\mathcal{J}\mathcal{S}_3^n h_n\rangle+\|\mathcal{S}_3^n h_n\|^2\big)\\
&\quad -\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)\\
&= (1-\gamma_n)\phi(u,h_n)+\gamma_n\phi(u,\mathcal{S}_3^n h_n)-\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)\\
&= (1-\gamma_n)\phi(u,h_n)+\gamma_n\kappa_n\phi(u,h_n)-\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)\\
&= \big(1+\gamma_n(\kappa_n-1)\big)\phi(u,h_n)-\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|). \qquad (3.23)
\end{aligned}$$

From (3.2) and (3.3), we have

$$\begin{aligned}
\phi(u,k_n) &\leq (1+\alpha_n(\kappa_n-1))\phi(u,t_n)\\
&\leq (1+\alpha_n(\kappa_n-1))(1+\beta_n(\kappa_n-1))\phi(u,w_n). \qquad (3.24)
\end{aligned}$$

By substituting (3.23) into (3.24) and using (3.5), we have

$$\begin{aligned}
\phi(u,k_n) &\leq (1+\alpha_n(\kappa_n-1))(1+\beta_n(\kappa_n-1))(1+\gamma_n(\kappa_n-1))\phi(u,h_n)\\
&\quad -(1+\alpha_n(\kappa_n-1))(1+\beta_n(\kappa_n-1))\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-J\mathcal{S}_3^n h_n\|)\\
&\leq \phi(u,h_n)+\varepsilon_n-\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\gamma_n(1-\gamma_n)\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)\\
\leq\;& \phi(u,h_n)-\phi(u,k_n)+\varepsilon_n\\
=\;& \|h_n\|^2-\|k_n\|^2+2\langle u,k_n-h_n\rangle+\varepsilon_n\\
\leq\;& (\|h_n\|-\|k_n\|)(\|h_n\|+\|k_n\|)+2\|u\|.\|k_n-h_n\|+\varepsilon_n\\
\leq\;& (\|h_n-k_n\|)(\|h_n\|+\|k_n\|)+2\|u\|.\|k_n-h_n\|+\varepsilon_n. \qquad (3.25)
\end{aligned}$$

It follows from (3.20), (3.25), $\lim_{n\to\infty}\varepsilon_n=0$ and $\liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0$ that

$$\lim_{n\to\infty}\varphi_r(\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|)=0.$$

By combining the above equality and the assumption $\varphi_r(0)=0$, we conclude that

$$\lim_{n\to\infty}\|\mathcal{J}h_n-\mathcal{J}\mathcal{S}_3^n h_n\|=0. \qquad (3.26)$$

Since $\mathcal{J}^{-1}$ is also uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty}\|h_n-\mathcal{S}_3^n h_n\|=0. \qquad (3.27)$$

Then, by combining $\lim_{n\to\infty}h_n=w^*$ and (3.27), we find that

$$\lim_{n\to\infty}\|\mathcal{S}_3^n h_n-w^*\|=0. \qquad (3.28)$$

Next, we have

$$\|\mathcal{S}_3^{n+1}h_n-w^*\|\leq\|\mathcal{S}_3^{n+1}h_n-\mathcal{S}_3^n h_n\|+\|\mathcal{S}_3^n h_n-w^*\|. \qquad (3.29)$$

By combining (3.28), (3.29) and the uniformly asymptotically regular property of $\mathcal{S}_3$, we have $\lim_{n\to\infty} \|\mathcal{S}_3^{n+1}h_n - w^*\| = 0$. Then, by the closedness of $\mathcal{S}_3$ and (3.28), we find that $\mathcal{S}_3 w^* = w^*$. Therefore, $w^* \in F(\mathcal{S}_3)$.

Next, we have $\|\mathcal{J}w_n - \mathcal{J}h_n\| = \gamma_n \|\mathcal{J}\mathcal{S}_3^n h_n - \mathcal{J}h_n\|$. By combining this with (3.27), we obtain

$$\lim_{n\to\infty} \|\mathcal{J}w_n - \mathcal{J}h_n\| = 0. \tag{3.30}$$

By the uniform continuous on bounded sets of $\mathcal{J}^{-1}$ and (3.30), we find that

$$\lim_{n\to\infty} \|w_n - h_n\| = 0. \tag{3.31}$$

By combining (3.31) and $\lim_{n\to\infty} h_n = w^*$, we get that

$$\lim_{n\to\infty} \|w_n - w^*\| = 0. \tag{3.32}$$

Therefore, we find that $\{w_n\}$ is bounded. Note that $\mathcal{S}_2$ is an asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping, by Remark 2.1.(2a) and $(u, w_n) \in E(\mathcal{G})$, we have

$$(\|u\| - \|\mathcal{S}_2^n w_n\|)^2 \le \phi(u, \mathcal{S}_2^n w_n) \le \kappa_n \phi(u, w_n) \le \kappa_n (\|u\| + \|w_n\|)^2 \le \kappa_n (\kappa + \|w_n\|)^2. \tag{3.33}$$

It follows from (3.33) and the boundedness of $\{w_n\}$ that $\{\mathcal{S}_2^n w_n\}$ is bounded. Put

$$\varepsilon = \max\{\sup_{n\in\mathbb{N}} \|w_n\|, \sup_{n\in\mathbb{N}} \|\mathcal{S}_2^n w_n\|\} = \max\{\sup_{n\in\mathbb{N}} \|\mathcal{J}w_n\|, \sup_{n\in\mathbb{N}} \|\mathcal{J}\mathcal{S}_2^n w_n\|\}.$$

Then $\mathcal{J}w_n \in B_\varepsilon$ and $\mathcal{S}_2^n w_n \in B_\varepsilon$ for all $n \in \mathbb{N}$. Since $\mathcal{W}$ is uniformly smooth and using Lemma 2.6, there exists a strictly increasing, continuous and convex function $\varphi_\varepsilon : [0, 2\varepsilon] \longrightarrow [0, \infty)$ such that $\varphi_\varepsilon(0) = 0$ and

$$\begin{aligned}
& \|(1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\|^2 \\
\le\ & (1-\beta_n)\|\mathcal{J}w_n\|^2 + \beta_n \|\mathcal{J}\mathcal{S}_2^n w_n\|^2 - \beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|) \\
=\ & (1-\beta_n)\|w_n\|^2 + \beta_n \|\mathcal{S}_2^n w_n\|^2 - \beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|).
\end{aligned}$$

This leads to

$$\begin{aligned}
& \phi(u, t_n) \\
=\ & \phi\big(u, \mathcal{J}^{-1}\big((1-\beta_n)Jw_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\big)\big) \\
=\ & \|u\|^2 - 2\langle p, (1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\rangle + \|\mathcal{J}^{-1}\big((1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\big)\|^2 \\
=\ & \|u\|^2 - 2\langle p, (1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\rangle + \|(1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\|^2 \\
\le\ & \|u\|^2 - 2\langle p, (1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2^n w_n\rangle + (1-\beta_n)\|w_n\|^2 + \beta_n \|\mathcal{S}_2^n w_n\|^2 \\
& -\beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|) \\
=\ & (1-\beta_n)\big(\|u\|^2 - 2\langle u, \mathcal{J}w_n\rangle + \|\mathcal{J}w_n\|^2\big) \\
& +\beta_n\big(\|u\|^2 - 2\langle u, \mathcal{J}\mathcal{S}_2^n w_n\rangle + \|\mathcal{J}\mathcal{S}_2^n w_n\|^2\big) - \beta_n(1-\beta_n)\varphi_\varepsilon(\|w_n - \mathcal{J}\mathcal{S}_2^n w_n\|) \\
=\ & (1-\beta_n)\phi(u, w_n) + \beta_n \phi(u, \mathcal{S}_2^n w_n) - \beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|) \\
\le\ & (1-\beta_n)\phi(u, w_n) + \beta_n \kappa_n \phi(u, w_n) - \beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|) \\
=\ & (1 + \beta_n(\kappa_n - 1))\phi(u, w_n) - \beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|).
\end{aligned} \tag{3.34}$$

From (3.2) and (3.4), we obtain

$$\phi(u, k_n) \le \big(1 + \alpha_n(\kappa_n - 1)\big)\phi(u, t_n) \tag{3.35}$$

and

$$\phi(u, w_n) \leq \big(1 + \gamma_n(\kappa_n - 1)\big)\phi(u, h_n). \tag{3.36}$$

By substituting (3.36) into (3.34), we get

$$\phi(u, t_n) \leq (1+\beta_n(\kappa_n-1))\big(1+\gamma_n(\kappa_n-1)\big)\phi(u, h_n)-\beta_n(1-\beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n-\mathcal{J}\mathcal{S}_2^n w_n\|). \tag{3.37}$$

By substituting (3.37) into (3.35) and (3.5), we have

$$\begin{aligned}\phi(u, k_n) &\leq \big(1 + \alpha_n(\kappa_n - 1)\big)(1 + \beta_n(\kappa_n - 1))\big(1 + \gamma_n(\kappa_n - 1)\big)\phi(u, h_n)\\ &\quad -\big(1 + \alpha_n(\kappa_n - 1)\big)a_n(1 - \alpha_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|)\\ &\leq \phi(u, h_n) + \gamma_n - \beta_n(1 - \beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|). \end{aligned} \tag{3.38}$$

Similarly to (3.25), from (3.38), we find that

$$\begin{aligned}&\beta_n(1 - \beta_n)\varphi_\varepsilon(\|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\|)\\ \leq\ &(\|h_n - k_n\|)(\|h_n\| + \|k_n\|) + 2\|u\|.\|k_n - h_n\| + \gamma_n.\end{aligned} \tag{3.39}$$

Then, it follows from (3.20), (3.39), $\lim_{n\to\infty} \gamma_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and the assumption $\varphi_\varepsilon(0) = 0$ that

$$\lim_{n\to\infty} \|\mathcal{J}w_n - \mathcal{J}\mathcal{S}_2^n w_n\| = 0.$$

Since J^{-1} is the uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \|w_n - \mathcal{S}_2^n w_n\| = 0. \tag{3.40}$$

Then, by combining this with (3.32), we conclude that

$$\lim_{n\to\infty} \|\mathcal{S}_2^n w_n - w^*\| = 0. \tag{3.41}$$

Furthermore, we have

$$\|\mathcal{S}_2^{n+1} w_n - w^*\| \leq \|\mathcal{S}_2^{n+1} w_n - \mathcal{S}_2^n w_n\| + \|\mathcal{S}_2^n w_n - w^*\|. \tag{3.42}$$

By (3.41), (3.42) and the uniformly asymptotically regular property of $\mathcal{S}_2$, we find that $\lim_{n\to\infty} \|\mathcal{S}_2^{n+1} w_n - w^*\| = 0$. Then, by the closedness of $\mathcal{S}_2$ and (3.41), we find that $\mathcal{S}_2 w^* = w^*$ and hence $w^* \in F(\mathcal{S}_2)$.

Next, we have $\|\mathcal{J}t_n - \mathcal{J}w_n\| = \beta_n\|\mathcal{J}\mathcal{S}_2^n w_n - \mathcal{J}w_n\|$. By combining this with (3.40), we get

$$\lim_{n\to\infty} \|\mathcal{J}t_n - \mathcal{J}w_n\| = 0. \tag{3.43}$$

It follows from the uniform continuous on bounded sets of $\mathcal{J}^{-1}$ and (3.30) that

$$\lim_{n\to\infty} \|t_n - w_n\| = 0. \tag{3.44}$$

By combining (3.32) and (3.44), we get that

$$\lim_{n\to\infty} \|t_n - w^*\| = 0. \tag{3.45}$$

This implies that $\{t_n\}$ is bounded. By using Remark 2.1.(2a), and the fact that $\mathcal{S}_1$ is an asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping and $(u, t_n) \in E(\mathcal{G})$, we have

$$(\|u\| - \|\mathcal{S}_1^n t_n\|)^2 \leq \phi(u, \mathcal{S}_1^n t_n) \leq \kappa_n\phi(u, t_n) \leq \kappa_n(\|u\| + \|t_n\|)^2 \leq \kappa_n(\kappa + \|t_n\|)^2. \tag{3.46}$$

It follows from (3.46) and the boundedness of $\{t_n\}$ that $\{\mathcal{S}_1^n t_n\}$ is bounded. Put

$$\lambda = \max\{\sup_{n\in\mathbb{N}} \|t_n\|, \sup_{n\in\mathbb{N}} \|\mathcal{S}_1^n t_n\|\} = \max\{\sup_{n\in\mathbb{N}} \|\mathcal{J} t_n\|, \sup_{n\in\mathbb{N}} \|\mathcal{J}\mathcal{S}_1^n t_n\|\}.$$

Then $\mathcal{J}t_n \in B_\lambda$ and $\mathcal{S}_1^n t_n \in B_\lambda$ for all $n \in \mathbb{N}$. Since $\mathcal{W}$ is uniformly smooth and using Lemma 2.6, there exists a strictly increasing, continuous and convex function $\varphi_\lambda : [0, 2\lambda] \longrightarrow [0,\infty)$ such that $\varphi_\lambda(0) = 0$ and

$$\begin{aligned}
& \|(1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\|^2 \\
\leq\ & (1-\alpha_n)\|\mathcal{J}t_n\|^2 + \alpha_n \|\mathcal{J}\mathcal{S}_2^n t_n\|^2 - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
=\ & (1-\alpha_n)\|t_n\|^2 + \alpha_n \|\mathcal{S}_1^n t_n\|^2 - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \phi(u, k_n) \\
=\ & \phi\big(u, \mathcal{J}^{-1}\big((1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\big)\big) \\
=\ & \|u\|^2 - 2\langle p, (1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\rangle + \|\mathcal{J}^{-1}\big((1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\big)\|^2 \\
=\ & \|u\|^2 - 2\langle p, (1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\rangle + \|(1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n u w_n\|^2 \\
\leq\ & \|u\|^2 - 2\langle p, (1-\alpha_n)\mathcal{J}t_n + \alpha_n \mathcal{J}\mathcal{S}_1^n t_n\rangle + (1-\alpha_n)\|t_n\|^2 + \alpha_n \|\mathcal{S}_1^n t_n\|^2 \\
& -\alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
=\ & (1-\alpha_n)\big(\|u\|^2 - 2\langle u, \mathcal{J}t_n\rangle + \|\mathcal{J}t_n\|^2\big) \\
& + a_n\big(\|u\|^2 - 2\langle u, \mathcal{J}\mathcal{S}_1^n t_n\rangle + \|\mathcal{J}\mathcal{S}_1^n t_n\|^2\big) - \alpha_n(1-\alpha_n)\varphi_\lambda(\|t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
=\ & (1-\alpha_n)\phi(u, t_n) + \alpha_n \phi(u, \mathcal{S}_1^n t_n) - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
\leq\ & (1-\alpha_n)\phi(u, t_n) + a_n \kappa_n \phi(u, t_n) - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
=\ & (1+\alpha_n(\kappa_n - 1))\phi(u, t_n) - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|). \qquad (3.47)
\end{aligned}$$

From (3.3) and (3.4), we obtain

$$\phi(u, t_n) \leq \big(1 + \beta_n(\kappa_n - 1)\big)\phi(u, w_n) \qquad (3.48)$$

and

$$\phi(u, w_n) \leq \big(1 + \gamma_n(\kappa_n - 1)\big)\phi(u, h_n). \qquad (3.49)$$

By substituting (3.49) into (3.48), we get

$$\phi(u, t_n) \leq (1 + \beta_n(\kappa_n - 1))\big(1 + \gamma_n(\kappa_n - 1)\big)\phi(u, h_n). \qquad (3.50)$$

By substituting (3.50) into (3.47) and from (3.5), we have

$$\begin{aligned}
\phi(u, k_n) \ \leq\ & \big(1 + \alpha_n(\kappa_n - 1)\big)(1 + \beta_n(\kappa_n - 1))\big(1 + \gamma_n(\kappa_n - 1)\big)\phi(u, h_n) \\
& -a_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \\
\leq\ & \phi(u, h_n) + \gamma_n - \alpha_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|). \qquad (3.51)
\end{aligned}$$

Similarly to (3.25), from (3.51), we find that

$$\begin{aligned}
& a_n(1-\alpha_n)\varphi_\lambda(\|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\|) \qquad (3.52) \\
\leq\ & (\|h_n - k_n\|)(\|h_n\| + \|k_n\|) + 2\|u\|.\|k_n - h_n\| + \gamma_n.
\end{aligned}$$

Then, it follows from (3.20), (3.52), $\lim_{n\to\infty} \gamma_n = 0$, $\liminf_{n\to\infty} a_n(1-\alpha_n) > 0$ and the assumption $\varphi_\varepsilon(0) = 0$ that

$$\lim_{n\to\infty} \|\mathcal{J}t_n - \mathcal{J}\mathcal{S}_1^n t_n\| = 0.$$

Since J^{-1} is the uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \|t_n - \mathcal{S}_1^n t_n\| = 0. \tag{3.53}$$

Then, by combining this with (3.45), we conclude that

$$\lim_{n\to\infty} \|\mathcal{S}_1^n t_n - w^*\| = 0. \tag{3.54}$$

Furthermore, we have

$$\|\mathcal{S}_1^{n+1} t_n - w^*\| \leq \|\mathcal{S}_1^{n+1} t_n - \mathcal{S}_1^n t_n\| + \|\mathcal{S}_1^n t_n - w^*\|. \tag{3.55}$$

By combining (3.54) and (3.55) with the uniformly asymptotically regular property of $\mathcal{S}_1$, we obtain $\lim_{n\to\infty} \|\mathcal{S}_1^{n+1} t_n - w^*\| = 0$. Then, by the closedness of $\mathcal{S}_1$ and (3.54), we find that $\mathcal{S}_1 w^* = w^*$ and hence $w^* \in F(\mathcal{S}_1)$. Therefore, $w^* \in \mathcal{F} = \bigcap_{i=1}^{3} F(\mathcal{S}_i)$.

Step 2. We claim that $w^* = \Pi_{\mathcal{F}}(h_1)$. Indeed, by Lemma 2.4 and $h_n = \Pi_{\mathcal{K}_n}(h_1)$, we get

$$\langle h_n - y, \mathcal{J}h_1 - \mathcal{J}h_n \rangle \geq 0 \quad \text{for all} \quad y \in \mathcal{K}_n. \tag{3.56}$$

Let $u \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{K}_n$, we have $u \in \mathcal{K}_n$. By choosing $y = u$ in (3.56), we get

$$\langle h_n - p, \mathcal{J}h_1 - \mathcal{J}h_n \rangle \geq 0. \tag{3.57}$$

Taking the limit in (3.57) as $n \to \infty$ and using $\lim_{n\to\infty} h_n = w^*$, we find that

$$\langle w^* - p, \mathcal{J}h_1 - \mathcal{J}w^* \rangle \geq 0.$$

By using Lemma 2.4, we conclude that $w^* = \Pi_{\mathcal{F}}(h_1)$. ∎

Since every quasi $\mathcal{G}$-ϕ-nonexpansive mapping is an asymptotically quasi $\mathcal{G}$-ϕ-nonexpansive mapping with $\kappa_n = 1$ for all $n \in \mathbb{N}$, from Theorem 3.4, we obtain the convergence of a hybrid iterative process for three quasi $\mathcal{G}$-ϕ-nonexpansive mappings in uniformly smooth and uniformly convex Banach space with directed graphs.

Corollary 3.5. *Assume that the assumptions* $(H1)$ *and* $(H2)$ *are satisfied. Let* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : V(\mathcal{G}) \longrightarrow V(\mathcal{G})$ *be three quasi* $\mathcal{G}$-ϕ*-nonexpansive mappings such that* $F(\mathcal{S}_i) \times F(\mathcal{S}_i) \subset E(\mathcal{G})$ *for all* $i = 1, 2, 3$ *and* $\mathcal{F} = \bigcap_{i=1}^{3} F(\mathcal{S}_i)$ *is nonempty,* $\{h_n\}$ *be generated by*

$$\begin{cases} h_1 \in \mathcal{K}, \mathcal{K}_1 = \mathcal{K} \\ w_n = \mathcal{J}^{-1}\big((1-\gamma_n)\mathcal{J}h_n + \gamma_n \mathcal{J}\mathcal{S}_3 h_n\big) \\ t_n = \mathcal{J}^{-1}\big((1-\beta_n)\mathcal{J}w_n + \beta_n \mathcal{J}\mathcal{S}_2 w_n\big) \\ k_n = \mathcal{J}^{-1}\big((1-\alpha_n)\mathcal{J}t_n + a_n \mathcal{J}\mathcal{S}_1 t_n\big) \\ \mathcal{K}_{n+1} = \big\{w \in \mathcal{K}_n : \phi(w, k_n) \leq \phi(w, h_n)\big\} \\ h_{n+1} = \Pi_{\mathcal{K}_{n+1}}(h_1), n \geq 1, \end{cases} \tag{3.58}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *such that* $(u, h_n), (u, t_n), (u, w_n) \in E(\mathcal{G})$ *for all* $u \in \mathcal{F}$*, and* $\liminf_{n\to\infty} a_n(1-\alpha_n) > 0, \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0, \liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$*. Then the sequence* $\{h_n\}$ *strongly converges to* $\Pi_{\mathcal{F}}(h_1)$*.*

Remark 3.6. (1) If $\mathcal{W}$ is a Hilbert space, then $\phi(h,k) = \|h-k\|^2$, $\mathcal{J}$ is an identify mapping, and the generalized projection becomes the metric projection. So, iteration (3.1) and iteration (3.58) become iteration (3.59) and iteration (3.60), respectively.

$$\begin{cases} h_1 \in \mathcal{K}, \mathcal{K}_1 = \mathcal{K} \\ w_n = (1-\gamma_n)h_n + \gamma_n \mathcal{S}_3^n h_n \\ t_n = (1-\beta_n)w_n + \beta_n \mathcal{S}_2^n w_n \\ k_n = (1-\alpha_n)t_n + a_n \mathcal{S}_1^n t_n \\ \mathcal{K}_{n+1} = \{w \in \mathcal{K}_n : \|w-k_n\|^2 \le \|w-h_n\|^2 + \gamma_n\} \\ h_{n+1} = P_{\mathcal{K}_{n+1}}(h_1), n \ge 1, \end{cases} \tag{3.59}$$

and

$$\begin{cases} h_1 \in \mathcal{K}, \mathcal{K}_1 = \mathcal{K} \\ w_n = (1-\gamma_n)h_n + \gamma_n \mathcal{S}_3 h_n \\ t_n = (1-\beta_n)w_n + \beta_n \mathcal{S}_2 w_n \\ k_n = (1-\alpha_n)t_n + a_n \mathcal{S}_1 t_n \\ \mathcal{K}_{n+1} = \{w \in \mathcal{K}_n : \|w-k_n\| \le \|w-h_n\|\} \\ h_{n+1} = P_{\mathcal{K}_{n+1}}(h_1), n \ge 1, \end{cases} \tag{3.60}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$, $P_{\mathcal{K}_{n+1}}(h_1)$ is the metric projection mapping from h_1 onto $\mathcal{K}_{n+1}$.

(2) The conclusions of Theorem 3.4 and Corollary 3.5 hold with iteration (3.59) and iteration (3.60).

Next, we give a numerical example to illustrate the convergence of the proposed iteration processes to common fixed point of three asymptotically $\mathcal{G}$-ϕ-nonexpansive mappings.

Example 3.7. Let $\mathcal{W} = \mathbb{R}$, $\mathcal{K} = [0,1]$ and $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})$ be a directed graph defined by $V(\mathcal{G}) = \mathcal{K}$ and $E(\mathcal{G}) = \{(h,k) \in \mathcal{K} \times \mathcal{K} : u \neq v \in [0,0.9] \text{ or } h = k \in \mathcal{K}\}$. Then $E(\mathcal{G})$ is coordinate-convex and $\{(h,h) : h \in V(\mathcal{G})\} \subset E(\mathcal{G})$. Define three mappings by $\mathcal{S}_1 h = \frac{h}{2}$, $\mathcal{S}_2 h = \frac{h^2}{2}$ and $\mathcal{S}_3 h = \frac{h^3}{9}$ for all $h \in \mathcal{K}$. Then $\mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) = \{0\}$ and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are three uniformly asymptotically regular, closed and asymptotically $\mathcal{G}$-ϕ-nonexpansive mappings with $\kappa_n = 1$ for all $n \in \mathbb{N}$. Let $a_n = \frac{4n+1}{5n+2}$, $\beta_n = \frac{3n+1}{4n+2}$ and $\gamma_n = \frac{2n+1}{3n+2}$ for al $n \in \mathbb{N}$ and $h_1 = 0.8 \in \mathcal{K}$. Numerical results of iteration processes (3.1) and (3.58) are presented by Table 1.

The numerical results in Table 1 prove for given mappings, iteration processes (3.1) and (3.58) converge to 0. Furthermore, iteration process (3.1) converges to 0 faster than iteration processes (3.58).

n	Value of h_n in iteration (3.1)	Value of h_n in iteration (3.58)
1	0.8000000	0.8000000
2	0.4388899	0.4308278
3	0.2274612	0.2287160
4	0.1168885	0.1207647
5	0.0597976	0.0636149
⋮	⋮	⋮
22	0.0000006	0.0000011
23	0.0000003	0.0000006
24	0.0000002	0.0000003
25	8.111D-08	0.0000002
⋮	⋮	⋮
54	1.251D-16	6.618D-16
55	0.	3.475D-16
56	0.	1.825D-16
57	0.	0.

Table 1. Numerical results of of iteration processes (3.1) and (3.58).

Finally, we apply Theorem 3.4 to approximate the solution to the system of the nonlinear integral equations.

Example 3.8. Let $\mathcal{W} = L_2([a,b])$ be the Hilbert space with norm $\|f\| = \sqrt{\int_a^b |f(h)|^2 dh}$ for $f \in \mathcal{W}$. We consider the system of the nonlinear integral equations as follows.

$$f(h) = G_i(h) + \int_a^b K_i(h,k,f(k))dk \tag{3.61}$$

where $h \in [a,b]$, $G_i : [a,b] \longrightarrow \mathbb{R}$ and $K_i : [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1,2,3$. Put

$$\mathcal{K} = \{f \in X : f(h) \geq 0 \text{ for all } h \in [a,b]\}$$

and

$$\mathcal{S}_i f(h) = G_i(h) + \int_a^b K_i(h,k,f(k))dk$$

for all $h \in [a,b]$ and $f \in \mathcal{W}$. For each $i = 1,2,3$, let us assume that the following conditions are satisfied.

(1) K_i is continuous on $[a,b]\times[a,b]\times\mathbb{R}$, G_i is integrable on $[a,b]$, and $G_i(h), K_i(h,k,l) \geq 0$ for all $h,k \in [a,b]$ and $l \in \mathbb{R}$.

(2) There exists $\mu_i : [a,b] \times [a,b] \longrightarrow [0,\infty)$ such that μ_i is continuous on $[a,b] \times [a,b]$, $\sup_{h\in[a,b]} \int_a^b \mu_i^2(h,k)dk \leq \frac{1}{b-a}$ and

$$|K_i(h,k,f(k))) - K_i(h,k,g(k))| \leq \mu_i(h,k)|f(k) - g(k)|$$

for all $h,k \in [a,b]$ and $f,g \in \mathcal{K}$.

(3) $\{f_n\}$ is a sequence generated by

$$\begin{cases} f_1 \in \mathcal{K}, \mathcal{K}_1 = \mathcal{K} \\ h_n = (1-\gamma_n) f_n + \gamma_n \mathcal{S}_3^n f_n, \\ \varphi_n = (1-\beta_n) h_n + \beta_n \mathcal{S}_2^n h_n \\ \psi_n = (1-\alpha_n)\varphi_n + a_n \mathcal{S}_1^n \varphi_n \\ \mathcal{K}_{n+1} = \{\Phi \in \mathcal{K}_n : \|\psi_n - \Phi\| \leq \|f_n - \Phi\|\} \\ f_{n+1} = P_{\mathcal{K}_{n+1}} f_1, n \geq 1, \end{cases} \tag{3.62}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ satisfying $\liminf_{n\to\infty} a_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$.

If system of nonlinear integral equations (3.61) has a solution $f \in \mathcal{K}$, then iteration process (3.62) converges to f.

In fact, let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a directed graph defined by $V(\mathcal{G}) = \mathcal{K}$ and $E(\mathcal{G}) = \{(f,g) : f, g \in \mathcal{K}\}$. Then $E(\mathcal{G})$ is coordinate-convex and $\{(f,f) : f \in V(\mathcal{G})\} \subset E(\mathcal{G})$. For $i = 1,2,3$, we define three mappings $\mathcal{S}_i : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$\mathcal{S}_i f(h) = G_i(h) + \int_a^b K_i(h, kf(k))dk$$

for all $h \in [a,b]$ and $f \in \mathcal{K}$. It follows from condition (1) that $\mathcal{S}_i$ is well-defined for each $i = 1,2,3$. Notice that $f \in \mathcal{K}$ is a solution of the systems of nonlinear integral equations (3.61) if only if $f \in \mathcal{K}$ is a common fixed point of $\mathcal{S}_i$.

Next, we will show that $\mathcal{S}_i$ is an asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping for each $i = 1,2,3$. Indeed, for all $(f,g) \in E(\mathcal{G})$, we have $f(h), g(h) \geq 0$ for all $h \in [a,b]$ and hence $f, g \in \mathcal{K}$. This gives that $\mathcal{S}f, \mathcal{S}g \in \mathcal{K}$ by condition (1). Therefore, $(\mathcal{S}f, \mathcal{S}g) \in E(\mathcal{G})$. Furthermore, for all $(f,g) \in E(\mathcal{G})$, we have $f(h), g(h) \geq 0$ for all $h \in [a,b]$ and hence by condition (2), we get

$$\begin{aligned} |\mathcal{S}_i f(h) - \mathcal{S}_i g(h)|^2 &= \left(\int_a^b |K_i(h,k,f(k)) - K_i(h,k,g(k))|dk \right)^2 \\ &\leq \left(\int_a^b \mu(h,k)|f(k) - g(k)|dk \right)^2 \\ &\leq \int_a^b \mu^2(h,k)dk \int_a^b |f(k) - g(k)|^2 dk \\ &\leq \frac{1}{b-a}\|f - g\|^2. \end{aligned}$$

This implies that

$$\|\mathcal{S}_i f - \mathcal{S}_i g\|^2 = \int_a^b |\mathcal{S}_i f(h) - \mathcal{S}_i g(h)|^2 dh \leq \int_a^b \frac{1}{b-a}\|f-g\|^2 dh = \|f-g\|^2.$$

This gives that $\|\mathcal{S}_i f - \mathcal{S}_i g\| \leq \|f - g\|$ and hence $\mathcal{S}_i$ is a $\mathcal{G}$-asymptotically nonexpansive mapping with $\kappa_n = 1$ for all $n \in \mathbb{N}$. Therefore, for each $i = 1,2,3$, we conclude that T_i is an asymptotically $\mathcal{G}$-ϕ-nonexpansive mapping with $\phi(h,k) = \|h-k\|^2$ and $\kappa_n = 1$ for all $n \in \mathbb{N}$.

Note that if $u \in \mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3)$, then u is a solution of the system of nonlinear integral equations (3.61) and hence $u \in \mathcal{K}$. This implies that $F(\mathcal{S}_i) \times F(\mathcal{S}_i) \subset E(\mathcal{G})$ and $(u, \varphi_n), (u, h_n), (u, f_n) \in E(\mathcal{G})$.

The following example guarantees the existence of the functions K_i and G_i satisfying all the conditions in Example 3.8.

Example 3.9. Let $\mathcal{W} = L_2([0,1])$ be a Hilbert space with inner product

$$\langle f, g\rangle = \int_0^1 f(h)g(h)dh$$

for all $f, g \in \mathcal{W}$ and norm $\|f\| = \sqrt{\int_0^1 |f(h)|^2 dh}$ for all $f \in \mathcal{W}$. We consider the following system of the nonlinear integral equations.

$$\begin{cases} f(h) = \Big(1 - \dfrac{\sqrt{15}}{\sqrt{28}}\Big)h^2 + \displaystyle\int_0^1 \dfrac{3\sqrt{15}h^2(1+k^2)f(k)}{\sqrt{28}(1+f(k))}dk \\ f(h) = h^2(1+\cos 1 - \sin 1) + \displaystyle\int_0^1 2h^2k^3 \sin f(k)dk, \end{cases} \tag{3.63}$$

for all $h \in [0,1]$ and $f \in \mathcal{W}$. Put $\mathcal{K} = \big\{f \in \mathcal{W} : f(h) \geq 0 \text{ for all } h \in [0,1]\big\}$,

$$G_1(h) = \Big(1 - \frac{\sqrt{15}}{\sqrt{28}}\Big)h^2,\ K_1(h,k,f(k)) = \frac{3\sqrt{15}h^2(1+k^2)f(k)}{\sqrt{28}(1+f(k))},$$

$$G_i(h) = h^2(1+\cos 1 - \sin 1), K_i(h,k,f(k)) = 2h^2k^3 \sin f(k) \text{ for } i = 2, 3,$$

and

$$\mathcal{S}_1 f(h) = \Big(1 - \frac{\sqrt{15}}{\sqrt{28}}\Big)h^2 + \int_0^1 \frac{3\sqrt{15}h^2(1+k^2)f(k)}{\sqrt{28}(1+f(k))}dk$$

and

$$\mathcal{S}_2 f(h) = \mathcal{S}_3 f(h) = h^2(1+\cos 1 - \sin 1) + \int_0^1 2h^2k^3 \sin f(k)dk$$

for all $h \in [0,1]$ and $f \in \mathcal{W}$. First, it is easy to see that condition (1) in Example 3.8 is satisfied. Next, we prove condition (2) in Example 3.8 is satisfied. Indeed, for all $f, g \in \mathcal{K}$, we have

$$\begin{aligned} |K_1(h,k,f(k)) - K_1(h,k,g(k))| &= \frac{3\sqrt{15}h^2(1+k^2)}{\sqrt{28}}\Big|\frac{f(k)}{1+f(k)} - \frac{g(k)}{1+g(k)}\Big| \\ &\leq \frac{3\sqrt{15}h^2(1+k^2)}{\sqrt{28}}|f(k) - g(k)|. \end{aligned}$$

By choosing $\mu_1(h,k) = \dfrac{3\sqrt{15}h^2(1+k^2)}{\sqrt{28}}$, we find that μ_1 is continuous on $[0,1] \times [0,1]$, $\sup\limits_{h\in[0,1]} \displaystyle\int_0^1 \mu_1^2(h,k)dk = \sup\limits_{h\in[0,1]} h^4 = 1$ and

$$|K_1(h,k,f(k)) - K_1(h,k,g(k))| \leq \mu_1(h,k)|f(k) - g(k)|.$$

Furthermore, for $i = 2, 3$, and $h, k \in [0, 1]$, we obtain

$$\begin{aligned} |K_i(h,k,f(k)) - K_i(h,k,g(k))| &= 2h^2k^3\Big|\sin f(k) - \sin g(k)\Big| \\ &\leq 2h^2k^3|f(k) - g(k)|. \end{aligned}$$

By choosing $\mu_i(h,k) = 2h^2k^3$, we get that μ_i is continuous on $[0,1] \times [0,1]$,

$$\sup_{h\in[0,1]} \int_0^1 \mu_i^2(h,k)dk = \sup_{h\in[0,1]} \frac{4h^4}{7} = \frac{4}{7} \leq 1$$

and

$$K_i(h,k,f(k)) - K_i(h,k,g(k))| \leq \mu_i(h,k)|f(k) - g(k)|.$$

From Example 3.8, we see that $f(h) = h^2$ for all $h \in [0,1]$ is a solution of the system nonlinear integral equations (3.63).

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