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A Novel Noor Iterative Technique for Mixed Type Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract The aim of this paper is to propose a novel Noor iteration technique for approximating a common fixed point of three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings in hyperbolic spaces. Then, a strong convergence theorem under mild conditions in a uniformly convex hyperbolic space is established. The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

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1. Introduction and Preliminaries

Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces have been obtained in literature, for example, see [4–7, 15, 17, 19, 25, 26, 37, 38, 49–53]. Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is easier working with Banach space due to its convex structures. However, metric space do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi [54] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [21, 30, 42]). Although the class of hyperbolic spaces defined by Kohlenbach [30] is slightly

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restrictive than the class of hyperbolic spaces introduced in [21], it is however, more general than the class of hyperbolic spaces introduced in [42]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [30]. Some other examples of this class of hyperbolic spaces includes Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and R-trees, see [8, 16, 21, 22, 30, 42].

The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. Fixed point theory and hence approximation techniques have been extended to hyperbolic spaces (see [1–3, 9, 43–45] and references therein).

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [30]. Recall that a hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is a metric space $(\mathcal{X}, d)$ together with a mapping $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ satisfying

$(H1) : d(z, \mathcal{H}(x, y, \beta)) \le (1 - \beta)\, d(z, x) + \beta d(z, y)$,

$(H2) : d(\mathcal{H}(x, y, \beta)), \mathcal{H}(x, y, \gamma) = |\beta - \gamma|\, d(x, y)$,

$(H3) : \mathcal{H}(x, y, \beta) = \mathcal{H}(y, x, (1 - \beta))$,

$(H4) : d(\mathcal{H}(x, z, \beta), \mathcal{H}(y, w, \beta)) \le (1 - \beta)\, d(x, y) + \beta d(z, w)$

for all $x, y, w, z \in \mathcal{X}$ and $\beta, \gamma \in [0, 1]$.

A subset $\mathcal{K}$ of a hyperbolic space $\mathcal{X}$ is convex if $\mathcal{H}(x, y, \beta) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\beta \in [0, 1]$.

Recall that a hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is said to be

(i) strictly convex [54] if for any $u, v \in \mathcal{X}$ and $\psi \in [0, 1]$, there exists a unique element $z \in \mathcal{X}$ such that $d(z, x) = \beta d(x, y)$ and $d(z, y) = (1 - \beta)d(x, y)$;

(ii) uniformly convex [48] if for all $x, y, w \in \mathcal{X}$, $r > 0$ and $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that $d(\mathcal{H}(x, y, \frac{1}{2}), x) \le (1 - \delta)r$ whenever $d(x, w) \le r, d(y, w) \le r$ and $d(x, y) \ge \epsilon r$.

Recall that a mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called modulus of uniform convexity. We call η-monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [31]).

Let $(\mathcal{X}, d)$ be a metric space, and let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. We denote the fixed point set of a mapping $\mathcal{T}$ by

$$F(\mathcal{T}) = \{x \in \mathcal{K} : \mathcal{T}x = x\}$$

and

$$d(x, F(\mathcal{T})) = \inf\{d(x, p) : p \in F(\mathcal{T})\}.$$

A self-mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be

(i) nonexpansive if $d(\mathcal{T}x, \mathcal{T}y) \le d(x, y)$ for all $x, y \in \mathcal{K}$.

(ii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \le k_n d(x, y)$ for all $x, y \in \mathcal{K}$ and $n \ge 1$.

(iii) uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$d(\mathcal{T}^n x, \mathcal{T}^n y) \le \mathcal{L} d(x, y)$$

for all $x, y \in \mathcal{K}$ and $n \ge 1$.

From the above definitions, one clearly sees that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \forall n \ge 1$. Both nonexpansive mappings and asymptotically nonexpansive mappings are Lipschitzian continuous. To be more precise, each nonexpansive mapping is $\mathcal{L}$-Lipschitzian and each asymptotically nonexpansive mapping is uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup_{n\in\mathbb{N}}\{k_n\}$.

Recently, various fixed-point iteration processes for nonexpansive mappings have been studied extensively by many authors [27, 33, 39, 47, 55].

In 1972, Goebel and Kirk [20] introduced the class of asymptotically nonexpansive self-mappings. They proved that if $\mathcal{K}$ is nonempty closed convex subset of a real uniformly convex Banach space and $\mathcal{T}$ is an asymptotically nonexpansive self-mapping on $\mathcal{K}$, then $\mathcal{T}$ has a fixed point.

In 1991, Schu [46] introduced the following modified Mann iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\mathcal{T}^n x_n, \quad n \geq 1, \tag{1.1}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schus iteration process (1.1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces; see, e.g., [12, 14, 32] and the references therein.

Recall that a subset K of space X is said to be a retract if there exists a continuous mapping $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ such that $\mathcal{P}x = x$, $\forall x \in \mathcal{K}$. $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. If P is a retraction, then $x = \mathcal{P}x$ for all x in the range of $\mathcal{P}$. We refer to [10, 22, 41] for more details.

For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping (see [13]) if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}x, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}y) \leq k_n d(x,y) \tag{1.2}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$. We denote by $(\mathcal{P}\mathcal{T})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping.

For asymptotically nonexpansive nonself-mappings Chidume, Ofoedu, and Zegeye [13] studied the following iterative sequence

$$x_{n+1} = \mathcal{P}((1-\alpha_n)x_n + \alpha_n\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}x_n) \tag{1.3}$$

to approximate some fixed point of $\mathcal{T}$. They obtained a convergence theorem under suitable conditions in real uniformly convex Banach spaces. If $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping. Hence, (1.3) reduces to (1.1).

In 2006, Wang [58] considered the following iteration process which is a generalization of (1.3) (see also [57]),

$$\begin{aligned} y_n &= \mathcal{P}((1-\beta_n)x_n + \beta_n\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n), \\ x_{n+1} &= \mathcal{P}((1-\alpha_n)x_n + \alpha_n\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}y_n), \ n \geq 1, \end{aligned} \tag{1.4}$$

where $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). They obtain a strong convergence theorem under weak restrictions imposed on the control parameters.

In 2012, Guo, Cho and Guo [23] further studied the following iteration scheme

$$\begin{aligned} x_n &= \mathcal{P}((1-\beta_n)\mathcal{S}_2^n x_n + \beta_n\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n), \\ x_{n+1} &= \mathcal{P}((1-\alpha_n)\mathcal{S}_1^n x_n + \alpha_n\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n), \ n \geq 1, \end{aligned} \tag{1.5}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). Weak and strong convergence theorems of common fixed points of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ were obtained.

Another classical iteration precess was introduced by Noor [34] which is formulated as follows: $x_1 = x \in \mathcal{K}$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n \mathcal{S}x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n \mathcal{S}z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \mathcal{S}y_n, \ n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1]. Such iterative method is called Noor iteration. Because of its simplicity, the method (1.6) has been widely utilized to solve the fixed point problem, and as a result, it has been enhanced by many works, as seen in [28, 35, 36, 40].

Glowinski and Le Tallec [18] employed three-step iterative approaches to find solutions for the problem of elastoviscoplasticity, eigenvalue computation and the theory of liquid crystals. In [18], it was shown that the three-step iterative process yields better numerical results than the estimated iterations in two and one steps. In 1998, Haubruge, Nguyen and Strodiot [24] studied the convergence analysis of three-step methods of Glowinski and Le Tallec [18] and applied these methods to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. As a result, we conclude that the three-step approach plays an important and substantial role in the solution of numerous problems in pure and applied sciences.

Let $\mathcal{K}$ be a nonempty closed convex subset of a real uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings. For an arbitrary $x_1 \in \mathcal{K}$, we suggest the following novel Noor iterative scheme for mixed type asypmtotically nonexpansive mappings

$$\begin{aligned} z_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \\ y_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, \beta_n)), \\ x_{n+1} &= \mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, T_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n, \gamma_n)), \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1).

In this paper, we, motivated by the above recent results, study the strong convergence of the novel Noor iteration scheme for three asymptotically nonexpansive self-mappings $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and three asymptotically nonexpansive nonself-mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ in the setting of uniformly convex hyperbolic spaces. The results presented in this paper extend and improve some recent results announced in the literature. To show our main convergence theorems, we shall need the following useful lemmas.

Lemma 1.1. [56] *Let $\{a_n\}$,$\{b_n\}$ and $\{c_n\}$ be sequences of non-negative real numbers such that $a_{n+1} \leq (1+b_n)a_n + c_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.*

Lemma 1.2. [29] *Let x_n and y_n be two sequences of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ such that, for $\mathcal{R} \in [0,\infty)$, $\limsup_{n\to\infty} d(x_n, a) \leq \mathcal{R}$, $\limsup_{n\to\infty} d(y_n, a) \leq \mathcal{R}$ and $\lim_{n\to\infty} d(\mathcal{H}(x_n, y_n, \alpha_n)) = \mathcal{R}$ where $\alpha_n \in [a, b]$ with $0 < a < b < 1$, then we have, $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

2. Main Results

In this section, we consider a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and prove a strong convergence theorem for $\mathcal{X}$, using the iterative scheme given in (1.7).

Lemma 2.1. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive selfmappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1,\infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonselfmappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\} \subset [1,\infty)$ such that, $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega = F(\mathcal{S}_1) \bigcap F(\mathcal{S}_2) \bigcap F(\mathcal{S}_3) \bigcap F(\mathcal{T}_1) \bigcap F(\mathcal{T}_2) \bigcap F(\mathcal{T}_3) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.7) where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$. Then $\lim_{n\to\infty} d(x_n, v)$ exists for any $v \in \Omega$.*

***Proof*.** Using (1.7) and setting $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$, we have

$$\begin{aligned} d(z_n, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), v) \\ &\leq d(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n), v) \\ &\leq (1-\alpha_n)d(\mathcal{S}_1^n x_n, v) + \alpha_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, v) \\ &\leq (1-\alpha_n)h_n d(x_n, v) + \alpha_n h_n d(x_n, v) \\ &= h_n d(x_n, v) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} d(y_n, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, \beta_n)), v) \\ &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n))), \beta_n)), v) \\ &\leq d(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \beta_n), v) \\ &\leq (1-\beta_n)d(\mathcal{S}_2^n x_n, v) + \beta_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), v) \\ &\leq (1-\beta_n)h_n^2 d(x_n, v) + \beta_n h_n^2 d(x_n, v) \\ &= h_n^2 d(x_n, v). \end{aligned} \tag{2.2}$$

Also,

$$\begin{aligned} d(x_{n+1}, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n, \gamma_n)), v) \\ &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \\ &\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n))), \beta_n))), \gamma_n)), v) \\ &\leq d(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \\ &\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \beta_n)), \gamma_n), v) \\ &\leq (1-\gamma_n)d(\mathcal{S}_3^n x_n, v) + \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n \\ &\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \beta_n)), v) \end{aligned}$$

$$\begin{aligned} &\leq (1-\gamma_n)h_n^3 d(x_n,v)+\gamma_n h_n^3 d(x_n,v)\\ &= (1+(h_n^3-1))d(x_n,v). \end{aligned} \tag{2.3}$$

By the hypothesis, $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2,3$. Therefore, $\sum_{n=1}^{\infty}(h_n^3-1)<\infty$ for $i=1,2,3$. Using Lemma 1.1, $\lim_{n\to\infty} d(x_n,v)$ exists. ■

Lemma 2.2. *Let $(\mathcal{X},d,\mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3:\mathcal{K}\to\mathcal{K}$ be three asymptotically nonexpansive selfmappings with $\{k_n^{(1)}\},\{k_n^{(2)}\},\{k_n^{(3)}\}\subset[1,\infty)$ and $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3:\mathcal{K}\to\mathcal{X}$ be three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\},\{l_n^{(2)}\},\{l_n^{(3)}\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2,3$, respectively and $\Omega=F(\mathcal{S}_1)\bigcap F(\mathcal{S}_2)\bigcap F(\mathcal{S}_3)\bigcap F(\mathcal{T}_1)\bigcap F(\mathcal{T}_2)\bigcap F(\mathcal{T}_3)\neq\emptyset$. Let $\{x_n\}$ be the sequence defined by (1.7) and the following conditions hold:*

(i) $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[\varepsilon,1-\varepsilon]$ forsome $\varepsilon\in(0,1)$,

(ii) $d(x,T_iy)\leq d(S_ix,T_iy)$ for all $x,y\in\mathcal{K}$ and $i=1,2,3$.

Then $\lim_{n\to\infty} d(x_n,S_ix_n)=\lim_{n\to\infty} d(x_n,T_ix_n)=0$ for $i=1,2,3$.

Proof. For any given $v\in\Omega$, $\lim_{n\to\infty} d(x_n,q)$ exists, by Lemma 2.1.

Taking $h_n=max\{k_n^{(1)},k_n^{(2)},k_n^{(3)},l_n^{(1)},l_n^{(2)},l_n^{(3)}\}$. Suppose that $\lim_{n\to\infty}(x_n,v)=c$. By (2.3) and $\sum_{n=1}^{\infty}(h_n^3-1)<\infty$, we have

$$\lim_{n\to\infty} d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n,\gamma_n)),v)=c \tag{2.4}$$

and

$$\lim_{n\to\infty}\sup d(\mathcal{S}_3^n x_n,v)\leq \lim_{n\to\infty}\sup h_n d(x_n,v)=c. \tag{2.5}$$

Taking lim sup on both sides of (2.2) we obtain,

$$\lim_{n\to\infty}\sup d(y_n,v)\leq c,$$

and so we have,

$$\lim_{n\to\infty}\sup d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n,v)\leq \lim_{n\to\infty}\sup d(y_n,v)\leq c. \tag{2.6}$$

Using (2.4), (2.5) and (2.6), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n)=0. \tag{2.7}$$

By the condition (ii) , we have

$$d(x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n)\leq d(\mathcal{S}_3^n x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n). \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\lim_{n\to\infty} d(x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n)=0. \tag{2.9}$$

In additon,

$$\begin{aligned} d(x_n,v) &\leq d(x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n)+d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n,v)\\ &\leq d(x_n,\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n)+h_n d(y_n,v). \end{aligned} \tag{2.10}$$

In inequality (2.10), taking infimum on both sides and applying (2.9), we obtain,

$$\lim_{n\to\infty}\inf d(y_n,v)\geq c.$$

Since $\lim_{n\to\infty}\sup d(y_n, v) \leq c$. Therefore, $\lim_{n\to\infty} d(y_n, v) = c$. Using the arguments in (2.2) and by $\sum_{n=1}^{\infty}(h_n^{(2)} - 1) < \infty$, we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, \beta_n), v) = c. \tag{2.11}$$

In additon,

$$\lim_{n\to\infty}\sup d(\mathcal{S}_2^n x_n, v) \leq \lim_{n\to\infty}\sup h_n d(x_n, v) = c. \tag{2.12}$$

Taking lim sup on both sides of (2.1), we have

$$\lim_{n\to\infty}\sup d(z_n, v) \leq c. \tag{2.13}$$

Using (2.13), we have

$$\lim_{n\to\infty}\sup d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, v) \leq \lim_{n\to\infty}\sup h_n d(z_n, v) \leq c. \tag{2.14}$$

Applying by Lemma 1.2, using (2.11), (2.12) and (2.14), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) = 0. \tag{2.15}$$

From condition (ii), we get

$$d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) \leq d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n). \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) = 0. \tag{2.17}$$

In addition,

$$\begin{aligned} d(x_n, v) &\leq d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, v) \\ &\leq d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) + h_n d(z_n, v). \end{aligned} \tag{2.18}$$

In the inequality (2.18), taking infimum on both sides and applying (2.17), we obtain $\lim_{n\to\infty}\inf d(z_n, v) \geq c$. Since $\lim_{n\to\infty}\sup d(z_n, v) \leq c$. Therefore, $\lim_{n\to\infty} d(z_n, v) = c$. Using the arguments in (2.1) and by $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ we have,

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n), v) = c. \tag{2.19}$$

In addition,

$$\lim_{n\to\infty}\sup d(\mathcal{S}_1^n x_n, v) \leq \lim_{n\to\infty}\sup h_n d(x_n, v) = c \tag{2.20}$$

and

$$\lim_{n\to\infty}\sup d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, v) \leq \lim_{n\to\infty}\sup h_n d(x_n, v) = c. \tag{2.21}$$

Applying by Lemma 1.2, using (2.19), (2.20) and (2.21), again we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) = 0. \tag{2.22}$$

From condition (ii), we get

$$d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) \leq d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n). \tag{2.23}$$

It follows from (2.22) and (2.23) that

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) = 0. \tag{2.24}$$

Using (1.7), we have

$$\begin{aligned} d(z_n, \mathcal{S}_1^n x_n) &\leq (1-\alpha_n) d(\mathcal{S}_1^n x_n, \mathcal{S}_1^n x_n) + \alpha_n d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) \\ &= \alpha_n d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n). \end{aligned}$$

It follows from (2.22) that

$$\lim_{n\to\infty} d(z_n, \mathcal{S}_1^n x_n) = 0. \tag{2.25}$$

Since

$$d(z_n, x_n) \leq d(z_n, \mathcal{S}_1^n x_n) + d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, x_n).$$

It follows from (2.22), (2.24) and (2.25) that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{2.26}$$

In addition,

$$d(x_n, \mathcal{S}_1^n x_n) \leq d(x_n, z_n) + d(z_n, \mathcal{S}_1^n x_n).$$

Following from (2.25) and (2.26), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1^n x_n) = 0. \tag{2.27}$$

From (1.7), we have

$$d(y_n, \mathcal{S}_2^n x_n) \leq \beta_n d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n). \tag{2.28}$$

Following from (2.15) and (2.28), we have

$$\lim_{n\to\infty} d(y_n, \mathcal{S}_2^n x_n) = 0. \tag{2.29}$$

Furthermore,

$$d(y_n, x_n) \leq d(y_n, \mathcal{S}_2^n x_n) + d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, x_n),$$

by using (2.15), (2.17) and (2.29), we have

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{2.30}$$

Since

$$d(x_n, \mathcal{S}_2^n x_n) \leq d(x_n, y_n) + d(y_n, \mathcal{S}_2^n x_n).$$

Using (2.29) and (2.30), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_2^n x_n) = 0. \tag{2.31}$$

Since

$$\begin{aligned} d(x_{n+1}, \mathcal{S}_3^n x_n) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \gamma_n)), \mathcal{S}_3^n x_n) \\ &\leq (1-\gamma_n) d(\mathcal{S}_3^n x_n, \mathcal{S}_3^n x_n) + \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \mathcal{S}_3^n x_n) \\ &\leq \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \mathcal{S}_3^n x_n). \end{aligned}$$

Using (2.7), we have

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{S}_3^n x_n) = 0. \tag{2.32}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} x_n) &\leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} x_n) \\ &\leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + h_n d(y_n, x_n). \end{aligned}$$

It follows from (2.7) and (2.30) that

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) = 0. \tag{2.33}$$

By condition (ii), we know that

$$d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) \le d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n).$$

Using (2.33), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) = 0. \tag{2.34}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) &\le d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \\ &\le d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + h_n d(z_n, x_n). \end{aligned}$$

Using (2.15) and (2.26), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) = 0. \tag{2.35}$$

Again by condition (ii), using (2.35), we also have

$$\begin{aligned} d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) &\le d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \\ &\to 0 \quad (as \;\; n\to\infty). \end{aligned} \tag{2.36}$$

Using (2.26), (2.32) and (2.33), we have

$$\begin{aligned} d(x_{n+1}, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}z_n) &\le d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) \\ &\quad + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}z_n) \\ &\le d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) + h_n d(x_n, z_n) \\ &\to 0 \quad (as \;\; n\to\infty). \end{aligned} \tag{2.37}$$

Since

$$d(\mathcal{S}_3^n x_n, x_n) \le d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) + d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n).$$

Using (2.33) and (2.44), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, x_n) = 0. \tag{2.38}$$

Since

$$d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \le d(\mathcal{S}_3^n x_n, x_n) + d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n).$$

It follows from (2.36) and (2.38) that

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) = 0. \tag{2.39}$$

In addition,

$$\begin{aligned} d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) &\le d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n, \\ &\quad \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) \\ &\le d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) + h_n d(x_n, z_n). \end{aligned}$$

Using (2.26), (2.32) and (2.39), we have

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) = 0. \tag{2.40}$$

Since

$$d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) \leq d(\mathcal{S}_3^n x_n, x_n) + d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n).$$

Using (2.24) and (2.38), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) = 0. \tag{2.41}$$

Moreover, we have

$$\begin{aligned} d(x_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) &\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \\ &\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) \\ &\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) + h_n d(x_n, z_n). \end{aligned}$$

It follows from (2.26), (2.32) and (2.41) that

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) = 0. \tag{2.42}$$

Again, since $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, x_n \in \mathcal{K}$ for $i = 1, 2, 3$ and $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are three asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) &= d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, \mathcal{T}_i(\mathcal{P}x_n)) \\ &\leq \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, \mathcal{P}x_n) \\ &\leq \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, x_n). \end{aligned} \tag{2.43}$$

For $i = 1, 2, 3$, using (2.37), (2.40) and(2.42) in (2.43), we have

$$\lim_{n\to\infty} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) = 0. \tag{2.44}$$

Since

$$d(x_{n+1}, z_n) \leq d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, x_n) + d(x_n, z_n),$$

from (2.17), (2.26) and (2.40), we have

$$\lim_{n\to\infty} d(x_{n+1}, z_n) = 0. \tag{2.45}$$

In addition, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, \mathcal{T}_i x_n) &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}) \\ &\quad + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) \\ &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + \max\{\sup_{n\geq 1} l_n^{(1)}, \sup_{n\geq 1} l_n^{(2)}, \sup_{n\geq 1} l_n^{(3)}\} d(x_n, z_{n-1}) \\ &\quad + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n). \end{aligned}$$

Thus, it follows from (2.24), (2.34), (2.36), (2.44) and (2.45), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_3 x_n) = 0.$$

The first part of the theorem is hence proved. We prove the next part of the theorem, ie.,

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

In fact, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, \mathcal{S}_i x_n) &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{S}_i x_n) \\ &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{S}_i^n x_n). \end{aligned}$$

Thus, it follows from (2.22), (2.24), (2.33), (2.34), (2.35) and (2.36) that

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

The proof is completed. ∎

Let $\{a_n\}$ be a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and ϑ exist with $\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^{\vartheta}} = \lambda$, then $\{a_n\}$ converges to a of order ϑ, with asymptotic error constant λ. If $\vartheta = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\vartheta = 2$, the sequence is quadratically convergent (see [11]).

The following example presents the condition (ii) in Lemma 2.2.

Example 2.1. [32] Let $\mathcal{X}$ be a real line with metric $d(x, y) = |x - y|$ and $\mathcal{K} = [-1, 1]$. Define $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ by $\mathcal{H}(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$. Then $(\mathcal{X}, d, \mathcal{H})$ is complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\mathcal{K}$ is a nonempty closed convex subset of $\mathcal{X}$. Define two mappings $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ by

$$\mathcal{T}x = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$\mathcal{S}x = \begin{cases} x, & \text{if } x \in [0, 1], \\ -x, & \text{if } x \in [-1, 0). \end{cases}$$

Clearly, $F(\mathcal{T}) = \{0\}$ and $F(\mathcal{S}) = \{x \in \mathcal{K}; 0 \leq x \leq 1\}$. Now, we show that $\mathcal{T}$ is nonexpansive. In fact, if $x, y \in [0, 1]$ or $x, y \in [-1, 0)$, then

$$d(\mathcal{T}x, \mathcal{T}y) = |\mathcal{T}x - \mathcal{T}y| = 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \leq |x - y| = d(x, y).$$

If $x \in [0, 1]$ and $y \in [-1, 0)$ or $x \in [-1, 0)$ and $y \in [0, 1]$, then

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\sin\frac{x}{2} + \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\leq |x + y| \\ &\leq |x - y| \\ &= d(x, y). \end{aligned}$$

That is, $\mathcal{T}$ is nonexpansive. It follows that $\mathcal{T}$ is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that $\mathcal{S}$ is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$. Next, to show that $\mathcal{S}$ and $\mathcal{T}$ satisfy the condition (ii) in Lemma 2.2, we have to consider the following cases:

Case 1. Let $x, y \in [0,1]$. It follows that

$$d(x,\mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x,\mathcal{T}y).$$

Case 2. Let $x, y \in [-1,0)$. It follows that

$$d(x,\mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| \leq |-x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x,\mathcal{T}y).$$

Case 3. Let $x \in [-1,0)$ and $y \in [0,1]$. It follows that

$$d(x,\mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| \leq |-x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x,\mathcal{T}y).$$

Case 4. Let $x \in [0,1]$ and $y \in [-1,0]$. It follows that

$$d(x,\mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x,\mathcal{T}y).$$

Hence the condition (ii) in Lemma 2.2 is satisfied. In addition, let $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$ and $\gamma_n = \frac{n}{4n+1}$, $\forall n \geq 1$. Consequently, the conditions of Lemma 2.2 are fulfilled. Thus, the convergence of the sequence $\{x_n\}$ generated by (1.7) to a point $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ can be recived. ■

Now, we present some numerical examples to illustrate the convergence and efficiency of the proposed algorithms. We choose $x_1 = 1$ and run our process within 100 iterations. All codes were written in Matlab 2022a. We obtain the iteration steps and its amplification factor of the proposed algorithms as shown in Table 1. For convenience, we call the iteration (1.7) the proposed iteration process.

Table 1. Numerical experiment of the proposed method for Example 2.1

The Proposed Iteration Process		
Iteration Number (n)	$\lvert x_n\rvert$	$\frac{\lvert x_{n+1}\rvert}{\lvert x_n\rvert}$
1	1.0000e+00	1.8283e-01
2	1.8283e-01	1.1064e-01
3	2.0229e-02	7.6918e-02
4	1.5559e-03	5.8824e-02
5	9.1526e-05	4.7619e-02
⋮	⋮	⋮
10	2.6686e-15	2.4390e-02
⋮	⋮	⋮
20	1.7026e-33	1.2346e-02
⋮	⋮	⋮
40	2.0079e-75	6.2112e-03
⋮	⋮	⋮
60	1.0911e-121	4.1494e-03
⋮	⋮	⋮
80	8.0992e-171	3.1153e-03
⋮	⋮	⋮
100	4.2888e-222	2.4938e-03

Table 1 show that the proposed method converges to zero. It can be concluded that the proposed method is linearly convergent and its amplification factor less than 0.003.

Next, we can prove a strong convergence theorem.

Theorem 2.3. *Let $\mathcal{K}$, $\mathcal{X}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the hypotheses of Lemma 2.2. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and $\mathcal{S}_i$, $\mathcal{T}_i$ for all $i = 1,2,3$ satisfy the condition (ii) in Lemma 2.2. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f(d(x,\Omega)) \leq d(x,\mathcal{S}_1x) + d(x,\mathcal{S}_2x) + d(x,\mathcal{S}_3x) + d(x,\mathcal{T}_1x) + d(x,\mathcal{T}_2x) + d(x,\mathcal{T}_3x)$$

for all $x \in \mathcal{K}$, where $d(x,\Omega) = \inf\{d(x,v) : v \in \Omega\}$. Then the sequence $\{x_n\}$ defined by algorithm (2.1) converges strongly to a common fixed point of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Proof. From Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, \mathcal{S}_i x_n) = 0 = \lim_{n\to\infty} d(x_n, \mathcal{T}_i x_n)$ for $i = 1,2,3$. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n,\Omega)) &\leq \lim_{n\to\infty} (d(x_n,\mathcal{S}_1x_n) + d(x_n,\mathcal{S}_2x_n) + d(x_n,\mathcal{S}_3x_n) \\ &\quad + d(x_n,\mathcal{T}_1x_n) + d(x_n,\mathcal{T}_2x_n) + d(x_n,\mathcal{T}_3x_n)) \\ &= 0.\end{aligned}$$

Thus $\lim_{n\to\infty} f(d(x_n,\Omega)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$. By Lemma 2.1, we obtain that $\lim_{n\to\infty} d(x_n,\Omega)$ exists. This implies that $\lim_{n\to\infty} d(x_n,\Omega) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$. Using (2.3), we have

$$d(x_{n+1}, v) \leq (1 + (h_n^3 - 1))d(x_n, v)$$

for each $n \geq 1$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$ and $v \in \Omega$. For any $m, n > n \geq 1$, we have

$$\begin{aligned} d(x_m, v) &\leq (1 + (h_{m-1}^3 - 1))d(x_{m-1}, v) \\ &\leq e^{h_{m-1}^3 - 1} d(x_{m-1}, v) \\ &\leq e^{h_{m-1}^3 - 1} e^{h_{m-2}^3 - 1} d(x_{m-2}, v) \\ &\vdots \\ &\leq e^{\sum_{i=n}^{m-1}} h_i^3 - 1 d(x_n, v) \\ &\leq M d(x_n, v), \end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^3 - 1)}$. So, for any $v \in \Omega$, we have

$$d(x_n, x_m) \leq d(x_n, v) + d(x_m, v) \leq (1 + M)d(x_n, v).$$

Taking the infimum over all $v \in \Omega$, we have

$$d(x_n, x_m) \leq (1 + M)d(x_n, \Omega).$$

Thus it follows from $\lim_{n\to\infty} d(x_n,\Omega) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space $\mathcal{X}$, the sequence $\{x_n\}$ converges strongly to some $v^* \in \mathcal{K}$. It is easy to prove that $F(\mathcal{S}_1)$, $F(\mathcal{S}_2)$, $F(\mathcal{S}_3)$, $F(\mathcal{T}_1)$, $F(\mathcal{T}_2)$ and $F(\mathcal{T}_3)$ are all closed, that is, Ω is closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty} d(x_n,\Omega) = 0$ gives that $d(v^*,\Omega) = 0$, we have $v^* \in \Omega$. The proof is completed. ■

Theorem 2.4. *Considering the assumption in Lemma 2.2 and if one of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ is completely continuous after that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a point in Ω.*

Proof. Let $\mathcal{S}_1$ be completely continuous. By Lemma 2.1, $\{x_n\}$ is bounded. This mean, there is a subsequence $\{\mathcal{S}_1 x_{n_j}\}$ of $\{\mathcal{S}_1 x_n\}$ such that $\{\mathcal{S}_1 x_{n_j}\}$ converges strongly to some $v^* \in \mathcal{K}$. Moreover, by Lemma 2.2, we have

$$\lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_1 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_2 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_3 x_n) = 0 \text{ and}$$

$$\lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_1 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_2 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_3 x_n) = 0,$$

which implies that,

$$\begin{aligned} d(x_{n_j}, v^*) &\leq d(x_{n_j}, \mathcal{S}_1 x_{n_j}) + d(\mathcal{S}_1 x_{n_j}, v^*) \\ &\to 0 \text{ (as } j \to \infty). \end{aligned}$$

Hence $\mathcal{S}_1 x_{n_j} \to v^* \in K$. Consequently,

$$d(v^*, \mathcal{S}_i v^*) = \lim_{n\to\infty} d(x_{n_j}, \mathcal{S}_i x_{n_j}) = 0.$$

Since $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are continuous, for $i = 1, 2, 3$. By Lemma 2.2, so we have

$$d(v^*, \mathcal{T}_i v^*) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_i x_{n_j}) = 0.$$

This implies that $v^* \in F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. From Lemma 2.1, we have $\lim_{n\to\infty} d(x_n, v^*)$ exists and so $\lim_{n\to\infty} d(x_n, v^*) = 0$. Thus $\{x\}$ converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ The proof is completed. ■

3. Conclusions

Authors constructed a novel Noor iteration technique to approximate a common fixed point for three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings in a uniformly convex hyperbolic space. An illustrative example is also provided as Example 2.1. The authors proved that strong convergence results were more substantial than the delta and weak convergence results.

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