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Strong Convergence of Modified SP-Iteration for Mixed Type Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract In this paper, a modified SP-iteration for asymptotically nonexpansive mappings is introduced and studied. We then establish a strong convergence theorem for a modified SP-iteration scheme for three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings under mild conditions in a uniformly convex hyperbolic spaces. The results presented here extend and improve some related results in the literature.

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1. Introduction

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. Iterative methods are popular tools to approximate fixed points of nonlinear mappings.

Structural properties of the underlying space, such as strict convexity and uniform convexity, are very much needed for the development of iterative fixed point theory in it. Hyperbolic spaces are general in nature and inherit rich geometrical structure suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory.

Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert

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and Banach spaces have been obtained in literature, for example, see [4–7, 12, 13, 18, 19, 27, 28, 42–46]. It is easier working with Banach space due to its convex structures. However, metric space do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi [47] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [15, 22, 31]). Although the class of hyperbolic spaces defined by Kohlenbach [22] is slightly restrictive than the class of hyperbolic spaces introduced in [15], it is however, more general than the class of hyperbolic spaces introduced in [31]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [22]. Some discussion of fixed point approximation in CAT(0) spaces and their essential roles are given in [40, 41]. Some other examples of this class of hyperbolic spaces includes Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and $\mathbb{R}$-trees, see [8, 11, 15, 16, 22, 31].

Goebel and Kirk [14], in 1972, introduced the class of asymptotically nonexpansive self-mappings, which is an important generalization of the class of nonexpansive self-mappings. In the last few decades investigations of fixed points by some iterative schemes for asymptotically nonexpansive mappings have attracted many mathematicians.

In 1991, Schu [36] introduced the following modified Mann iteration process

$$x_{n+1} = (1-\beta_n)x_n + \beta_n \mathcal{T}^n x_n, \quad n \geq 1, \tag{1.1}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schu's iteration process (1.1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces ([9, 26, 30, 32, 36, 37, 48]).

In 2003, Chidume, Ofoedu, and Zegeye [10] introduced the concept of asymptotically nonexpansive nonself-mappings. Also, they studied the following iterative sequence

$$x_{n+1} = \mathcal{P}((1-\beta_n)x_n + \beta_n \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} x_n), \tag{1.2}$$

to approximate some fixed point of $\mathcal{T}$ under suitable conditions.

If $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping so that (1.2) reduces to (1.1).

In 2006, Wang [50] considered the following iteration process which is a generalization of (1.2),

$$\begin{aligned} y_n &= \mathcal{P}((1-\alpha_n)x_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} x_n), \\ x_{n+1} &= \mathcal{P}((1-\beta_n)x_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} y_n), \ n \geq 1, \end{aligned} \tag{1.3}$$

where $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). Meanwhile, the results of [50] generalized the results of [10].

The projection type Ishikawa iteration process for approximating common fixed points of two asymptotically nonexpansive nonself-mappings was defined and constructed by

Thianwan [49] in a uniformly convex Banach space as follows:

$$y_n = \mathcal{P}((1-\alpha_n)x_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n),$$
$$x_{n+1} = \mathcal{P}((1-\beta_n)y_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}y_n),\ n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in [0,1). Note that Thianwan process (1.4) and Wang process (1.3) are independent neither reduces to the other.

In 2012, Guo, Cho and Guo [17] studied the following iteration scheme:

$$y_n = \mathcal{P}((1-\alpha_n)\mathcal{S}_2^n x_n + \alpha_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n),$$
$$x_{n+1} = \mathcal{P}((1-\beta_n)\mathcal{S}_1^n x_n + \beta_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}y_n),\ n \geq 1, \tag{1.5}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1) to approximate common fixed points of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ under proper conditions.

The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. Fixed point theory and hence approximation techniques have been extended to hyperbolic spaces (see [1–3, 33–35] and references therein).

Recently, Jayashree and Eldred [20] introduced and studied the following mixed type iteration scheme in a uniformly convex hyperbolic space and prove some strong convergence theorems for mixed type asymptotically nonexpansive mappings:

$$y_n = \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n, \alpha_n)),$$
$$x_{n+1} = \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}y_n, \beta_n)),\ n \geq 1, \tag{1.6}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ be two asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ be two asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1).

2. Besic Definitions and Relevant Results

Throughout this paper, our study is in hyperbolic space introduced by Kohlenbach [22].

Definition 2.1. A hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is metric space $\mathcal{X}, d$ together with $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ satisfying

(i) $d(u, \mathcal{H}(x,y,\alpha)) \leq (1-\alpha)d(u,x) + \alpha d(u,y)$,
(ii) $d(\mathcal{H}(x,y,\alpha), \mathcal{H}(x,y,\beta)) = |\alpha - \beta| d(x,y)$,
(iii) $\mathcal{H}(x,y,\alpha) = \mathcal{H}(y,x,1-\alpha)$,
(iv) $d(\mathcal{H}(x,z,\alpha), \mathcal{H}(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$

for all $w,x,y,z \in \mathcal{X}$ and $\alpha, \beta \in [0,1]$.

Example 2.2. [39] Let $\mathcal{X}$ be a real Banach space which is equipped with norm $||\cdot||$. Define the function $d : \mathcal{X} \times \mathcal{X} \times [0,\infty)$ by

$$d(x,y) = ||x-y||.$$

Then, we have that $(\mathcal{X}, d, \mathcal{H})$ is a hyperbolic space with mapping $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ defined by $\mathcal{H}(x,y,\alpha) = (1-\alpha)x + \alpha y$.

A subset $\mathcal{K}$ of a hyperbolic space $\mathcal{X}$ is convex if $\mathcal{H}(x, y, \alpha) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\alpha \in [0, 1]$.

Definition 2.3. [47] A hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is said to be strictly convex if for any $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$, there exists a unique element $z \in \mathcal{X}$ such that $d(z, x) = \alpha d(x, y)$ and $d(z, y) = (1 - \alpha)d(x, y)$.

Definition 2.4. [38] A hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is said to be uniformly convex if for any $w, x, y \in \mathcal{X}$, $r > 0$ and $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that

$$d(\mathcal{H}(x, y, \frac{1}{2}), x) \leq (1 - \delta)r$$

whenever $d(x, w) \leq r, d(y, w) \leq r$ and $d(x, y) \geq \epsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [24]).

In the sequel, let $(\mathcal{X}, d)$ be a metric space, and let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. We shall denote the fixed point set of a mapping $\mathcal{T}$ by $F(\mathcal{T}) = \{x \in \mathcal{K} : \mathcal{T}x = x\}$ and $d(x, F(\mathcal{T})) = \inf \{d(x, z) : z \in F(\mathcal{T})\}$.

A self-mapping $\mathcal{T}$ is said to be nonexpansive if $d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y)$ for all $x, y \in \mathcal{K}$. $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d(\mathcal{T}^n x, \mathcal{T}^n y) \leq k_n d(x, y) \tag{2.1}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$. $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq \mathcal{L} d(x, y)$ for all $x, y \in \mathcal{K}$ and $n \geq 1$.

It follows that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \forall n \geq 1$. Moreover, each asymptotically nonexpansive mapping is a uniformly $\mathcal{L}$ -Lipschitzian mapping with $\mathcal{L} = \sup_{n \in \aleph} \{k_n\}$. However, the converse of these statements is not true, in general.

Note that, a subset $\mathcal{K}$ of $\mathcal{X}$ is said to be a retract if there exists a continuous mapping $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ such that $\mathcal{P}x = x$ for all $x \in \mathcal{K}$. For more information on nonexpansive retracts and retractions, we refer the reader to ([16, 23]).

For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping (see [10]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} x, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} y) \leq k_n d(x, y) \tag{2.2}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$.

We denote by $(\mathcal{P}\mathcal{T})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping, so that (2.2) reduces to (2.1).

In addition, if $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is asymtotically nonexpansive in light of (2.2) and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is a nonexpansive retraction, then $\mathcal{P}\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is asymtotically nonexpansive in light of (2.1) (see also (2.3)) . Indeed, for all $x, y \in \mathcal{K}$ and $n \geq 1$, by (2.2), it follows that

$$\begin{aligned} d((\mathcal{P}\mathcal{T})^n x, (\mathcal{P}\mathcal{T})^n y) &= d(\mathcal{P}\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} x, \mathcal{P}\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} y) \\ &\leq d(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} x, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} y) \\ &\leq k_n d(x, y). \end{aligned}$$

Therefore, we now introduce the following definition.

Definition 2.5. For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d((\mathcal{P}\mathcal{T})^n x, (\mathcal{P}\mathcal{T})^n y) \leq k_n d(x, y), \tag{2.3}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$.

Lemma 2.6. [48] *Assume $\{s_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences non-negative real numbers such that*

$$s_{n+1} \leq (1 + b_n)s_n + c_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 2.7. [21] *Assume that $\{x_n\}$ and $\{y_n\}$ be two sequence of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ such that, for $R \in [0, \infty)$, $\lim_{n\to\infty} \sup d(x_n, a) \leq R$, $\lim_{n\to\infty} \sup d(y_n, a) \leq R$ and*

$$\lim_{n\to\infty} d(\mathcal{H}(x_n, y_n, \mu_n), a) = R,$$

where $\mu_n \in [a, b]$ with $0 < a \leq b < 1$, then we have, $\lim_{n\to\infty} d(x_n, y_n) = 0$.

3. Main Results

In this section, we suggest a modified SP-iteration for mixed type asypmtotically nonexpansive mappings and establish the strong convergence theorem in a uniformly convex hyperbolic space.

Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings. We will denote the set of common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ by Ω, that is, $\Omega := F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. The following iteration process is a translation of the SP-iteration scheme introduced in [29] from Banach spaces to hyperbolic spaces. The SP-iteration is equivalent to Mann, Ishikawa, Noor iterations and converges faster than the others for the class of continuous and non-decreasing functions (see [29]).

$$\begin{cases} x_1 \in \mathcal{K}, \\ z_n = \mathcal{H}(\mathcal{S}_3^n x_n, (\mathcal{P}\mathcal{T}_3)^n x_n, \alpha_n), \\ y_n = \mathcal{H}(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n, \beta_n), \\ x_{n+1} = \mathcal{H}(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n, \gamma_n), \quad n \leq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$.

The following lemmas are needed.

Lemma 3.1. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three* *asymptotically nonexpansive nonself-mappings with* $\{l_n^{(1)}\},\{l_n^{(2)}\},\{l_n^{(3)}\}\subset[1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ *and* $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ *for* $i=1,2,3$, *respectively and* $\Omega\neq\emptyset$. *Assume that* $\{\alpha_n\},\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequence in* $[0,1)$. *From* $x_1\in\mathcal{K}$, *define the sequence* $\{x_n\}$ *using* (3.1). *Then* $\lim_{n\to\infty} d(x_n,z)$ *exists for any* $z\in\Omega$.

Proof. Let $z\in\Omega$ and setting $h_n=max\{k_n^{(1)},k_n^{(2)},k_n^{(3)},l_n^{(1)},l_n^{(2)},l_n^{(3)}\}$. From (3.1), we have

$$\begin{aligned} d(z_n,z) &= d(\mathcal{H}(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n,\alpha_n),z)\\ &\leq (1-\alpha_n)d(\mathcal{S}_3^n x_n,z)+\alpha_n d((\mathcal{PT}_3)^n x_n,z)\\ &\leq (1-\alpha_n)h_n d(x_n,z)+\alpha_n h_n d(x_n z)\\ &= h_n d(x_n,z) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(y_n,z) &= d(\mathcal{H}(\mathcal{S}_2^n z_n,(\mathcal{PT}_2)^n z_n,\beta_n),z)\\ &\leq (1-\beta_n)d(\mathcal{S}_2^n z_n,z)+\beta_n d((\mathcal{PT}_2)^n z_n,z)\\ &\leq (1-\beta_n)h_n d(z_n,z)+\beta_n h_n d(z_n,z)\\ &= h_n d(z_n,z)\\ &\leq h_n^2 d(x_n,z). \end{aligned} \tag{3.3}$$

Using (3.3), we have

$$\begin{aligned} d(x_{n+1},z) &= d(\mathcal{H}(\mathcal{S}_1^n y_n,(\mathcal{PT}_1)^n y_n,\gamma_n),z)\\ &\leq (1-\gamma_n)d(\mathcal{S}_1^n y_n,z)+\gamma_n d((\mathcal{PT}_1)^n y_n,z)\\ &\leq (1-\gamma_n)h_n d(y_n,z)+\gamma_n h_n d(y_n,z)\\ &= h_n d(y_n,z)\\ &\leq h_n^3 d(x_n,z)\\ &= (1+(h_n^3-1))d(x_n,z). \end{aligned} \tag{3.4}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2,3$, we have $\sum_{n=1}^{\infty}(h_n^{(3)}-1)<\infty$. Using Lemma 2.6, $\lim_{n\to\infty} d(x_n,z)$ exists. ∎

Lemma 3.2. *Let* $(\mathcal{X},d,\mathcal{H})$ *be a uniformly convex hyperbolic space and* $\mathcal{K}$ *a nonempty closed convex subset of* $\mathcal{X}$. *Let* $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3:\mathcal{K}\to\mathcal{K}$ *be three asymptotically nonexpansive self-mappings with* $\{k_n^{(1)}\},\{k_n^{(2)}\},\{k_n^{(3)}\}\subset[1,\infty)$ *and* $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3:\mathcal{K}\to\mathcal{X}$ *be three asymptotically nonexpansive nonself-mappings with* $\{l_n^{(1)}\},\{l_n^{(2)}\},\{l_n^{(3)}\}\subset[1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ *and* $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ *for* $i=1,2,3$, *respectively and* $\Omega\neq\emptyset$. *Assume* $\{x_n\}$ *be a sequence defined by* (3.1) *and the following conditions hold:*

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[\varepsilon,1-\varepsilon]$ *for some* $\varepsilon\in(0,1)$,

(ii) $d(x,\mathcal{T}_i y)\leq d(\mathcal{S}_i x,\mathcal{T}_i y)$ *for all* $x,y\in\mathcal{K}$ *and* $i=1,2,3$.

Then $\lim_{n\to\infty} d(x_n,\mathcal{S}_i x_n)=\lim_{n\to\infty} d(x_n,(\mathcal{PT}_i)x_n)=0$ *for* $i=1,2,3$.

Proof. Let $z\in\Omega$ and setting $h_n=max\{k_n^{(1)},k_n^{(2)},k_n^{(3)},l_n^{(1)},l_n^{(2)},l_n^{(3)}\}$. From Lemma 3.1, we have $\lim_{n\to\infty} d(x_n,z)$ exists. Suppose that $\lim_{n\to\infty} d(x_n,z)=c$, letting $n\to\infty$ in the inequality (3.4), we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n y_n,(\mathcal{PT}_1)^n y_n,\gamma_n),z)=c. \tag{3.5}$$

In addition, using (3.3), we obtain $d(\mathcal{S}_1^n y_n, z) \leq h_n^3 d(x_n, z)$. Taking the lim sup on both sides in this inequality, we have

$$\lim_{n\to\infty} \sup d(S_1^n y_n, z) \leq c. \tag{3.6}$$

Taking the lim sup in (3.3), we get $\lim_{n\to\infty} \sup d(y_n, z) \leq c$. Thus

$$\lim_{n\to\infty} \sup d((\mathcal{P}\mathcal{T}_1)^n y_n, z) \leq \lim_{n\to\infty} \sup h_n d(y_n, z) = c. \tag{3.7}$$

By (3.5),(3.6),(3.7) and Lemma 2.7, we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) = 0. \tag{3.8}$$

By the condition (ii), we have

$$\lim_{n\to\infty} d(y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) \leq \lim_{n\to\infty} d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n). \tag{3.9}$$

Using (3.9), we have

$$\lim_{n\to\infty} d(y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) = 0. \tag{3.10}$$

From (3.4), we obtain

$$\begin{aligned} d(x_{n+1}, z) &\leq d(\mathcal{H}(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n, \gamma_n), z) \\ &\leq (1-\gamma_n) d(\mathcal{S}_1^n y_n, z) + \gamma_n d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) + \gamma_n d(\mathcal{S}_1^n y_n, z) \\ &= d(\mathcal{S}_1^n y_n, z) + \gamma_n d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) \\ &\leq h_n d(y_n, z) + \gamma_n d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n). \end{aligned} \tag{3.11}$$

Taking the lim inf on both sides in the inequality (3.11), using (3.8), $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $\lim_{n\to\infty} d(x_{n+1}, z) = c$, we have

$$\lim_{n\to\infty} \inf d(y_n, z) \geq c. \tag{3.12}$$

Since $\lim_{n\to\infty} \sup d(y_n, z) \leq c$, by (3.12), we have

$$\lim_{n\to\infty} d(y_n, z) = c.$$

Again, letting $n \to \infty$ in the inequality (3.3), we have

$$\lim_{n\to\infty} d(H(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n, \beta_n), z) = c. \tag{3.13}$$

In addition, using (3.2), we obtain $d(\mathcal{S}_2^n z_n, z) \leq h_n^2 d(x_n, z)$. Taking the lim sup on both sides in this inequality, we have

$$\lim_{n\to\infty} \sup d(\mathcal{S}_2^n z_n, z) \leq c. \tag{3.14}$$

Taking the lim sup in (3.2), we get $\lim_{n\to\infty} \sup d(z_n, z) \leq c$. Thus

$$\lim_{n\to\infty} \sup d((\mathcal{P}\mathcal{T}_2)^n z_n, z) \leq \lim_{n\to\infty} \sup h_n d(z_n, z) = c. \tag{3.15}$$

By (3.13), (3.14), (3.15) and Lemma 2.7, we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) = 0. \tag{3.16}$$

By the condition (ii), we have

$$\lim_{n\to\infty} d(z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) \leq \lim_{n\to\infty} d(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n),$$

and thus

$$\lim_{n\to\infty} d(z_n,(\mathcal{PT}_2)^n z_n)=0. \tag{3.17}$$

From (3.3), we obtain

$$\begin{aligned} d(y_n,z) &\le d(\mathcal{H}(\mathcal{S}_2^n z_n,(\mathcal{PT}_2)^n z_n,\beta_n),z)\\ &\le (1-\beta_n)d(S_2^n z_n,z)+\beta_n d(S_2^n z_n,(\mathcal{PT}_2)^n z_n)+\beta_n d(\mathcal{S}_2^n z_n,z)\\ &= d(\mathcal{S}_2^n z_n,z)+\beta_n d(\mathcal{S}_2^n z_n,(\mathcal{PT}_2)^n z_n)\\ &\le h_n d(z_n,z)+\beta_n d(\mathcal{S}_2^n z_n,(\mathcal{PT}_2)^n z_n). \end{aligned} \tag{3.18}$$

Taking the lim inf on both sides in this inequality (3.18), using (3.16), $\sum_{n=1}^{\infty}(h_n-1)<\infty$ and $\lim_{n\to\infty} d(y_n,z)=c$, we have

$$\lim_{n\to\infty}\inf d(z_n,z)\ge c. \tag{3.19}$$

Since $\lim_{n\to\infty}\sup d(z_n,z)\le \lim_{n\to\infty}\sup h_n d(x_n,z)\le c$, by (3.19), we have

$$\lim_{n\to\infty} d(z_n,z)=c.$$

Letting $n\to\infty$ in the inequality (3.2), we have

$$\begin{aligned} c=\lim_{n\to\infty} d(z_n,z) &\le \lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n,\alpha_n),z)\\ &\le \lim_{n\to\infty} d(x_n,z)=c, \end{aligned}$$

and so

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n,\alpha_n),z)=c. \tag{3.20}$$

Moreover, we obtain

$$\lim_{n\to\infty}\sup d(\mathcal{S}_3^n x_n,z)\le \lim_{n\to\infty}\sup h_n d(x_n,z)=c, \tag{3.21}$$

and

$$\lim_{n\to\infty}\sup d((\mathcal{PT}_3)^n x_n,z)\le \lim_{n\to\infty}\sup h_n d(x_n,z)=c. \tag{3.22}$$

Following (3.20), (3.21), (3.22) and Lemma 2.7, we get

$$\lim_{n\to\infty} d(S_3^n x_n,(\mathcal{PT}_3)^n x_n)=0. \tag{3.23}$$

Next, we show that

$$\lim_{n\to\infty} d(x_n,(\mathcal{PT}_1)x_n)=\lim_{n\to\infty} d(x_n,(\mathcal{PT}_2)x_n)=\lim_{n\to\infty} d(x_n,(\mathcal{PT}_3)x_n)=0.$$

Indeed, by the condition (ii), we have

$$d(x_n,(\mathcal{PT}_3)^n x_n)\le d(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n). \tag{3.24}$$

By (3.23) and (3.24), which implies that

$$\lim_{n\to\infty} d(x_n,(\mathcal{PT}_3)^n x_n)=0. \tag{3.25}$$

Using (3.1), we have

$$\begin{aligned} d(z_n,\mathcal{S}_3^n x_n) &\le (1-\alpha_n)d(\mathcal{S}_3^n x_n,\mathcal{S}_3^n x_n)+\alpha_n d(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n)\\ &= \alpha_n d(\mathcal{S}_3^n x_n,(\mathcal{PT}_3)^n x_n). \end{aligned}$$

Following from (3.23),

$$\lim_{n\to\infty} d(z_n, \mathcal{S}_3^n x_n) = 0. \tag{3.26}$$

In addition, we have

$$d(z_n, x_n) \le d(z_n, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, (\mathcal{P}\mathcal{T}_3)^n x_n) + d((\mathcal{P}\mathcal{T}_3)^n x_n, x_n). \tag{3.27}$$

It follows from (3.23), (3.25), (3.26) and (3.27) that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.28}$$

Furthermore,

$$d(\mathcal{S}_2^n z_n, z_n) \le d(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) + d((\mathcal{P}\mathcal{T}_2)^n z_n, z_n),$$

by using (3.16) and (3.17), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n z_n, z_n) = 0. \tag{3.29}$$

It follows from (3.1), (3.17) and (3.29) that

$$\begin{aligned} d(y_n, z_n) &= d(\mathcal{H}(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n, \beta_n), z_n) \\ &\le (1-\beta_n) d(\mathcal{S}_2^n z_n, z_n) + \beta_n d((\mathcal{P}\mathcal{T}_2)^n z_n, z_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.30}$$

And then, from (3.28) and (3.30), we have

$$\begin{aligned} d(y_n, x_n) &\le d(y_n, z_n) + d(z_n, x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.31}$$

By the condition (ii), we know that

$$d(x_n, (\mathcal{P}\mathcal{T}_1)^n x_n) \le d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_1)^n x_n). \tag{3.32}$$

Since

$$\begin{aligned} d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_1)^n x_n) &\le d(\mathcal{S}_1^n x_n, \mathcal{S}_1^n y_n) + d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_1)^n y_n, (\mathcal{P}\mathcal{T}_1)^n x_n) \\ &\le h_n d(x_n, y_n) + d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) \\ &\quad + h_n d(y_n, x_n). \end{aligned} \tag{3.33}$$

Using (3.8) and (3.31) in (3.33), we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_1)^n x_n) = 0. \tag{3.34}$$

It follows from (3.32) and (3.34) that

$$\lim_{n\to\infty} d(x_n, (\mathcal{P}\mathcal{T}_1)^n x_n) = 0. \tag{3.35}$$

From (3.17) and (3.28), we have

$$\begin{aligned} d(x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) &\le d(x_n, z_n) + d(z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) + d((\mathcal{P}\mathcal{T}_2)^n z_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\le d(x_n, z_n) + d(z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) + h_n d(z_n, x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.36}$$

Using (3.16), (3.17) and (3.28), we have

$$\begin{aligned} d(\mathcal{S}_2^n x_n, x_n) &\leq d(\mathcal{S}_2^n x_n, \mathcal{S}_2^n z_n) + d(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_2)^n z_n, z_n) + d(z_n, x_n) \\ &\leq h_n d(x_n, z_n) + d(\mathcal{S}_2^n z_n, (\mathcal{P}\mathcal{T}_2)^n z_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_2)^n z_n, z_n) + d(z_n, x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.37}$$

It follows from (3.36) and (3.37) that

$$\begin{aligned} d(\mathcal{S}_2^n x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) &\leq d(\mathcal{S}_2^n x_n, x_n) + d(x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.38}$$

Using (3.8), we have

$$\begin{aligned} d(x_{n+1}, \mathcal{S}_1^n y_n) &= d(\mathcal{H}(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n, \gamma_n), \mathcal{S}_1^n y_n) \\ &\leq (1-\gamma_n) d(\mathcal{S}_1^n y_n, \mathcal{S}_1^n y_n) + \gamma_n d((\mathcal{P}\mathcal{T}_1)^n y_n, \mathcal{S}_1^n y_n) \\ &= \gamma_n d((\mathcal{P}\mathcal{T}_1)^n y_n, \mathcal{S}_1^n y_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.39}$$

By (3.8) and (3.39), we have

$$\begin{aligned} d(x_{n+1}, (\mathcal{P}\mathcal{T}_1)^n y_n) &\leq d(x_{n+1}, \mathcal{S}_1^n y_n) + d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_1)^n y_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.40}$$

Using (3.30) and (3.40), we have

$$\begin{aligned} d(x_{n+1}, (\mathcal{P}\mathcal{T}_1)^n z_n) &\leq d(x_{n+1}, (\mathcal{P}\mathcal{T}_1)^n y_n) + d((\mathcal{P}\mathcal{T}_1)^n y_n, (\mathcal{P}\mathcal{T}_1)^n z_n) \\ &\leq d(x_{n+1}, (\mathcal{P}\mathcal{T}_1)^n y_n) + h_n d(y_n, z_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.41}$$

Moreover, from (3.34) and (3.35), we have

$$\begin{aligned} d(\mathcal{S}_1^n x_n, x_n) &\leq d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_1)^n x_n) + d((\mathcal{P}\mathcal{T}_1)^n x_n, x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.42}$$

Using (3.36) and (3.42), we have

$$\begin{aligned} d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) &\leq d(\mathcal{S}_1^n x_n, x_n) + d(x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.43}$$

It follows from (3.31) and (3.43) that

$$\begin{aligned} d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_2)^n x_n) &\leq d(\mathcal{S}_1^n y_n, \mathcal{S}_1^n x_n) + d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\leq h_n d(y_n, x_n) + d(\mathcal{S}_1^n x_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.44}$$

Using (3.28), (3.39) and (3.44), we have

$$\begin{aligned} d(x_{n+1}, (\mathcal{P}\mathcal{T}_2)^n z_n) &\leq d(x_{n+1}, \mathcal{S}_1^n y_n) + d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_2)^n x_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_2)^n x_n, (\mathcal{P}\mathcal{T}_2)^n z_n) \\ &\leq d(x_{n+1}, \mathcal{S}_1^n y_n) + d(\mathcal{S}_1^n y_n, (\mathcal{P}\mathcal{T}_2)^n x_n) + h_n d(x_n, z_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.45}$$

In addition, using (3.25), (3.28), (3.31), (3.39) and (3.42), we have

$$\begin{aligned} d(x_{n+1}, (\mathcal{P}\mathcal{T}_3)^n z_n) &\leq d(x_{n+1}, \mathcal{S}_1^n y_n) + d(\mathcal{S}_1^n y_n, \mathcal{S}_1^n x_n) + d(\mathcal{S}_1^n x_n, x_n) \\ &\quad + d(x_n, (\mathcal{P}\mathcal{T}_3)^n x_n) + d((\mathcal{P}\mathcal{T}_3)^n x_n, (\mathcal{P}\mathcal{T}_3)^n z_n) \\ &\leq d(x_{n+1}, \mathcal{S}_1^n y_n) + h_n d(y_n, x_n) + d(\mathcal{S}_1^n x_n, x_n) \\ &\quad + d(x_n, (\mathcal{P}\mathcal{T}_3)^n x_n) + h_n d(x_n, z_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.46}$$

Since $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} z_{n-1}, x_n \in \mathcal{K}$ for $i = 1, 2, 3$, and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are three asymptotically nonexpansive nonselt-mappings, we have

$$\begin{aligned} d((\mathcal{P}\mathcal{T}_i)^n z_{n-1}, (\mathcal{P}\mathcal{T}_i) x_n) &= d((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} z_{n-1}, (\mathcal{P}\mathcal{T}_i) x_n) \\ &\leq max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d((\mathcal{P}\mathcal{T}_i)^{n-1} z_{n-1}, x_n). \end{aligned} \tag{3.47}$$

For $i = 1, 2, 3$, using (3.41), (3.45), (3.46) and (3.47), we obtain

$$\lim_{n\to\infty} d((\mathcal{P}\mathcal{T}_i)^n z_{n-1}, (\mathcal{P}\mathcal{T}_i) x_n) = 0. \tag{3.48}$$

Using (3.17) and (3.45), we have

$$\begin{aligned} d(x_{n+1}, z_n) &\leq d(x_{n+1}, (\mathcal{P}\mathcal{T}_2)^n z_n) + d((\mathcal{P}\mathcal{T}_2)^n z_n, z_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.49}$$

Moreover, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, (\mathcal{P}\mathcal{T}_i) x_n) &\leq d(x_n, (\mathcal{P}\mathcal{T}_i)^n x_n) + d((\mathcal{P}\mathcal{T}_i)^n x_n, (\mathcal{P}\mathcal{T}_i)^n z_{n-1}) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n z_{n-1}, (\mathcal{P}\mathcal{T}_i) x_n) \\ &\leq d(x_n, (\mathcal{P}\mathcal{T}_i)^n x_n) + max\{\sup_{n\geq 1} l_1^{(1)}, \sup_{n\geq 1} l_2^{(2)}, \sup_{n\geq 1} l_3^{(3)}\} d(z_{n-1}, x_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n z_{n-1}, (\mathcal{P}\mathcal{T}_i) x_n). \end{aligned}$$

Therefore, it follows from (3.25), (3.35), (3.36), (3.48) and (3.49) that

$$\lim_{n\to\infty} d(x_n, (\mathcal{P}\mathcal{T}_1) x_n) = \lim_{n\to\infty} d(x_n, (\mathcal{P}\mathcal{T}_2) x_n) = \lim_{n\to\infty} d(x_n, (\mathcal{P}\mathcal{T}_3) x_n) = 0.$$

Lastly, we prove that

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

In fact, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, \mathcal{S}_i x_n) &\leq d(x_n, (\mathcal{P}\mathcal{T}_i)^n x_n) + d((\mathcal{P}\mathcal{T}_i)^n x_n, \mathcal{S}_i x_n) \\ &\leq d(x_n, (\mathcal{P}\mathcal{T}_i)^n x_n) + d((\mathcal{P}\mathcal{T}_i)^n x_n, \mathcal{S}_i^n x_n). \end{aligned}$$

So, it follows from (3.23), (3.25), (3.34), (3.35), (3.36) and (3.38) that

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

This completes the proof. ■

Example 3.3. [25] Let $\mathcal{X}$ be a real line with metric $d(x, y) = |x - y|$ and $\mathcal{K} = [-1, 1]$. Define $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ by $\mathcal{H}(x, y, \alpha) := \alpha x + (1 - \alpha) y$ for all $x, y \in \mathcal{X}$ and $\alpha \in [0, 1]$. Then $(\mathcal{X}, d, \mathcal{H})$ is complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\mathcal{K}$ is a nonempty closed convex subset of $\mathcal{X}$. Define two mappings $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ by

$$\mathcal{T}x = \begin{cases} -2\sin\dfrac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\dfrac{x}{2}, & \text{if } x \in [-1,0) \end{cases}$$

and

$$\mathcal{S}x = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Clearly, $F(\mathcal{T}) = \{0\}$ and $F(\mathcal{S}) = \{x \in \mathcal{K}; 0 \leq x \leq 1\}$. Now, we show that $\mathcal{T}$ is nonexpansive. In fact, if $x, y \in [0,1]$ or $x, y \in [-1,0)$, then

$$d(\mathcal{T}x, \mathcal{T}y) = |\mathcal{T}x - \mathcal{T}y| = 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \leq |x-y| = d(x,y).$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\sin\frac{x}{2} + \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\leq |x+y| \\ &\leq |x-y| \\ &= d(x,y). \end{aligned}$$

That is, $\mathcal{T}$ is nonexpansive. It follows that $\mathcal{T}$ is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that $\mathcal{S}$ is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$. Next, to show that $\mathcal{S}$ and $\mathcal{T}$ satisfy the condition (ii) in Lemma 3.2, we have to consider the following cases:

Case 1. Let $x, y \in [0,1]$. It follows that

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 2. Let $x, y \in [-1,0)$. It follows that

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| \leq |-x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 3. Let $x \in [-1,0)$ and $y \in [0,1]$. It follows that

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| \leq |-x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 4. Let $x \in [0,1]$ and $y \in [-1,0]$. It follows that

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Hence the condition (ii) in Lemma 3.2 is satisfied. In addition, we take $\alpha_n = \dfrac{n}{2n+1}$, $\beta_n = \dfrac{n}{3n+1}$ and $\gamma_n = \dfrac{n}{4n+1}$, $\forall n \geq 1$. Therefore, the conditions of Lemma 3.2 are fulfilled. Thus, the convergence of the sequence $\{x_n\}$ generated by (3.1) to a point $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ can be received. ■

Now, we provide some numerical examples to illustrate the convergence behavior of iteration (1.6) comparing with iteration (3.1). All program computation are performed on an Hp Laptop Intel(R) Core(TM) i7-1165G7, 16.00 GB RAM. We choose the starting point at $x_1 = 1$ and the stop criterion is defined by $||x_n - 0|| < 10^{-15}$. The convergence performance of both iteration are shown in the following Table 1 and Figure 1.

TABLE 1. Computational result for all setting in Example 3.3

	Iteration (1.6)	Iteration (3.1)
No of Iter.	26	10
CPU time (sec)	0.0035	0.0027

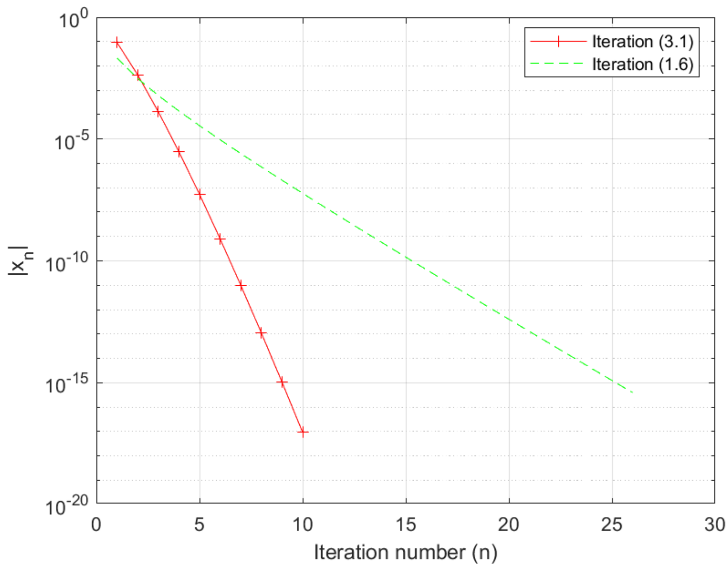

FIGURE 1. The value of $\{x_n\}$ generated by iteration (1.6) and iteration (3.1)

Under the same condition settings shown in Example 3.3, by Table 1 and Figure 1, our proposed iteration (3.1) has a better performance in both the time taken by CPU-runtime to reach the convergence and the number of iterations when comparing with iteration (1.6).

Next, we can prove a strong convergence theorem.

Theorem 3.4. *Let $\mathcal{K}$, $\mathcal{X}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the hypotheses of Lemma 3.2. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and $\mathcal{S}_i$, $\mathcal{T}_i$ for all $i = 1,2,3$ satisfy the condition (ii) in Lemma 3.2. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f(d(x,\Omega)) \leq d(x,\mathcal{S}_1x)+d(x,\mathcal{S}_2x)+d(x,\mathcal{S}_3x)+d(x,(\mathcal{P}\mathcal{T}_1)x)+d(x,(\mathcal{P}\mathcal{T}_2)x)+d(x,(\mathcal{P}\mathcal{T}_3)x)$$

for all $x \in \mathcal{K}$, where $d(x,\Omega) = \inf\{d(x,z) : z \in \Omega\}$. Then the sequence $\{x_n\}$ defined by algorithm (3.1) converges strongly to a common fixed point of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Proof. From Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, \mathcal{S}_i x_n) = 0 = \lim_{n\to\infty} d(x_n, (\mathcal{P}\mathcal{T}_i)x_n)$ for $i = 1, 2, 3$. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n,\Omega)) &\le \lim_{n\to\infty}(d(x_n,\mathcal{S}_1x_n)+d(x_n,\mathcal{S}_2x_n)+d(x_n,\mathcal{S}_3x_n)\\ &\quad +d(x_n,(\mathcal{P}\mathcal{T}_1)x_n)+d(x_n,(\mathcal{P}\mathcal{T}_2)x_n)+d(x_n,(\mathcal{P}\mathcal{T}_3)x_n))\\ &= 0.\end{aligned}$$

Thus $\lim_{n\to\infty} f(d(x_n,\Omega)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$. By Lemma 3.1, we obtain that $\lim_{n\to\infty} d(x_n,\Omega)$ exists. This implies that $\lim_{n\to\infty} d(x_n,\Omega) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$. Using (3.4), we have

$$d(x_{n+1}, z) \le (1 + (h_n^3 - 1))d(x_n, z)$$

for each $n \ge 1$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$ and $z \in \Omega$. For any $m, n > n \ge 1$, we have

$$\begin{aligned}d(x_m,z) &\le (1+(h_{m-1}^3-1))d(x_{m-1},z)\\ &\le e^{h_{m-1}^3-1}d(x_{m-1},z)\\ &\le e^{h_{m-1}^3-1}e^{h_{m-2}^3-1}d(x_{m-2},z)\\ &\vdots\\ &\le e^{\sum_{i=n}^{m-1}(h_i^3-1)}d(x_n,z)\\ &\le Md(x_n,z),\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^3-1)}$. So, for any $z \in \Omega$, we have

$$d(x_n, x_m) \le d(x_n, z) + d(x_m, z) \le (1 + M)d(x_n, z).$$

Taking the infimum over all $z \in \Omega$, we have

$$d(x_n, x_m) \le (1 + M)d(x_n, \Omega).$$

Thus it follows from $\lim_{n\to\infty} d(x_n,\Omega) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space $\mathcal{X}$, the sequence $\{x_n\}$ converges strongly to some $z^* \in \mathcal{K}$. It is easy to prove that $F(\mathcal{S}_1)$, $F(\mathcal{S}_2)$, $F(\mathcal{S}_3)$, $F(\mathcal{T}_1)$, $F(\mathcal{T}_2)$ and $F(\mathcal{T}_3)$ are all closed, that is, Ω is closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty} d(x_n,\Omega) = 0$ gives that $d(z^*,\Omega) = 0$, we have $z^* \in \Omega$. The proof is completed. ■

If $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are self-mappings, then $\mathcal{P}$ becomes the identity mapping. By using to the same ideas and techniques as in Lemma 3.1, Lemma 3.2 and Theorem 3.4, we can obtain a strong convergence theorem for asymptotically nonexpansive mappings in a uniformly convex hyperbolic space. Therefore we can state the following result without proofs.

Theorem 3.5. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1,\infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega \ne \emptyset$.*

Assume $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$ *and* $\mathcal{S}_i, \mathcal{T}_i$ *for all* $i = 1, 2, 3$ *satisfy the condition (ii) in Lemma 3.2. If there is a nondecreasing function* $f : [0, \infty) \to [0, \infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$ *for all* $r \in (0, \infty)$ *such that*

$$f(d(x, \Omega)) \leq d(x, \mathcal{S}_1 x) + d(x, \mathcal{S}_2 x) + d(x, \mathcal{S}_3 x) + d(x, \mathcal{T}_1 x) + d(x, \mathcal{T}_2 x) + d(x, \mathcal{T}_3 x)$$

for all $x \in \mathcal{K}$*, where* $d(x, \Omega) = \inf\{d(x, z) : z \in \Omega\}$*. Then the sequence* $\{x_n\}$ *defined by*

$$\begin{aligned} z_n &= \mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3^n x_n, \alpha_n), \\ y_n &= \mathcal{H}(\mathcal{S}_2^n z_n, \mathcal{T}_2^n z_n, \beta_n), \\ x_{n+1} &= \mathcal{H}(\mathcal{S}_1^n y_n, \mathcal{T}_1^n y_n, \gamma_n) \end{aligned}$$

converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *ans* $\mathcal{T}_3$*.*

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