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# Characterization of Some Regular Relational Hypersubstitutions for Algebraic Systems

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**Abstract** An algebraic system is a structure which consists of a nonempty set together with a sequence of operations and a sequence of relations which are defined on the set. The concept of a relational hypersubstitution for algebraic systems is a canonical extension of the concept of a hypersubstitution for universal algebras. Such relational hypersubstitutions are mappings which map operation symbols to terms and map relation symbols to relational terms preserving arities. The set of all relational hypersubstitutions for algebraic systems together with an associative binary operation, which was defined in [D. Phusanga, J. Koppitz, The monoid of hypersubstitutions for algebraic systems, Announcements of Union of Scientists Silven 33 (1) (2018) 119–126], forms a monoid. The concept of the special regular elements are important role in semigroup theory. In this paper, we characterize the set of all completely regular, left regular and right regular elements of this monoid of type $((m),(n))$. The results show that the set of all completely regular elements and the set of all left(right) regular elements of this monoid are the same.

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## 1. Introduction

The notion of a hypersubstitution of a given type $\tau$ in universal algebras was introduced by Denecke et. al [9]. They used the concept of a hypersubstitution for the characterization of solid varieties of type $\tau$. A solid variety is a variety which is closed under the following operation: taking a universal algebra $(A,(f_i^A)_{i\in I})$ of type $\tau=(m_i)_{i\in I}$ with the universe $A$ and a family $(f_i^A)_{i\in I}$ of $m_i$-ary operation $f_i^A$ on $A$ for $i\in I$. Then we replace the operation $f_i^A$ by any $m_i$-ary term operation $t_i^A$, for $i\in I$, and obtain a new universal algebra $(A,(t_i^A)_{i\in I})$, which also belongs to the variety. A hypersubstitution of a given type $\tau=(m_i)_{i\in I}$ is a mapping which maps the $m_i$-ary operation symbol $f_i^A$ to an $m_i$-ary term, for $i\in I$. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type $\tau$. Then

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$Hyp(\tau)$ together with an associative binary operation $\circ_h$ and the identity hypersubstitution forms a monoid, see more detail in [9, 17]. There are several published papers on algebraic properties of this monoid and its submonoids. For example see [4, 5, 8].

On the other hand, we consider the algebraic systems in the sense of Mal′cev. Let $I, J$ be indexed sets. Let $f_i$ be an operation symbol with the arity $m_i$ and $\gamma_j$ a relation symbol with the arity $n_j$, for $i \in I$ and $j \in J$.

**Definition 1.1.** [13] An algebraic system of type $(\tau, \tau')$ is a triple $(A, (f_i)_{i\in I}, (\gamma_j)_{j\in J})$ consisting of a nonempty set $A$, a sequence $(f_i)_{i\in I}$ of $m_i$-ary operations defined on $A$ and a sequence $(\gamma_j)_{j\in J}$ of $n_j$-ary relations on $A$, where $\tau = (m_i)_{i\in I}$ is a sequence of the arities of each operation $f_i$ and $\tau' = (n_j)_{j\in J}$ is a sequence of the arities of each relation $\gamma_j$. The pair $(\tau, \tau')$ is called the type of an algebraic system.

There were first attempts to define a concept of a hypersubstitution for algebraic systems. The concept of such a hypersubstitution, introduced in [10], does not be practicable enough. Another attempt to define a hypersubstitution for algebraic systems was done in [11], but also this concept has not proven to be impractical. Five years later, Phusanga and Koppitz [16] introduced a new concept that generalizes the notion of a hypersubstitution of type $\tau$ for universal algebras in a canonical way. Such hypersubstitution is called the *relational hypersubstitution* for algebraic systems of type $(\tau, \tau')$, that name was first used in [12]. Since any hypersubstitution for universal algebras of type $\tau$ assigns an $m_i$-ary operation $f_i$ to an $m_i$-ary term for $i \in I$, it seems quite naturally that any relational hypersubstitution for algebraic systems of type $(\tau, \tau')$ assigns an $m_i$-ary operation symbol $f_i$ to an $m_i$-ary term for $i \in I$, and assigns an $n_j$-ary relation symbol $\gamma_j$ to an $n_j$-ary relational term for $j \in J$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau')$ equipped with an associative binary operation forms a monoid, see [16]. The algebraic properties of such monoid were studied intensively for several types. For examples, all idempotent elements, all regular elements and the order of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau') = ((2), (2))$ were characterized by Phusanga and Koppitz [16]. The order of linear relational hypersubstitutions for algebraic systems of type $(\tau, \tau') = ((m), (n))$ was determined by Phusanga and Lekkoksung [12]. In this present paper, we characterize the set of all completely regular, left regular and right regular elements of this monoid of type $(\tau, \tau') = ((m), (n))$, where $m, n \in \mathbb{N}$.

## 2. Monoid of All Relational Hypersubstitutions for Algebraic Systems

In this section, we recall the concept of relational hypersubstitutions for algebraic systems of type $((m), (n))$ and give some useful results that will be used in the next sections. To define the concept of relational hypersubstitutions for algebraic systems of a given type, we firstly introduce the notion of terms and formulas.

Let $X := \{x_1, x_2, ...\}$ be a countably infinite set of variables. We often refer to these variables as *letters*, to $X$ as an *alphabet*, and also refer to the set $X_m := \{x_1, ..., x_m\}$ as a set of $m$ variables. Let $\{f_i : i \in I\}$ be the set of $m_i$-ary operation symbols indexed by $I$, where $m_i \geq 1$ is a natural number. An $m$-ary term of type $\tau = (m_i)_{i\in I}$, for simply an *$m$-ary term*, is defined inductively as follows:

(i) The variables $x_1, ..., x_m$ are $m$-ary terms of type $\tau$.

(ii) If $t_1, ..., t_{m_i}$ are $m$-ary terms of type $\tau$, then $f_i(t_1, ..., t_{m_i})$ is an $m$-ary term of type $\tau$.

Let $W_\tau(X_m)$ be the set of all $m$-ary terms of type $\tau$ and let $W_\tau(X) := \bigcup_{m\geq 1} W_\tau(X_m)$ be the set of all terms of type $\tau$. Clearly, every $m$-ary term of type $\tau$ is also an $n$-ary term of type $\tau$, where $n \geq m$.

**Example 2.1.** Let $\tau = (2)$ be a type with a binary operation symbol $f$. These are some examples of binary terms of type (2): $x_1$, $x_2$, $f(x_1, x_1)$, $f(x_2, x_2)$, $f(x_1, x_2)$, $f(x_2, x_1)$, $f(f(x_1, x_2), x_1)$, $f(f(x_2, x_1), f(x_1, x_2))$, $f(f(x_2, x_1), f(f(x_1, x_2), x_1))$.

To define formulas of type $(\tau, \tau')$, we need the logical connections $\vee$, $\neg$ and the equation symbol $\approx$. Let $\{\gamma_j : j \in J\}$ be the set of $n_j$-ary relation symbols indexed by $J$.

**Definition 2.2.** An $n$-ary quantifier free formula of type $(\tau, \tau') = ((m_i)_{i\in I}, (n_j)_{j\in J})$, for simply *$n$-ary formula*, is defined in the following way:

(i) If $t_1, t_2$ are $n$-ary terms of type $\tau$, then the equation $t_1 \approx t_2$ is an $n$-ary quantifier free formula of type $(\tau, \tau')$.
(ii) If $j \in J$ and $t_1, ..., t_{n_j}$ are $n$-ary terms of type $\tau$, then $\gamma_j(t_1, ..., t_{n_j})$ is an $n$-ary quantifier free formula of type $(\tau, \tau')$.
(iii) If $F$ is an $n$-ary quantifier free formula of type $(\tau, \tau')$, then $\neg F$ is an $n$-ary quantifier free formula of type $(\tau, \tau')$.
(vi) If $F_1$ and $F_2$ are $n$-ary quantifier free formulas of type $(\tau, \tau')$, then $F_1 \vee F_2$ is an $n$-ary quantifier free formula of type $(\tau, \tau')$.

Let $F_{(\tau,\tau')}(X_n)$ be the set of all $n$-ary formulas of type $(\tau, \tau')$ and let $F_{(\tau,\tau')}(X) := \bigcup_{n\in\mathbb{N}} F_{(\tau,\tau')}(X_n)$ be the set of all formulas of type $(\tau, \tau')$. Let $rF_{(\tau,\tau')}(X_n)$ be the set of all $n$-ary formulas of type $(\tau, \tau')$ of the form (ii) in Definition 2.2. We call such formulas in this set the *$n$-ary relational terms of type* $(\tau, \tau')$. Let $rF_{(\tau,\tau')}(X) := \bigcup_{n\in\mathbb{N}} rF_{(\tau,\tau')}(X_n)$ be the set of all relational terms of type $(\tau, \tau')$.

A relational hypersubstitution for algebraic systems of type $(\tau, \tau') = ((m_i)_{i\in I}, (n_j)_{j\in J})$ is a mapping

$$\sigma : \{f_i : i \in I\} \cup \{\gamma_j : j \in J\} \to W_\tau(X) \cup rF_{(\tau,\tau')}(X)$$

with $\sigma(f_i) \in W_\tau(X_{m_i})$ and $\sigma(\gamma_j) \in rF_{(\tau,\tau')}(X_{n_j})$. We denote by $Relhyp(\tau, \tau')$ the set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau')$. To defined a binary operation on this set, we firstly recall the concept of superposition of terms and superposition of relational terms.

For any $m, n \in \mathbb{N}$. Let $t, t_1, ..., t_{m_i}, s_1, ..., s_{n_j} \in W_\tau(X_m), u_1, ..., u_m \in W_\tau(X_n)$ and $F = \gamma_j(s_1, ..., s_{n_j}) \in rF_{(\tau,\tau')}(X_m)$. We define inductively the concept of superposition of terms $S^m_n : W_\tau(X_m) \times (W_\tau(X_n))^m \to W_\tau(X_n)$ by the following steps:

(i) If $t = x_k$ for $1 \leq k \leq n$, then $S^m_n(t, u_1, ..., u_m) := u_k$.
(ii) If $t = f_i(t_1, ..., t_{m_i})$, then

$$S^m_n(t, u_1, ..., u_m) := f_i(S^m_n(t_1, u_1, ..., u_m), ..., S^m_n(t_{m_i}, u_1, ..., u_m)).$$

Next, we define the superposition operation of relational terms

$$R^m_n : (W_\tau(X_m) \cup rF_{(\tau,\tau')}(X_m)) \times (W_\tau(X_n))^m \to W_\tau(X_n) \cup rF_{(\tau,\tau')}(X_n)$$

by the following steps:

(i) $R^m_n(t, u_1, ..., u_m) := S^m_n(t, u_1, ..., u_m)$,
(ii) $R^m_n(F, u_1, ..., u_m) := \gamma_j(S^m_n(s_1, u_1, ..., u_m), ..., S^m_n(s_{n_j}, u_1, ..., u_m))$.

To define the binary operation on $Relhyp(\tau,\tau')$, we need the concept of the extension $\widehat{\sigma}$ of $\sigma$ which is defined by:
For any $\sigma \in Relhyp(\tau,\tau')$, we define a mapping

$$\widehat{\sigma} : W_\tau(X) \cup rF_{(\tau,\tau')}(X) \to W_\tau(X) \cup rF_{(\tau,\tau')}(X)$$

by the following steps:

(i) $\widehat{\sigma}[x_k] := x_k$, for $k \in \mathbb{N}$,
(ii) $\widehat{\sigma}[f_i(t_1,...,t_{m_i})] := R_m^{m_i}(\sigma(f_i), \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_{m_i}]) = S_m^{m_i}(\sigma(f_i), \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_{m_i}])$,
(iii) $\widehat{\sigma}[\gamma_j(s_1,...,s_{n_j})] := R_n^{n_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], ..., \widehat{\sigma}[s_{n_j}])$.

Then an associative binary operation $\circ_h$ on $Relhyp(\tau,\tau')$ is defined by $\sigma \circ_h \rho := \widehat{\sigma} \circ \rho$, where $\circ$ denotes the usual composition of mappings, for all $\sigma, \rho \in Relhyp(\tau,\tau')$. Let $\sigma_{id}$ be the identity relational hypersubstitution for algebraic systems of type $(\tau,\tau')$ which maps an $m_i$-ary operation symbol $f_i$ to a term $f_i(x_1,...,x_{m_i})$ and maps an $n_j$-ary relation symbol $\gamma_j$ to a relational term $\gamma_j(x_1,...,x_{n_j})$. Then the structure $\underline{Relhyp(\tau,\tau')} = (Relhyp(\tau,\tau'), \circ_h, \sigma_{id})$ forms a monoid, see more detail in [11, 12, 16].

In this present paper, we study on the monoid $Relhyp((m),(n))$. Let $f$ be an $m$-ary operation symbol and $\gamma$ an $n$-ary relation symbol. For any $t \in W_{(m)}(X_m)$ and $F \in rF_{((m),(n))}(X_n)$, we denote

(i) $\sigma_{t,F}$ := the relational hypersubstitution for algebraic systems of type $((m),(n))$ which maps $f$ to a term $t$ and maps $\gamma$ to a relational term $F$,
(ii) $var(t)$ := the set of all variables occur in a term $t$,
(iii) $var(F)$ := the set of all variables occur in a relational term $F$,
(iv) $I(t)$ := the set of all indices of variables occur in a term $t$,
(v) $I(F)$ := the set of all indices of variables occur in a relational term $F$,
(vi) $\pi(t)$ := the term such that each $x_k \in var(t)$ is replaced by $x_{\pi(k)}$ where $\pi$ is a bijective map on $I(t)$.

**Example 2.3.** Let $\tau = (4)$ be a type with a quaternary operation symbol $f$. Let $t = f(f(x_4,x_3,x_4,x_3),x_1,x_3,x_4) \in W_{(4)}(X_4)$. Then $var(t) = \{x_1,x_3,x_4\}$ and $I(t) = \{1,3,4\}$. Let $\pi : I(t) \to I(t)$ by $\pi(1) = 3, \pi(3) = 4$ and $\pi(4) = 1$. Then $\pi(t) = f(f(x_{\pi(4)},x_{\pi(3)},x_{\pi(4)},x_{\pi(3)}),x_{\pi(1)},x_{\pi(3)},x_{\pi(4)}) = f(f(x_1,x_4,x_1,x_4),x_3,x_4,x_1)$.

**Remark 2.4.** $var(s) = var(\pi(s))$ and $\pi^{-1}(\pi(s)) = s = \pi(\pi^{-1}(s))$.

**Lemma 2.5.** *Let $t \in W_{(m)}(X_m)$ and $var(t) = \{x_{i_1},...,x_{i_p}\}$. Let $\pi$ be a bijective map on $I(t)$. If $y_1,...,y_m \in W_{(m)}(X_m)$ such that $y_{i_1} = x_{\pi(i_1)},...,y_{i_p} = x_{\pi(i_p)}$, then $S_m^m(t,y_1,...,y_m) = \pi(t)$.*

*Proof.* The proof is straightforward. ∎

**Lemma 2.6.** [6] *Let $\sigma_{t,F}, \sigma_{w,H} \in Relhyp((m),(n))$. Then $var((\sigma_{t,F} \circ_h \sigma_{w,H})(f)) \subseteq var(\sigma_{w,H}(f))$ and $var((\sigma_{t,F} \circ_h \sigma_{w,H})(\gamma)) \subseteq var(\sigma_{w,H}(\gamma))$.*

In 2015, Wongpinit and Leeratanavalee [18] introduced the notion of the $i-most$ of terms.

**Definition 2.7.** For $m \in \mathbb{N}$. Let $t \in W_{(m)}(X_m)$ and $1 \leq i \leq m$. Then $i-most(t)$ is defined inductively by:

(i) If $t$ is a variable, then $i-most(t) := t$.
(ii) If $t = f(t_1,...,t_m)$, then $i-most(t) := i-most(t_i)$.

**Example 2.8.** Let $\tau = (3)$ be a type with a ternary operation symbol $f$.
Let $t = f(f(f(x_3, x_1, x_1), x_3, x_2), f(x_1, x_1, x_2), x_2) \in W_{(3)}(X_3)$. Then
$1 - most(t) = 1 - most(f(f(x_3, x_1, x_1), x_3, x_2)) = 1 - most(f(x_3, x_1, x_1)) = x_3$,
$2 - most(t) = 2 - most(f(x_1, x_1, x_2)) = x_1$ and
$3 - most(t) = 3 - most(x_2) = x_2$.

**Remark 2.9.** It is easy to see that $\widehat{\sigma}_{x_i}[t] = i - most(t)$ and $\widehat{\sigma}_{x_i,F}[t] = i - most(t)$.

**Lemma 2.10.** [7] *Let* $\sigma_{t,F} \in Relhyp((m), (n))$ *and* $s \in W_{(m)}(X_m)$. *If* $i - most(t) = x_j$, *then* $i - most(\widehat{\sigma}_{t,F}[s]) = j - most(s)$.

**Lemma 2.11.** [7] *Let* $s, t \in W_{(m)}(X_m)$ *and* $F, H \in rF_{((m),(n))}(X_n)$ *with* $s = f(s_1, ..., s_m)$ *and* $H = \gamma(h_1, ..., h_n)$ *where* $s_1, ..., s_m \in W_{(m)}(X_m)$ *and* $h_1, ..., h_n \in W_{(m)}(X_n)$. *Let* $t = \widehat{\sigma}_{t,F}[s]$ *and* $F = \widehat{\sigma}_{t,F}[H]$ *with* $x_k \in var(t)$ *and* $x_l \in var(F)$. *Then*

(i) *if* $t = x_i \in X_m$, *then* $i - most(s_k) = x_k$ *and* $i - most(h_l) = x_l$,
(ii) *if* $t \in W_{(m)}(X_m) \setminus X_m$, *then* $s_k = x_k$ *and* $h_l = x_l$.

**Definition 2.12.** [1] Let $t \in W_{(m)}(X_m)$, a *subterm* of $t$, is defined inductively by the following:

(i) Every variable $x \in var(t)$ is a subterm of $t$.
(ii) If $t = f(t_1, ..., t_m)$, then $t$ itself, $t_1, ..., t_m$ and all subterms of $t_i$, $1 \leq i \leq m$, are subterms of $t$.

We denote the set of all subterms of $t$ by $sub(t)$.

**Definition 2.13.** [3] Let $t \in W_{(m)}(X_m) \setminus X_m$, where $t = f(t_1, ..., t_m)$ for some $t_1, ..., t_m \in W_{(m)}(X_m)$. For each $s \in sub(t)$ such that $s \neq t$. The set $seq^t(s)$ of sequences of $s$ in $t$ is defined by

$$seq^t(s) := \{(i_1, ..., i_n) : n \in \mathbb{N} \text{ and } s = \pi_{i_n} \circ ... \circ \pi_{i_1}(t)\},$$

where $\pi_{i_j} : W_{(m)}(X_m) \setminus X_m \to W_{(m)}(X_m)$ by the form $\pi_{i_j}(f(t_1, ..., t_m)) = t_{i_j}$. Maps $\pi_{i_j}$ are defined for $j = 1, ..., n$.

## 3. All Completely Regular Elements in $Relhyp((m), (n))$

In this section, we determine the set of all completely regular elements of the monoid of all relational hypersubstitutions for algebraic systems of type $((m), (n))$. We recall first the definitions of regular and completely regular.

**Definition 3.1.** [14] An element $a$ of a semigroup $S$ is called *regular* if there exists $x \in S$ such that $axa = a$.

**Definition 3.2.** [14] An element $a$ of a semigroup $S$ is called *completely regular* if there exists $x \in S$ such that $axa = a$ and $xa = ax$.

Clearly, every completely regular element of $S$ is always a regular element of $S$, but any regular element of $S$ does not need to be a completely regular element of $S$. It follows that the set of all completely regular elements of $S$ is a subset of the set of all regular elements of $S$. Let $\sigma_{t,F} \in Relhyp((m), (n))$, we denote

$R_X := \{\sigma_{t,F} \mid t = x_i$ and $F = \gamma(s_1, ..., s_n)$ with $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $j - most(s_{b'_k}) = x_{b_k}$ for all $k = 1, ..., l$ and for some distinct $b'_1, ..., b'_l \in \{1, ..., n\}$ where $j \in \{1, ..., m\}\}$,

$R_T := \{\sigma_{t,F} \mid t = f(t_1, ..., t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$ for all $i = 1, ..., k, j = 1, ..., l$, and for some distinct $a'_1, ..., a'_k \in \{1, ..., m\}$ and some distinct $b'_1, ..., b'_l \in \{1, ..., n\}\}$.

In [7], Daengsaen and Leeratanavalee showed that $R_X \cup R_T$ is the set of all regular elements of $Relhyp((m),(n))$. Next, we will define some subsets of $R_X \cup R_T$ and prove that it is the set of all completely regular elements of $Relhyp((m),(n))$. Let $\sigma_{t,F} \in Relhyp((m),(n))$, we denote

$CR(R_X) := \{\sigma_{t,F} \mid t = x_i$ and $F = \gamma(s_1, ..., s_n)$ with $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $i - most(s_{b_j}) = x_{\phi(b_j)}$ for all $j = 1, ..., l$ where $\phi$ is a bijective map on $\{b_1, ..., b_l\}\}$,

$CR(R_T) := \{\sigma_{t,F} \mid t = f(t_1, ..., t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a_i} = x_{\pi(a_i)}$ and $s_{b_j} = x_{\phi(b_j)}$ for all $i = 1, ..., k, j = 1, ..., l$, where $\pi$ is a bijective map on $\{a_1, ..., a_k\}$ and $\phi$ is a bijective map on $\{b_1, ..., b_l\}\}$,

$CR(Relhyp((m),(n))) := CR(R_X) \cup CR(R_T)$.

Clearly, $CR(R_X) \subseteq R_X$ and $CR(R_T) \subseteq R_T$.

**Lemma 3.3.** *Let $\sigma_{t,F} \in R_T$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$. Then there exists $\sigma_{u,H} \in Relhyp((m),(n))$ and $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$ if and only if $u = f(u_1, ..., u_m)$ and $H = \gamma(h_1, ..., h_n)$ with $u_{a_i} = x_{a'_i}$ and $h_{b_j} = x_{b'_j}$ for all $i = 1, ..., k$ and $j = 1, ..., l$.*

*Proof.* Let $\sigma_{t,F} \in R_T$ with $t = f(t_1, ..., t_m)$ and $F = \gamma(s_1, ..., s_n)$ where $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$. There exist some distinct integers $a'_1, ..., a'_k \in \{1, ..., m\}$ and some distinct integers $b'_1, ..., b'_l \in \{1, ..., n\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$ for all $i = 1, ..., k, j = 1, ..., l$. Suppose that there exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$. Then

$$\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] = (\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F})(f) = \sigma_{t,F}(f) = t,$$

$$\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]] = (\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma) = F.$$

If $u = x_p \in X_m$ for some $p \in \{1, ..., m\}$, then

$$t = \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] = \widehat{\sigma}_{t,F}[p - most(t)] = p - most(t) \in X_m.$$

This is a contradiction with $t \in W_{(m)}(X_m) \setminus X_m$. Let $u = f(u_1, ..., u_m)$ and $H = \gamma(h_1, ..., h_n)$. Since $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and by Lemma 2.11(ii), we have $\widehat{\sigma}_{u,H}[t] = f(w_1, ..., w_m)$ with $w_{a_i} = x_{a_i}$ for all $i = 1, ..., k$. Consider

$$f(w_1, ..., w_m) = \widehat{\sigma}_{u,H}[t] = S^m_m(f(u_1, ..., u_m), \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]),$$

then $x_{a_i} = w_{a_i} = S^m_m(u_{a_i}, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m])$ for all $i = 1, ..., k$. If there exists $i \in \{1, ..., k\}$ such that $u_{a_i} \in W_{(m)}(X_m) \setminus X_m$, then $x_{a_i} = S^m_m(u_{a_i}, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \in W_{(m)}(X_m) \setminus X_m$. This is a contradiction. So $u_{a_i} \in X_m$ for all $i \in \{1, ..., k\}$. Since $t_{a'_i} = x_{a_i}$ for all $i \in \{1, ..., k\}$, we have $\widehat{\sigma}_{u,H}[t_{a'_i}] = x_{a_i}$ for all $i \in \{1, ..., k\}$. It follows that we can choose $u_{a_i} = x_{a'_i}$ for all $i = 1, ..., k$. For case $h_{b_j} = x_{b'_j}$ for all $j = 1, ..., l$, the proof is similar. Conversely, it is obvious. ∎

**Proposition 3.4.** *For any $\sigma_{t,F} \in CR(R_X)$, $\sigma_{t,F}$ is a completely regular element in $Relhyp((m),(n))$.*

*Proof.* Let $\sigma_{t,F} \in CR(R_X)$. Then $t = x_i \in X_m$ and $F = \gamma(s_1, ..., s_n) \in rF_{((m),(n))}(X_n)$ with $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $i - most(s_{b_j}) = x_{\phi(b_j)}$ for all $j = 1, ..., l$ where $\phi$ is a bijective map on $\{b_1, ..., b_l\}$. Let $\sigma_{u,H} \in Relhyp((m),(n))$ with $u = x_i$ and $H = \gamma(h_1, ..., h_n)$ such that $var(H) = var(F)$ and $h_p = (\phi^{-1} \circ \phi^{-1})(s_p)$ for all $p = 1, ..., m$.

We will show that $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$ and $\sigma_{t,F} \circ_h \sigma_{u,H} = \sigma_{u,H} \circ_h \sigma_{t,F}$. For each $b_j \in \{b_1, ..., b_l\} \subseteq \{1, ..., m\}$, we have $i - most(h_{b_j}) = i - most((\phi^{-1} \circ \phi^{-1})(s_{b_j})) = x_{\phi^{-1}(\phi^{-1}(\phi(b_j)))} = x_{\phi^{-1}(b_j)}$. First, we show that $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$. Consider

$$\begin{aligned}
(\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{x_i,F}[\widehat{\sigma}_{x_i,H}[F]] \\
&= \widehat{\sigma}_{x_i,F}[\widehat{\sigma}_{x_i,H}[\gamma(s_1, ..., s_n)]] \\
&= \widehat{\sigma}_{x_i,F}[R^n_n(\gamma(h_1, ..., h_n), \widehat{\sigma}_{x_i,H}[s_1], ..., \widehat{\sigma}_{x_i,H}[s_n])] \\
&= \widehat{\sigma}_{x_i,F}[R^n_n(\gamma(h_1, ..., h_n), i - most(s_1), ..., i - most(s_n))] \\
&= \widehat{\sigma}_{x_i,F}[\gamma(\phi(h_1), ..., \phi(h_n))] \\
&\quad (\text{since } var(H) = var(F) \text{ and } i - most(s_{b_j}) = x_{\phi(b_j)}) \\
&= R^n_n(F, \widehat{\sigma}_{x_i,F}[\phi(h_1)], ..., \widehat{\sigma}_{x_i,F}[\phi(h_n)]) \\
&= R^n_n(F, i - most(\phi(h_1)), ..., i - most(\phi(h_n))) \\
&= F \\
&\quad (\text{since } i - most(\phi(h_{b_j})) = x_{\phi(\phi^{-1}(b_j))} = x_{b_j}) \\
&= \sigma_{t,F}(\gamma).
\end{aligned}$$

Clearly, $(\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F})(f) = x_i = \sigma_{t,F}(f)$. So $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$. Next, we show that $\sigma_{t,F} \circ_h \sigma_{u,H} = \sigma_{u,H} \circ_h \sigma_{t,F}$. Consider

$$(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) = R^n_n(\gamma(s_1, ..., s_n), i - most(h_1), ..., i - most(h_n)) = \gamma(w_1, ..., w_n),$$

where $w_p = S^n_n(s_p, i - most(h_1), ..., i - most(h_n))$ for all $p = 1, ..., n$ and

$$(\sigma_{u,H} \circ_h \sigma_{t,F})(\gamma) = R^n_n(\gamma(h_1, ..., h_n), i - most(s_1), ..., i - most(s_n)) = \gamma(v_1, ..., v_n),$$

where $v_p = S^n_n(h_p, i - most(s_1), ..., i - most(s_n))$ for all $p = 1, ..., n$.
Since $i - most(h_{b_j}) = x_{\phi^{-1}(b_j)}$ for all $x_{b_j} \in var(s_p) \subseteq var(F)$ and by Lemma 2.5, we have $w_p = S^n_n(s_p, i - most(h_1), ..., i - most(h_n)) = \phi^{-1}(s_p)$ for all $p = 1, ..., n$. Then

$$\begin{aligned}
v_p &= S^n_n(h_p, i - most(s_1), ..., i - most(s_n)) \\
&= S^n_n(\phi^{-1}(\phi^{-1}(s_p)), i - most(s_1), ..., i - most(s_n)) \\
&= \phi(\phi^{-1}(\phi^{-1}(s_p))) \\
&\quad (\text{since } i - most(s_{b_j}) = x_{\phi(b_j)} \text{ and by Lemma 2.5}) \\
&= \phi^{-1}(s_p) \\
&= w_p
\end{aligned}$$

for all $p = 1, 2, ..., n$. It implies that $(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) = \gamma(w_1, ..., w_n) = \gamma(v_1, ..., v_n) = (\sigma_{u,H} \circ_h \sigma_{t,F})(\gamma)$. Clearly, $(\sigma_{t,F} \circ_h \sigma_{u,H})(f) = x_i = (\sigma_{u,H} \circ_h \sigma_{t,F})(f)$. So $\sigma_{t,F} \circ_h \sigma_{u,H} = \sigma_{u,H} \circ_h \sigma_{t,F}$. Therefore $\sigma_{t,F}$ is a completely regular element in $Relhyp((m), (n))$. ∎

**Proposition 3.5.** *For any $\sigma_{t,F} \in CR(R_T)$, $\sigma_{t,F}$ is a completely regular element in $Relhyp((m), (n))$.*

*Proof.* Let $\sigma_{t,F} \in CR(R_T)$. Then $t = f(t_1, ..., t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a_i} = x_{\pi(a_i)}$ and $s_{b_j} = x_{\phi(b_j)}$ for all $i = 1, ..., k, j = 1, ..., l$, where $\pi$ is a bijective map on $\{a_1, ..., a_k\}$ and $\phi$ is a bijective map on $\{b_1, ..., b_l\}$. Pick $\sigma_{u,H} \in Relhyp((m), (n))$ where $u = f(u_1, ..., u_m)$ and $H = \gamma(h_1, ..., h_n)$ with $var(u) = var(t)$ and $var(H) = var(F)$ such that $u_{\pi(a_1)} = x_{a_1}, ..., u_{\pi(a_k)} = x_{a_k}$ and $h_{\phi(b_1)} = x_{b_1}, ..., h_{\phi(b_l)} = x_{b_l}$. For any $i \in \{1, ..., m\} \setminus \{a_1, ..., a_k\}$ and $j \in \{1, ..., n\} \setminus$

$\{b_1, ..., b_l\}$, we choose $u_i = (\pi^{-1} \circ \pi^{-1})(t_i)$ and $h_j = (\phi^{-1} \circ \phi^{-1})(s_j)$. By Lemma 3.3, we have $\sigma_{t,F} \circ_h \sigma_{u,H} \circ_h \sigma_{t,F} = \sigma_{t,F}$. Next, we will show that $\sigma_{t,F} \circ_h \sigma_{u,H} = \sigma_{u,H} \circ_h \sigma_{t,F}$. Consider

$$(\sigma_{t,F} \circ_h \sigma_{u,H})(f) = S^m_m(f(t_1, ..., t_m), \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) = f(w_1, ..., w_m)$$

where $w_p = S^m_m(t_p, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m])$ for all $p = 1, ..., m$,

$$(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) = R^n_n(\gamma(s_1, ..., s_n), \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n]) = \gamma(v_1, ..., v_n)$$

where $v_q = S^n_n(s_q, \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n])$ for all $q = 1, ..., n$,

$$(\sigma_{u,H} \circ_h \sigma_{t,F})(f) = S^m_m(f(u_1, ..., u_m), \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) = f(\widetilde{w}_1, ..., \widetilde{w}_m)$$

where $\widetilde{w}_p = S^m_m(u_p, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m])$ for all $p = 1, ..., m$,

$$(\sigma_{u,H} \circ_h \sigma_{t,F})(\gamma) = R^n_n(\gamma(h_1, ..., h_n), \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n]) = \gamma(\widetilde{v}_1, ..., \widetilde{v}_n)$$

where $\widetilde{v}_q = S^n_n(h_q, \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n])$ for all $q = 1, ..., n$.

Case 1 : $i \in \{a_1, ..., a_k\}$ and $j \in \{b_1, ..., b_l\}$. Since $u_{\pi(a_i)} = x_{a_i}$, we have $u_{a_i} = u_{\pi(\pi^{-1}(a_i))} = x_{\pi^{-1}(a_i)}$. Since $t_{a_i} = x_{\pi(a_i)}$, we have $t_{\pi^{-1}(a_i)} = x_{\pi(\pi^{-1}(a_i))} = x_{a_i}$. Then

$$\begin{aligned}
\widetilde{w}_{a_i} &= S^m_m(u_{a_i}, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \\
&= S^m_m(x_{\pi^{-1}(a_i)}, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \\
&= \widehat{\sigma}_{u,H}[t_{\pi^{-1}(a_i)}] \\
&= x_{a_i}.
\end{aligned}$$

So

$$\begin{aligned}
w_{a_i} &= S^m_m(t_{a_i}, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) \\
&= S^m_m(x_{\pi(a_i)}, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) \\
&= \widehat{\sigma}_{t,F}[u_{\pi(a_i)}] \\
&= x_{a_i} \\
&= \widetilde{w}_{a_i}
\end{aligned}$$

for all $i = 1, ..., k$. Similarly, $v_{b_j} = \widetilde{v}_{b_j}$ for all $j = 1, ..., l$.

Case 2 : $i \in \{1, ..., m\} \setminus \{a_1, ..., a_k\}$ and $j \in \{1, ..., n\} \setminus \{b_1, ..., b_l\}$. Since $u_{a_1} = x_{\pi^{-1}(a_1)}, ...,$ $u_{a_k} = x_{\pi^{-1}(a_k)}$, we obtain $\widehat{\sigma}_{t,F}[u_{a_r}] = x_{\pi^{-1}(a_r)}$ for all $x_{a_r} \in var(t_i) \subseteq var(u)$. By Lemma 2.5, we have $w_i = S^m_m(t_i, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) = \pi^{-1}(t_i)$ for all $i \in \{1, ..., m\} \setminus \{1, ..., k\}$. Then

$$\begin{aligned}
\widetilde{w}_i &= S^m_m(u_i, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \\
&= S^m_m(\pi^{-1}(\pi^{-1}(t_i)), \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \\
&\quad (\text{since } \widehat{\sigma}_{u,H}[t_{a_r}] = x_{\pi(a_r)} \text{ for all } x_{a_r} \in var(\pi^{-1}(\pi^{-1}(t_i))) \text{ and by Lemma 2.5}) \\
&= \pi(\pi^{-1}(\pi^{-1}(t_i))) \\
&= \pi^{-1}(t_i) \\
&= w_i
\end{aligned}$$

for all $i \in \{1, ..., m\} \setminus \{a_1, ..., a_k\}$. Similarly, $v_j = \widetilde{v}_j$ for all $j \in \{1, ..., n\} \setminus \{b_1, ..., b_l\}$. It follows that $f(w_1, ..., w_m) = f(\widetilde{w}_1, ..., \widetilde{w}_m)$, $\gamma(v_1, ..., v_n) = \gamma(\widetilde{v}_1, ..., \widetilde{v}_n)$ and so $\sigma_{t,F} \circ_h \sigma_{u,H} = \sigma_{u,H} \circ_h \sigma_{t,F}$. Therefore $\sigma_{t,F}$ is a completely regular element in $Relhyp((m), (n))$.

■

**Lemma 3.6.** *If $\sigma_{t,F} \in R_X \setminus CR(R_X)$, then $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$ for all $\sigma_{u,H} \in Relhyp((m),(n))$.*

*Proof.* Let $\sigma_{t,F} \in R_X \setminus CR(R_X)$ with $t = x_i \in X_m$ and $F = \gamma(s_1,...,s_n)$ such that $var(F) = \{x_{b_1},...,x_{b_l}\}$. Then there exists $x_{b_k} \in var(F)$ such that $i - most(s_p) \neq x_{b_k}$ for all $p = b_1,...,b_l$. Since $\sigma_{t,F} \in R_X$, there exist $j \in \{1,...,n\}$ and some distinct integers $b'_1,...,b'_l \in \{1,...,n\}$ such that $j - most(s_{b'_r}) = x_{b_r}$ for all $r = 1,...,l$. Let $\sigma_{u,H} \in Relhyp((m),(n))$. We show that $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$. Assume that $\sigma_{t,F} = \sigma_{u,H} \circ_h \sigma^2_{t,F}$. First, consider

$$(\sigma^2_{t,F})(\gamma) = \widehat{\sigma}_{x_i,F}[F] = R^n_n(\gamma(s_1,...,s_n), i - most(s_1),...,i - most(s_n)) = \gamma(v_1,...,v_n)$$

where $v_p = S^n_n(s_p, i - most(s_1),...,i - most(s_n))$ for all $p = 1,...,n$. Next, consider

$$\begin{aligned}
\gamma(s_1,...,s_n) &= \sigma_{t,F}(\gamma) \\
&= (\sigma_{u,H} \circ_h \sigma^2_{t,F})(\gamma) \\
&= \widehat{\sigma}_{u,H}[\sigma^2_{t,F}(\gamma)] \\
&= \widehat{\sigma}_{u,H}[\gamma(v_1,...,v_n)] \\
&= R^n_n(\gamma(h_1,...,h_n), \widehat{\sigma}_{u,H}[v_1],...,\widehat{\sigma}_{u,H}[v_n]) \quad \text{where } H = \gamma(h_1,...,h_n) \\
&= \gamma(S^n_n(h_1, \widehat{\sigma}_{u,H}[v_1],...,\widehat{\sigma}_{u,H}[v_n]),...,S^n_n(h_n, \widehat{\sigma}_{u,H}[v_1],...,\widehat{\sigma}_{u,H}[v_n])).
\end{aligned}$$

So $s_p = S^n_n(h_p, \widehat{\sigma}_{u,H}[v_1],...,\widehat{\sigma}_{u,H}[v_n])$ for all $p = 1,...,n$. Without loss of generality, let $j - most(h_{b'_k}) = x_\alpha$ and $j - most(u) = x_\beta$ for some $x_\alpha \in var(H)$ and $x_\beta \in var(u)$. Then

$$\begin{aligned}
x_{b_k} &= j - most(s_{b'_k}) \\
&= S^n_n(j - most(h_{b'_k}), j - most(\widehat{\sigma}_{u,H}[v_1]),...,j - most(\widehat{\sigma}_{u,H}[v_n])) \\
&= S^n_n(x_\alpha, j - most(\widehat{\sigma}_{u,H}[v_1]),...,j - most(\widehat{\sigma}_{u,H}[v_n])) \\
&= j - most(\widehat{\sigma}_{u,H}[v_\alpha]) \\
&= \beta - most(v_\alpha) \\
&\quad (\text{since } j - most(u) = x_\beta \text{ and by Lemma 2.10}) \\
&= S^n_n(\beta - most(s_\alpha), i - most(s_1),...,i - most(s_n)).
\end{aligned}$$

Since $i - most(s_p) \neq x_{b_k}$ for all $p = b_1,...,b_l$, we obtain that $\beta - most(s_\alpha) \notin \{x_{b_1},...,x_{b_l}\} = var(F)$. This is a contradiction with $\beta - most(s_\alpha) \in var(f(s_1,...,s_m)) = var(F)$. Therefore $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$ for all $\sigma_{u,H} \in Relhyp((m),(n))$. ∎

**Lemma 3.7.** *If $\sigma_{t,F} \in R_T \setminus CR(R_T)$, then $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$ for all $\sigma_{u,H} \in Relhyp((m),(n))$.*

*Proof.* Let $\sigma_{t,F} \in R_T \setminus CR(R_T)$ with $t = f(t_1,...,t_m)$ and $F = \gamma(s_1,...,s_n)$ such that $var(t) = \{x_{a_1},...,x_{a_k}\}$ and $var(F) = \{x_{b_1},...,x_{b_l}\}$. Since $\sigma_{t,F} \in R_T$, there exist some distinct integers $a'_1,...,a'_k \in \{1,...,m\}$ and $b'_1,...,b'_l \in \{1,...,n\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$ for all $i = 1,...,k, j = 1,...,l$. Let $\sigma_{u,H} \in Relhyp((m),(n))$. We will show that $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$. Assume that $\sigma_{t,F} = \sigma_{u,H} \circ_h \sigma^2_{t,F}$. If $u = x_\alpha \in X_m$, then $t = \sigma_{t,F}(f) = (\sigma_{u,H} \circ_h \sigma^2_{t,F})(f) = \widehat{\sigma}_{x_\alpha,H}[\sigma^2_{t,F}(f)] = \alpha - most(\sigma^2_{t,F}(f)) \in X_m$. This is a contradiction with $t \in W_{(m)}(X_m) \setminus X_m$. Let $u = f(u_1,...,u_m) \in W_{(m)}(X_m) \setminus X_m$. Since $\sigma_{t,F} \in R_T \setminus CR(R_T)$, we can consider into two cases.
(1) There exists $x_{a_i} \in var(t)$ such that $t_p \neq x_{a_i}$ for all $p = a_1,...,a_k$.
(2) There exists $x_{b_j} \in var(F)$ such that $s_p \neq x_{b_j}$ for all $p = b_1,...,b_l$.

Case 1 : There exists $x_{a_i} \in var(t)$ such that $t_p \neq x_{a_i}$ for all $p = a_1, ..., a_k$. Consider $(\sigma^2_{t,F})(f) = (\sigma_{t,F} \circ_h \sigma_{t,F})(f) = S^m_m(f(t_1, ..., t_m), \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m]) = f(w_1, ..., w_m)$ where $w_p = S^m_m(t_p, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m])$ for all $p = 1, ..., m$. Then

$$\begin{aligned} f(t_1, ..., t_m) &= \sigma_{t,F}(f) \\ &= (\sigma_{u,H} \circ_h \sigma^2_{t,F})(f) \\ &= \widehat{\sigma}_{u,H}[\sigma^2_{t,F}(f)] = \widehat{\sigma}_{u,H}[f(w_1, ..., w_m)] \\ &= S^m_m(f(u_1, ..., u_m), \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m]) \\ &= f(S^m_m(u_1, \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m]), ..., S^m_m(u_m, \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m])). \end{aligned}$$

So $t_p = S^m_m(u_p, \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m])$ for all $p = 1, ..., m$. If $u_{a'_i} = f(u_{a'_{i1}}, ..., u_{a'_{im}}) \in W_{(m)}(X_m) \setminus X_m$, then $x_{a_i} = t_{a'_i} = S^m_m(u_{a'_i}, \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m]) = S^m_m(f(u_{a'_{i1}}, ..., u_{a'_{im}}), \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m]) \in W_{(m)}(X_m) \setminus X_m$, which is impossible. Let $u_{a'_i} = x_\beta \in X_m$. Then $x_{a_i} = t_{a'_i} = S^m_m(u_{a'_i}, \widehat{\sigma}_{u,H}[w_1], ..., \widehat{\sigma}_{u,H}[w_m]) = \widehat{\sigma}_{u,H}[w_\beta]$. Since $u \in W_{(m)}(X_m) \setminus X_m$, it follows that $w_\beta = x_{a_i}$. Then $x_{a_i} = w_\beta = S^m_m(t_\beta, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m])$. If $t_\beta \in W_{(m)}(X_m) \setminus X_m$, then $x_{a_i} = S^m_m(t_\beta, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m]) \in W_{(m)}(X_m) \setminus X_m$, which is impossible. So $t_\beta \in X_m$. Since $t_p \neq x_{a_i}$ for all $p = a_1, ..., a_k$, we have $\widehat{\sigma}_{t,F}[t_p] \neq x_{a_i}$ for all $p = a_1, ..., a_k$. It follows that $t_\beta \notin \{x_{a_1}, ..., x_{a_k}\} = var(t)$. This is a contradiction with $t_\beta \in var(t)$. Therefore $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$ for all $\sigma_{u,H} \in Relhyp((m), (n))$.
Case 2 : The proof is similar to Case 1. ∎

**Theorem 3.8.** [14] *An element $a$ of a semigroup $S$ is completely regular if and only if $a \in a^2Sa^2$.*

**Theorem 3.9.** *$CR(Relhyp((m), (n)))$ is the set of all completely regular elements in $Relhyp((m), (n))$.*

*Proof.* Since $CR(Relhyp((m), (n))) = CR(R_X) \cup CR(R_T)$, every element in $CR(Relhyp((m), (n)))$ is completely regular. Let $\sigma_{t,F}$ be a regular element and $\sigma_{t,F} \notin CR(Relhyp((m), (n)))$. Then $\sigma_{t,F} \in R_X \setminus CR(R_X)$ or $\sigma_{t,F} \in R_T \setminus CR(R_T)$. By Lemma 3.6 and Lemma 3.7, we obtain that $\sigma_{t,F} \neq \sigma_{u,H} \circ_h \sigma^2_{t,F}$ for all $\sigma_{u,H} \in Relhyp((m), (n))$. It follows that $\sigma_{t,F} \neq \sigma^2_{t,F} \circ_h \sigma_{v,G} \circ_h \sigma^2_{t,F}$, where $\sigma_{u,H} = \sigma^2_{t,F} \circ_h \sigma_{v,G}$, for all $\sigma_{v,G} \in Relhyp((m), (n))$. By Theorem 3.8, $\sigma_{t,F}$ is not a completely regular element in $Relhyp((m), (n))$. Therefore $CR(Relhyp((m), (n)))$ is the set of all completely regular elements in $Relhyp((m), (n))$. ∎

## 4. All Right(Left) Regular Elements in $Relhyp((m), (n))$

In this section, we characterize the set of all right(left) regular elements in $Relhyp((m), (n))$. First, we recall the definition of a right(left) regular element of semigroups.

**Definition 4.1.** An element $x$ of a semigroup $S$ is called *right(left) regular* if there exists $y \in S$ such that $xxy = x$ $(yxx = x)$.

Clearly, in semigroup, every completely regular element is both a right regular and left regular element. Indeed, if $x$ is a completely regular element in a semigroup $S$, then there exists $y \in S$ such that $xyx = x$ and $xy = yx$. Hence $x = xyx = xxy$ and $x = xyx = yxx$.

Next, we characterize the set of all right regular elements in $Relhyp((m), (n))$ as the following lemmas.

**Lemma 4.2.** *Let* $t = x_i \in X_m$ *and* $F = \gamma(s_1, ..., s_n) \in rF_{((m),(n))}(X_n)$ *with* $var(F) = \{x_{b_1}, ..., x_{b_l}\}$. *Then* $\sigma_{t,F}$ *is right regular if and only if* $\sigma_{t,F} \in CR(R_X)$.

*Proof.* Let $\sigma_{t,F}$ be right regular. There exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_h \sigma_{t,F} \circ_h \sigma_{u,H}$ where $u \in W_{(m)}(X_m)$ and $H = \gamma(h_1, ..., h_n) \in rF_{((m),(n))}(X_n)$. First, we consider

$$\begin{aligned}(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{x_i,F}[\sigma_{u,H}(\gamma)] \\ &= \widehat{\sigma}_{x_i,F}[H] \\ &= R_n^n(F, i-most(h_1), ..., i-most(h_n)) \\ &= \gamma(w_1, ..., w_n)\end{aligned}$$

where $w_p = S_n^n(s_p, i-most(h_1), ..., i-most(h_n))$ for all $p = 1, ..., n$. Then

$$\begin{aligned}F &= \sigma_{t,F}(\gamma) \\ &= (\sigma_{t,F} \circ_h \sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) \\ &= \widehat{\sigma}_{t,F}[(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma)] \\ &= \widehat{\sigma}_{x_i,F}[\gamma(w_1, ..., w_n)] \\ &= R_n^n(F, i-most(w_1), ..., i-most(w_n)).\end{aligned}$$

Since $var(F) = \{x_{b_1}, ..., x_{b_l}\}$, it follows that $x_{b_j} = i-most(w_{b_j})$ for all $j = 1, ..., l$. Then $x_{b_j} = i-most(w_{b_j}) = S_n^n(i-most(s_{b_j}), i-most(h_1), ..., i-most(h_n))$. Without loss of generality, we may assume that $i-most(s_{b_j}) = x_{b'_j} \in var(F)$ for some $b'_j \in \{b_1, ..., b_l\}$. Then $x_{b_j} = i-most(h_{b'_j})$ for all $j = 1, ..., l$ and we obtain a subset $\{b'_1, ..., b'_l\}$ of $\{b_1, ..., b_l\}$. Next, we show that all elements of a subset $\{b'_1, ..., b'_l\}$ are distinct. Assume that there exist some integers $j, k \in \{1, ..., l\}$ and $j \neq k$ such that $b'_j = b'_k$. Then $x_{b_j} = i-most(h_{b'_j}) = i-most(h_{b'_k}) = x_{b_k}$. This is a contradiction with $x_{b_j} \neq x_{b_k} \in var(F)$. It implies that $\{b_1, ..., b_l\} = \{b'_1, ..., b'_l\}$. Define a bijective map $\phi : \{b_1, ..., b_l\} \to \{b'_1, ..., b'_l\}$ such that $\phi(b_j) = b'_j$ for all $j = 1, ..., l$. Then $i-most(s_{b_j}) = x_{b'_j} = x_{\phi(b_j)}$ for all $j = 1, ..., l$. Therefore $\sigma_{t,F} \in CR(R_X)$. Conversely, let $\sigma_{t,F} \in CR(R_X)$. Then $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is right regular. ■

**Lemma 4.3.** *Let* $t = f(t_1, ..., t_m) \in W_{(m)}(X_m) \backslash X_m$ *and* $F = \gamma(s_1, ..., s_n) \in rF_{((m),(n))}(X_n)$ *with* $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ *and* $var(F) = \{x_{b_1}, ..., x_{b_l}\}$. *Then* $\sigma_{t,F}$ *is right regular if and only if* $\sigma_{t,F} \in CR(R_T)$.

*Proof.* Let $\sigma_{t,F}$ be right regular. There exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_h \sigma_{t,F} \circ_h \sigma_{u,H}$ where $u \in W_{(m)}(X_m)$ and $H = \gamma(h_1, ..., h_n) \in rF_{((m),(n))}(X_n)$. If $u \in X_m$ then $t = \sigma_{t,F}(f) = (\sigma_{t,F} \circ_h \sigma_{t,F} \circ_h \sigma_{u,H})(f) = u \in X_m$, which is impossible. Let $u = f(u_1, ..., u_m) \in W_{(m)}(X_m) \setminus X_m$. Then

$$(\sigma_{t,F} \circ_h \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[u] = S_m^m(t, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) = f(w_1, ..., w_m)$$

where $w_p = S_m^m(t_p, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m])$ for all $p = 1, ..., m$ (1)
and $(\sigma_{t,F} \circ_h \sigma_{u,H})(\gamma) = \widehat{\sigma}_{t,F}[H] = R_n^n(F, \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n]) = \gamma(g_1, ..., g_n)$
where $g_q = S_n^n(s_q, \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n])$ for all $q = 1, ..., n$. (2)

First, we consider (1). Then

$$\begin{aligned} t &= \sigma_{t,F}(f) \\ &= (\sigma_{t,F} \circ_h \sigma_{t,F} \circ_h \sigma_{u,H})(f) \\ &= \widehat{\sigma}_{t,F}[f(w_1,...,w_m)] \\ &= S^m_m(t, \widehat{\sigma}_{t,F}[w_1],...,\widehat{\sigma}_{t,F}[w_m]). \end{aligned}$$

Since $var(t) = \{x_{a_1},...,x_{a_k}\}$, it follows that $x_{a_i} = \widehat{\sigma}_{t,F}[w_{a_i}]$ for all $i = 1,...,k$. Since $t \in W_{(m)}(X_m) \setminus X_m$, it implies that $w_{a_i} = x_{a_i}$ for all $i = 1,...,k$. Hence $x_{a_i} = w_{a_i} = S^m_m(t_{a_i}, \widehat{\sigma}_{t,F}[u_1],...,\widehat{\sigma}_{t,F}[u_m])$. If $t_{a_i} \in W_{(m)}(X_m) \setminus X_m$, then $x_{a_i} = S^m_m(t_{a_i}, \widehat{\sigma}_{t,F}[u_1],...,\widehat{\sigma}_{t,F}[u_m]) \in W_{(m)}(X_m) \setminus X_m$, which is impossible. So $t_{a_i} = x_{a'_i}$ for some $a'_i \in \{a_1,...,a_k\}$. Then $x_{a_i} = S^m_m(t_{a_i}, \widehat{\sigma}_{t,F}[u_1],...,\widehat{\sigma}_{t,F}[u_m]) = \widehat{\sigma}_{t,F}[u_{a'_i}]$.
It implies that $u_{a'_i} = x_{a_i}$ for all $i = 1,...,k$. By using this process, we obtain all distinct integers $a'_1,...,a'_k \in \{a_1,...,a_k\}$, i.e., $\{a'_1,...,a'_k\} = \{a_1,...,a_k\}$. Define a bijective map $\pi : \{a_1,...,a_k\} \to \{a'_1,...,a'_k\}$ by $\pi(a_i) = a'_i$ for all $i = 1,...,k$. So $t_{a_i} = x_{a'_i} = x_{\pi(a_i)}$ for all $i = 1,...,k$. Similarly, if we consider (2) then we can show that there exists a bijective map $\phi$ on $\{b_1,...,b_l\}$ such that $s_{b_j} = x_{\phi(b_j)}$ for all $j = 1,...,l$. Therefore $\sigma_{t,F} \in CR(R_T)$. Conversely, let $\sigma_{t,F} \in CR(R_T)$. Then $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is right regular. ∎

Finally, we determine the set of all left regular elements in $Relhyp((m),(n))$ as follows.

**Lemma 4.4.** *Let $t = x_i \in X_m$ and $F = \gamma(s_1,...,s_n) \in rF_{((m),(n))}(X_n)$ with $var(F) = \{x_{b_1},...,x_{b_l}\}$. Then $\sigma_{t,F}$ is left regular if and only if $\sigma_{t,F} \in CR(R_X)$.*

*Proof.* Let $\sigma_{t,F}$ be left regular. There exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} = \sigma_{u,H} \circ_h \sigma_{t,F} \circ_h \sigma_{t,F}$ where $u \in W_{(m)}(X_m)$ and $H = \gamma(h_1,...,h_n) \in rF_{((m),(n))}(X_n)$. Assume that $\sigma_{t,F} \notin CR(R_X)$. Then there exists $x_{b_j} \in var(F)$ such that $i - most(s_{b_k}) \neq x_{b_j}$ for all $k = 1,...,l$. Consider

$$(\sigma_{t,F} \circ_h \sigma_{t,F})(\gamma) = \widehat{\sigma}_{x_i,F}[\sigma_{x_i,F}(\gamma)] = \widehat{\sigma}_{x_i,F}[F] = R^n_n(F, i - most(s_1),...,i - most(s_n)).$$

Then every variable $x_{b_k}$ in a relational term $F$ is replaced by $i - most(s_{b_k})$. But $x_{b_j} \neq i - most(s_{b_k})$ for all $k = 1,...,l$, so $x_{b_j} \notin var((\sigma_{t,F} \circ_h \sigma_{t,F})(\gamma))$. Since $var((\sigma_{u,H} \circ_h \sigma_{t,F} \circ_h \sigma_{t,F})(\gamma)) = var((\sigma_{u,H} \circ_h (\sigma_{t,F} \circ_h \sigma_{t,F}))(\gamma)) \subseteq var((\sigma_{t,F} \circ_h \sigma_{t,F})(\gamma))$, it follows that $x_{b_j} \notin var((\sigma_{u,H} \circ_h \sigma_{t,F} \circ_h \sigma_{t,F})(\gamma))$. This contradicts with $x_{b_j} \in var(F) = var(\sigma_{t,F}(\gamma)) = var((\sigma_{u,H} \circ_h \sigma_{t,F} \circ_h \sigma_{t,F})(\gamma))$. Therefore $\sigma_{t,F} \in CR(R_X)$. Conversely, let $\sigma_{t,F} \in CR(R_X)$. Then $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is left regular. ∎

**Lemma 4.5.** *Let $t = f(t_1,...,t_m) \in W_{(m)}(X_m) \setminus X_m$ and $F = \gamma(s_1,...,s_n) \in rF_{((m),(n))}(X_n)$ with $var(t) = \{x_{a_1},...,x_{a_k}\}$ and $var(F) = \{x_{b_1},...,x_{b_l}\}$. Then $\sigma_{t,F}$ is left regular if and only if $\sigma_{t,F} \in CR(R_T)$.*

*Proof.* Let $\sigma_{t,F}$ be left regular. There exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} = \sigma_{u,H} \circ_h \sigma_{t,F} \circ_h \sigma_{t,F}$ where $u \in W_{(m)}(X_m)$ and $H = \gamma(h_1,...,h_n) \in rF_{((m),(n))}(X_n)$. Consider

$$\begin{aligned} (\sigma_{t,F} \circ_h \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] \\ &= \widehat{\sigma}_{t,F}[t] \\ &= S^m_m(t, \widehat{\sigma}_{t,F}[t_1],...,\widehat{\sigma}_{t,F}[t_m]) \\ &= f(w_1,...,w_m) \end{aligned}$$

where $w_i = S_m^m(t_i, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_n])$ for all $i = 1, ..., m$ and

$$\begin{aligned}(\sigma_{t,F} \circ_h \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] \\ &= \widehat{\sigma}_{t,F}[F] \\ &= R_n^n(F, \widehat{\sigma}_{t,F}[s_1], ..., \widehat{\sigma}_{t,F}[s_n]) \\ &= \gamma(z_1, ..., z_n)\end{aligned}$$

where $z_i = S_n^n(s_i, \widehat{\sigma}_{t,F}[s_1], ..., \widehat{\sigma}_{t,F}[s_n])$ for all $i = 1, ..., n$. We want to show that $\sigma_{t,F} \in CR(R_T)$. Assume that $\sigma_{t,F} \notin CR(R_T)$. We can consider into 4 cases.

(1) $t_{a_1}, ..., t_{a_k} \in var(t)$ such that $t_p = t_q$ for some $p, q \in \{a_1, ..., a_k\}$ and $p \neq q$.

(2) There exists $p \in \{a_1, ..., a_k\}$ such that $t_p \in W_{(m)}(X_m) \setminus X_m$.

(3) $s_{b_1}, ..., s_{b_l} \in var(F)$ such that $s_p = s_q$ for some $p, q \in \{b_1, ..., b_l\}$ and $p \neq q$.

(4) There exists $p \in \{b_1, ..., b_l\}$ such that $s_p \in W_{(m)}(X_m) \setminus X_m$.

Case (1) : $t_{a_1}, ..., t_{a_k} \in var(t)$ such that $t_p = t_q$ for some $p, q \in \{a_1, ..., a_k\}$ and $p \neq q$. Then there exist at least one element of $var(t)$ which is not an element of the set $\{t_{a_1}, ..., t_{a_k}\}$, say $x_{a_i}$. So $x_{a_i} \notin var(\widehat{\sigma}_{t,F}[t_j])$ for all $j = a_1, ..., a_k$. Since $(\sigma_{t,F} \circ_h \sigma_{t,F})(f) = S_m^m(t, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m])$, we have to replace every variable $x_j$ of a term $t$ by $\widehat{\sigma}_{t,F}[t_j]$. But $x_{a_i} \notin var(\widehat{\sigma}_{t,F}[t_j])$ for all $j = a_1, ..., a_k$, so $x_{a_i} \notin var((\sigma_{t,F} \circ_h \sigma_{t,F})(f))$. Since $var((\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f)) \subseteq var(\sigma_{t,F}^2(f))$, it follows that $x_{a_i} \notin var((\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f)) = var(\sigma_{t,F}(f)) = var(t)$. This is a contradiction with $x_{a_i} \in var(t)$.

Case (2) : There exists $p \in \{a_1, ..., a_k\}$ such that $t_p \in W_{(m)}(X_m) \setminus X_m$.

Give $A := \{p \mid p \in \{a_1, ..., a_k\}$ such that $t_p \in W_{(m)}(X_m) \setminus X_m\}$. Then $A \neq \emptyset$. If $A = I(t)$ then $t_p \in W_{(m)}(X_m) \setminus X_m$ for all $p = a_1, ..., a_k$. It implies that $\widehat{\sigma}_{t,F}[t_p] \in W_{(m)}(X_m) \setminus X_m$ for all $p = a_1, ..., a_k$. Since $w_i = S_m^m(t_i, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_n])$ for all $i = 1, ..., m$, we have to replace every variable $x_{a_j}$ of a term $t_i$ by $\widehat{\sigma}_{t,F}[t_{a_j}]$. Thus the structure of a subterm $w_i = S_m^m(t_i, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_n])$ is longer than a subterm $t_i$ for all $i = 1, ..., m$. Hence, for each $j = 1, ..., k$, if we pick the smallest sequence $(c)$ of $seq^t(x_{a_j}) = seq^{\sigma_{t,F}(f)}(x_{a_j})$, then we always obtain the smallest sequence $(c')$ of $seq^{f(w_1,...,w_m)}(x_{a_j}) = seq^{(\sigma_{t,F}^2)(f)}(x_{a_j})$ such that $|c'| > |c|$. It implies that the smallest sequence of $seq^{(\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f)}(x_{a_j})$ is longer than the smallest sequence of $seq^t(x_{a_j})$. Since $(\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f) = \sigma_{t,F}(f) = t$, the smallest sequence of $seq^{(\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f)}(x_{a_j})$ and the smallest sequence of $seq^t(x_{a_j})$ are the same, which is a contradiction. So $A \subset I(t)$. Next, consider $I(t) \setminus A \neq \emptyset$. There are 2 subcases.

Case (2.1) : There exists $p \in I(t) \setminus A$ such that $t_p = x_i$ but $i \in A$. That is $t_i \in W_{(m)}(X_m) \setminus X_m$. Then $\widehat{\sigma}_{t,F}[t_i] \in W_{(m)}(X_m) \setminus X_m$. Give $\widetilde{I}(t) = \{p \in I(t) \mid t_p \neq t_q$ for all $q = 1, ..., m\}$. Then $\widetilde{I}(t) \setminus A \subseteq I(t) \setminus A$. Consider $(\sigma_{t,F} \circ_h \sigma_{t,F})(f) = S_m^m(t, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m]) = f(w_1, ..., w_m)$. Then $w_p = S_m^m(t_p, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m]) = S_m^m(x_i, \widehat{\sigma}_{t,F}[t_1], ..., \widehat{\sigma}_{t,F}[t_m]) = \widehat{\sigma}_{t,F}[t_i] \in W_{(m)}(X_m) \setminus X_m$.

Give $B = \{p \mid p \in \{a_1, ..., a_k\}$ such that $w_p \in var(t)$ and $w_p \neq w_q$ for all $q = 1, ..., m\}$. Since $var(f(w_1, ..., w_m)) = var((\sigma_{t,F} \circ_h \sigma_{t,F})(f)) \subseteq var(t)$, $|B| \leq |\widetilde{I}(t) \setminus A| - 1$ (delete $t_p = x_i$ such that $w_p \in W_{(m)}(X_m) \setminus X_m$). Since $t = \sigma_{t,F}(f) = (\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f) = \widehat{\sigma}_{u,H}[f(w_1, ..., w_m)]$, we have $var(t) \subseteq var(f(w_1, ..., w_m))$. It follows that $|\widetilde{I}(t) \setminus A| \leq |B| \leq |\widetilde{I}(t) \setminus A| - 1$. This is a contradiction.

Case (2.2) : $t_p = x_{\pi(p)}$ for all $p \in I(t) \setminus A$, where $\pi$ is a mapping on $I(t) \setminus A$. Then for each $i \in A$, we have $x_i \notin \{t_p \mid p \in I(t) \setminus A\}$ and $t_i \in W_{(m)}(X_m) \setminus X_m$. There are 2 subcases.

Case (2.2.1) : $x_i \notin var(t_q)$ for all $q \in A$. The proof is similar to Case (1), we have $x_i \notin var((\sigma_{u,H} \circ_h \sigma_{t,F}^2)(f)) = var(\sigma_{t,F}(f)) = var(t)$. This is a contradiction.

Case (2.2.2) : $x_i \in var(t_q)$ for some $q \in A$. The proof is similar to case $A = I(t)$ and we also get a contradiction. The proof of Case (3) and Case (4) are similar to Case (1) and Case (2), respectively. Therefore $\sigma_{t,F} \in CR(R_T)$.

Conversely, let $\sigma_{t,F} \in CR(R_T)$. Then $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is left regular. ∎

By all previous lemmas, we can conclude that the set of all completely regular elements and the set of all right(left) regular elements in $Relhyp((m),(n))$ are the same.

**Theorem 4.6.** *$CR(Relhyp((m),(n)))$ is the set of all right(left) regular elements in $Relhyp((m),(n))$.*

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