Thai Journal of Mathematics
Volume 21 Number 2 (2023)
Pages 315–333
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

International Conference on Digital Image Processing and Machine Learning 2022

A Novel Iterative Procedure with Perturbations for Two Generalized Asymptotically Quasi–Nonexpansive Nonself–Mappings in Banach Spaces

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Abstract The goal of this note is to propose a new projection type of two-step iterative procedure with perturbations for finding the common fixed point of two nonself generalized asymptotically quasi–nonexpansive mappings in Banach spaces. A sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is also established in a real uniformly convex Banach space as well as strong convergence theorems in a nonempty closed convex subset of a real Banach space. Our results generalize and improve several relevant results of the existing literature.

MSC: 47H09; 47H10; 54H25; 58C30

Keywords: generalized asymptotically quasi–nonexpansive–nonself mapping; uniformly covex Banach space; strong convergence; completely continuous; common fixed point

Submission date: 29.12.2022 / Acceptance date: 13.06.2023

1. Introduction and Preliminaries

Let C be a nonempty closed convex subset of real normed linear space X. A self-mapping $T : C \to C$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$. A mapping $T : C \to C$ is said to be uniformly L–Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

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It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L–Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

A self-mapping $T : C \to C$ is called *generalized asymptotically nonexpansive* see [23] if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1, k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + \delta_n \tag{1.3}$$

for all $x, y \in C$ and $n \geq 1$. $T : C \to C$ is said to be *generalized asymptotically quasi–nonexpansive* if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1, k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n p\| \leq k_n \|x - p\| + \delta_n \tag{1.4}$$

for all $x \in C, p \in F(T)$ ($F(T)$ denote the set of fixed points of T) and $n \geq 1$.

It is clear from the definition that a generalized asymptotically quasi–nonexpansive mapping is to unify various definitions of classes of mappings associated with the class of generalized asymptotically nonexpansive mapping, asymptotically nonexpansive type, asymptotically nonexpansive mappings, and nonexpansive mappings. However, the converse of each of above statement may be not true. The example shows that a generalized asymptotically quasi–nonexpansive mapping is not an asymptotically quasi–nonexpansive mapping; see [23].

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations, for example, asymptotically nonexpansive mappings, etc., have been studied by a number of authors (see, e.g., [4, 5, 15, 17, 20, 23, 26]) and references cited therein.

Approximation of fixed points remains a widely used technique to prove the existence of solutions of ordinary as well as partial differential equations. In recent years, a multitude of iterative procedures has been developed and utilized to approximate the fixed points of various classes of mappings. Indeed, the Mann and Ishikawa iteration procedures are two basic iteration schemes which now form the foundation of iterative fixed point theory.

In an attempt to construct a convergent sequence of iterates involving a nonexpansive mapping, Mann [15] defined an iteration method as (for any $x_1 \in C$)

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \; n \geq 1, \tag{1.5}$$

where $\alpha_n \in (0, 1)$.

In 1974, with a view to approximate the fixed point of pseudo–contractive mappings in Hilbert spaces, Ishikawa [11] introduced a new iteration procedure as (for $x_1 \in K$)

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in (0, 1)$.

Obviously the iterative schemes and deals with one self–mapping only.

Iterative techniques for approximating fixed points have been investigated by various authors (e.g., [12, 18, 19, 26, 28–30, 32]) using the Mann iteration scheme or Ishikawa iteration scheme. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an up–to date account of literature on this topic, we refer the readers to Berinde [1].

In 1986, Das and Debata [6] introduced and studied the case of two mapping in iteration processes. This success can be rich source of inspiration for many authors, see for example, Takahashi and Tamura [25] and Khan and Takahashi [13]. For approximating the common fixed points, the two mappings case has its own importance as it has a direct link with the minimization problem, see for example Takahashi [24].

Being an important generalization of the class of nonexpansive self–mappings, in 1972, Goebel and Kirk [10] introduced the class of asymptotically nonexpansive self–mappings, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self–mapping on C, then T has a fixed point.

In 1991, Schu [21] introduced the following modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \tag{1.7}$$

to approximate fixed points of asymptotically nonexpansive self–mappings in Hilbert space. Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self–mappings in Hilbert space or Banach spaces (see [17, 20, 21, 26]).

In most of these papers, the well known Mann iteration process (1.5) has been studied and the operator T has been assumed to map C into itself. The convexity of C then ensures that the sequence $\{x_n\}$ generated by (1.5) is well defined. If, however, C is a proper subset of the real Banach space X and T maps C into X (as is the case in many applications), then the sequence given by (1.5) may not be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P\colon X \to C$ in the recursion formula (1.5) as follows: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n PTx_n, \ n \geq 1. \tag{1.8}$$

For nonexpansive nonself–mappings, some authors (see [8, 16, 21, 22, 31]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space.

The concept of asymptotically nonexpansive nonself–mappings was introduced by Chidume, Ofoedu and Zegeye [3] in 2003 as the generalization of asymptotically nonexpansive self–mappings. The asymptotically nonexpansive nonself–mapping is defined as follows:

Let C be a nonempty subset of a real normed linear space X. Let $P\colon\ X \to C$ be a nonexpansive retraction of X onto C. A nonself–mapping $T : C \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\ \infty), k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\| \tag{1.9}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be *uniformly L–Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x-y\| \tag{1.10}$$

for all $x, y \in C$ and $n \geq 1$.

In [3], they studied the following iterative sequence: $x_1 \in C$,

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \tag{1.11}$$

to approximate some fixed point of T under suitable conditions.

If T is a self–mapping, then P becomes the identity mapping so that (1.9) and (1.10) reduce to (1.1) and (1.2) respectively. (1.11) reduces to (1.7).

In 2006, Wang [31] generalizes the iteration process (1.11) as follows: $x_1 \in C$,

$$y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n)\ ,\ n \geq 1, \tag{1.12}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself–mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$. He proved strong convergence of the sequence $\{x_n\}$ defined by (1.12) to a common fixed point of T_1 and T_2 under proper conditions. Meanwhile, the results of [31] generalized the results of [3].

The generalized asymptotically nonexpansive nonself and generalized asymptotically quasi–nonexpansive nonself–mappings are defined by Deng and Liu [8] as follows:

Let C be a nonempty subset of a real normed linear space X. Let P: $X \to C$ be a nonexpansive retraction of X onto C. A nonself–mapping T: $C \to X$ is called *generalized asymptotically nonexpansive* if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1, k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| + \delta_n \tag{1.13}$$

for all $x, y \in C$ and $n \geq 1$. $T : C \to X$ is said to be *generalized asymptotically quasi–nonexpansive* if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1, k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}p\| \leq k_n\|x - p\| + \delta_n \tag{1.14}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

If T is a self–mapping, then P becomes the identity mappings so that (1.13) and (1.14) reduces to (1.3) and (1.4), respectively.

Deng and Liu [8] studied the following iterative sequence which can be viewed as an extension for iterative schemes of Wang [31] : $x_i \in C$ ($i = 0, 1, 2, \ldots, q$ and $q \in \mathbb{N}$ is a fixed number),

$$y_n = P(\alpha_n x_n + \overline{\beta}_n T_2(PT_2)^{n-1}x_n + \overline{\gamma}_n v_n)\ ,\ n = 0, 1, 2,\ \ldots,$$
$$x_{n+1} = P(\alpha_n x_n + \beta_n T_1(PT_1)^{n-1}y_{n-q} + \gamma_n u_n)\ , n = q,\ q+1,\ q+2,\ \ldots, \tag{1.15}$$

where $T_1, T_2 : C \to X$ are generalized asymptotically quasi–nonexpansive nonself–mappings, $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\overline{\alpha}_n\}, \{\overline{\beta}_n\}$ and $\{\overline{\gamma}_n\}$ are real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \overline{\alpha}_n + \overline{\beta}_n + \overline{\gamma}_n = 1$ for all $n \geq 0$. They also provide the strong convergence theorem in a real uniformly convex Banach space.

In 2009, the projection type Ishikawa iteration for two asymptotically nonexpansive nonself–mappings was defined and constructed by Thianwan [27]. The scheme is defined as follows:

$$y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n)\ ,\ n \geq 1, \tag{1.16}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$. He studied the scheme for two asymptotically nonexpansive nonself–mappings and proved strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to a common fixed point of T_1, T_2 under suitable conditions in a uniformly convex Banach space.

Note that Thianwan process (1.12) and Wang process (1.12) are independent: neither reduces to the other.

If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.16) reduces to (1.11). It also can be reduces to Schu process (1.7).

We note that, in applications, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. This leads us, in this paper, to introduce and study a new class of two–step iterative scheme with perturbations for solving the fixed point problem for generalized asymptotically quasi–nonexpansive nonself–mappings. It is given as follows.

Let X be a normed space, C a nonempty convex subset of X, $P\colon X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2 : C \to X$ are given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} y_n &= P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n), \\ x_{n+1} &= P((1-\alpha_n-\lambda_n)T_2(PT_2)^{n-1}y_n + \alpha_n T_1(PT_1)^{n-1}y_n + \lambda_n u_n),\ n \geq 1, \end{aligned} \tag{1.17}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are appropriate real sequences in $[0,1)$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. We then prove its strong convergence under some suitable conditions in Banach spaces.

Now, we recall some well known concepts and results.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) \ = \inf\{1 - \|\frac{1}{2}(x+y)\| \ : \ \|x\| = 1,\ \|y\| = 1,\ \epsilon = \|x-y\|\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset C of X is said to be a retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P.

A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [7]) that

(1) If X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset$ X is an optimal set with interior, then C is a nonexpansive retract of X.

(2) A subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that two mappings $S, T : C \to X$ where C is a subset of a normed space X, are said to satisfy condition A' (see [9]) if there exists a nondecreasing function $f : [0,\ \infty) \to [0,\ \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\ \infty)$ such that either

$$\|x - Sx\| \geq f(d(x,\ F)) \ \text{or} \ \|x - Tx\| \geq f(d(x,\ F))$$

for all $x \in C$, where $d(x,\ F) = \inf\{\|x-q\| \ : \ q \in F = F(S) \cap F(T)\}$.

Note that condition A' reduces to condition (A) (see [26]) when $S = T$. Maiti and Ghosh [14] and Tan and Xu [26] have approximated fixed points of a nonexpansive mapping T by Ishikawa iterates under the condition (A).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1. (see [26]). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$, *then*

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) In particular, if $\{a_n\}$ *has a sequence* $\{a_{n_k}\}$ *converging to* 0, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. (see [21]). *Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Main Results

In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\}, \{\delta_n^{(i)}\} \subset [1, \ \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty, \sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in $[0, 1)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.17). If $q \in F$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in F$, by boundedness of the sequences $\{u_n\}$ and $\{v_n\}$, so we can put

$$M = \max\{\sup_{n\geq 1} \|u_n - q\|, \sup_{n\geq 1} \|v_n - q\|\}.$$

Setting $k_n^{(1)} = 1 + r_n^{(1)}$, $k_n^{(2)} = 1 + r_n^{(2)}$. Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ $(i = 1, 2)$, so $\sum_{n=1}^{\infty} r_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} r_n^{(2)} < \infty$. Using (1.17), we have

$$\begin{aligned}
\|y_n - q\| &= \|P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n) - P(q)\| \\
&\leq \|(1-\beta_n-\gamma_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q) + \gamma_n(v_n - q)\| \\
&\leq (1-\beta_n-\gamma_n)\|x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}x_n - q\| + \gamma_n\|v_n - q\| \\
&\leq (1-\beta_n-\gamma_n)\|x_n - q\| + \beta_n(1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M \\
&= (1-\beta_n-\gamma_n)\|x_n - q\| + (\beta_n + \beta_n r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M \\
&\leq \|x_n - q\| + r_n^{(2)}\|x_n - q\| + \delta_n^{(2)} + \gamma_n M \\
&= (1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M, \qquad (2.1)
\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1}-q\| &= \|P((1-\alpha_n-\lambda_n)T_2(PT_2)^{n-1}y_n+\alpha_nT_1(PT_1)^{n-1}y_n+\lambda_nu_n)-P(q)\|\\ &\le \|(1-\alpha_n-\lambda_n)T_2(PT_2)^{n-1}(y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)+\lambda_n(u_n-q)\|\\ &\le (1-\alpha_n-\lambda_n)\|T_2(PT_2)^{n-1}y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\|+\lambda_n\|u_n-q\|\\ &\le (1-\alpha_n-\lambda_n)(1+r_n^{(2)})\|y_n-q\|+\alpha_n(1+r_n^{(1)})\|y_n-q\|+\delta_n^{(2)}+\delta_n^{(1)}\\ &\quad +\lambda_n\|u_n-q\|\\ &\le (1-\alpha_n-\lambda_n)(1+r_n^{(2)})\|y_n-q\|+\alpha_n(1+r_n^{(1)})\|y_n-q\|+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM\\ &\le (1+r_n^{(2)})\|y_n-q\|+(r_n^{(1)})\|y_n-q\|+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM\\ &= (1+r_n^{(2)}+r_n^{(1)})\|y_n-q\|+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM\\ &\le (1+r_n^{(2)}+r_n^{(1)})((1+r_n^{(2)})\|x_n-q\|+\delta_n^{(2)}+\gamma_nM)+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM\\ &= (1+r_n^{(2)}+r_n^{(1)})(1+r_n^{(2)})\|x_n-q\|+(1+r_n^{(2)}+r_n^{(1)})\delta_n^{(2)}\\ &\quad +(1+r_n^{(2)}+r_n^{(1)})\gamma_nM+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM\\ &= (1+r_n^{(2)}+r_n^{(2)}++r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)})\|x_n-q\|+\epsilon_n^{(1)}\end{aligned}$$

where $\epsilon_n^{(1)}=(1+r_n^{(2)}+r_n^{(1)})\delta_n^{(2)}+(1+r_n^{(2)}+r_n^{(1)})\gamma_nM+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM$ and we note here that $\sum_{n=1}^{\infty}\epsilon_n^{(1)}<\infty$ since $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, $\sum_{n=1}^{\infty}r_n^{(1)}<\infty$, $\sum_{n=1}^{\infty}r_n^{(2)}<\infty$, $\sum_{n=1}^{\infty}\delta_n^{(1)}<\infty$ and $\sum_{n=1}^{\infty}\delta_n^{(2)}<\infty$. Since $\sum_{n=1}^{\infty}(r_n^{(2)}+r_n^{(2)}+r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)})<\infty$ we obtained by Lemma 1.1(i) that $\lim_{n\to\infty}\|x_n-q\|$ exists. This completes the proof. ■

Lemma 2.2. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1,T_2:C\to X$ be two uniformly L–Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\},\{\delta_n^{(i)}\}\subset[1,\ \infty)$ $(i=1,2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$, $\sum_{n=1}^{\infty}\delta_n^{(i)}<\infty$ and $F=F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\},\{\beta_n\}$ are real sequences in $[\epsilon,1-\epsilon]$ for some $\epsilon\in(0,1)$, $\{\gamma_n\},\{\lambda_n\}\subset[0,1)$ such that $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\{u_n\},\{v_n\}$ are bounded sequences in C. From an arbitrary $x_1\in C$, define the sequence $\{x_n\}$ by (1.17). Then $\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|x_n-T_2x_n\|=0$.*

Proof. Let $q\in F$. Setting $k_n^{(1)}=1+r_n^{(1)}, k_n^{(2)}=1+r_n^{(2)}$. By Lemma 2.1 we see that $\lim_{n\to\infty}\|x_n-q\|$ exists. It follows that $\{x_n\}$ and $\{y_n\}$ are bounded. Also, $\{u_n-y_n\}$ and $\{v_n-x_n\}$ are bounded. Now we set

$$C=\max\{\sup_{n\ge1}\|u_n-y_n\|,\sup_{n\ge1}\|v_n-x_n\|\}.$$

Assume that $\lim_{n\to\infty} \|x_n - q\| = c$. In addition,

$$\|y_n - q\| \leq (1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M, \tag{2.2}$$

where the notation M is taken from Lemma 2.1. Taking the lim sup on both sides in the inequality (2.2), we have

$$\limsup_{n\to\infty} \|y_n - q\| \leq c. \tag{2.3}$$

Note that

$$\begin{aligned}
\|T_2(PT_2)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)\| &\leq \|T_2(PT_2)^{n-1}y_n - q\| \\
&\quad + \lambda_n\|u_n - T_2(PT_2)^{n-1}y_n\| \\
&\leq \|T_2(PT_2)^{n-1}y_n - q\| + \lambda_n\|u_n - y_n\| \\
&\quad + \lambda_n\|T_2(PT_2)^{n-1}y_n - y_n\| \\
&\leq \|T_2(PT_2)^{n-1}y_n - q\| + \lambda_n\|u_n - y_n\| \\
&\quad + \lambda_n\|T_2(PT_2)^{n-1}y_n - q\| \\
&\quad + \lambda_n\|y_n - q\| \\
&\leq (1 + r_n^{(2)})\|y_n - q\| + \delta_n^{(2)} \\
&\quad + \lambda_n\|u_n - y_n\| + \lambda_n(1 + r_n^{(2)})\|y_n - q\| \\
&\quad + \delta_n^{(2)} + \lambda_n\|y_n - q\| \\
&\leq (1 + r_n^{(2)})\|y_n - q\| + \delta_n^{(2)} + \lambda_n C \\
&\quad + \lambda_n(1 + r_n^{(2)})\|y_n - q\| + \delta_n^{(2)} \\
&\quad + \lambda_n\|y_n - q\|,
\end{aligned}$$

taking the lim sup on both sides in this inequality, using (2.3) and we note here that $\sum_{n=1}^{\infty} r_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n^{(2)} < \infty$, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)\| \leq c. \tag{2.4}$$

In addition,

$$\begin{aligned}
\|T_1(PT_1)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)\| &\leq \|T_1(PT_1)^{n-1}y_n - q\| \\
&\quad + \lambda_n\|u_n - T_2(PT_2)^{n-1}y_n\| \\
&\leq (1 + r_n^{(1)})\|y_n - q\| + \delta_n^{(1)} \\
&\quad + \lambda_n\|u_n - y_n\| \\
&\quad + \lambda_n\|T_2(PT_2)^{n-1}y_n - q\| \\
&\quad + \lambda_n\|y_n - q\| \\
&\leq (1 + r_n^{(1)})\|y_n - q\| + \delta_n^{(1)} \\
&\quad + \lambda_n C + \lambda_n(1 + r_n^{(2)})\|y_n - q\| \\
&\quad + \delta_n^{(2)} + \lambda_n\|y_n - q\|,
\end{aligned}$$

taking the lim sup on both sides in this inequality, using (2.3) and we note here that $\sum_{n=1}^{\infty} r_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} r_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty} \delta_n^{(2)} < \infty$, we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)\| \leq c. \tag{2.5}$$

In addition,

$$\begin{aligned}\|x_{n+1} - q\| \leq \|&(1-\alpha_n-\lambda_n)(T_2(PT_2)^{n-1}y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q) \\ &+ \lambda_n(u_n - q)\| \\ &\leq (1 + r_n^{(2)} + r_n^{(2)} + +r_n^{(2)}r_n^{(2)} + r_n^{(1)} + r_n^{(1)}r_n^{(2)})\|x_n - q\| + \epsilon_n^{(1)},\end{aligned} \tag{2.6}$$

where the notation $\epsilon_n^{(1)}$ is taken from Lemma 2.1.
Since $\sum_{n=1}^{\infty}(r_n^{(2)}+r_n^{(2)}+r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)}) < \infty$, $\sum_{n=1}^{\infty}\epsilon_n^{(1)} < \infty$ and $\lim_{n\to\infty}\|x_{n+1}-q\| = c$, letting $n \to \infty$ in the inequality (2.6), we have

$$\lim_{n\to\infty} \|(1-\alpha_n-\lambda_n)(T_2(PT_2)^{n-1}y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q) + \lambda_n(u_n - q)\| = c \tag{2.7}$$

From

$$\begin{aligned}&\|(1-\alpha_n)(T_2(PT_2)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)) + \alpha_n(T_1(PT_1)^{n-1}y_n - q \\ &\qquad + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n))\| = \\ &\qquad \|(1-\alpha_n-\lambda_n)(T_2(PT_2)^{n-1}y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q) + \lambda_n(u_n - q)\|\end{aligned}$$

and (2.7), we have

$$\begin{aligned}\lim_{n\to\infty} \|&(1-\alpha_n)(T_2(PT_2)^{n-1}y_n - q + \lambda_n(u_n - T_2(PT_2)^{n-1}y_n)) + \alpha_n(T_1(PT_1)^{n-1}y_n - q \\ &+\lambda_n(u_n - T_2(PT_2)^{n-1}y_n))\| = c.\end{aligned} \tag{2.8}$$

By using (2.4), (2.5), (2.8) and Lemma 1.2, we have

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - T_2(PT_2)^{n-1}y_n\| = 0. \tag{2.9}$$

In addition,

$$\begin{aligned}\|T_2(PT_2)^{n-1}x_n - q + \gamma_n(v_n - x_n)\| &\leq \|T_2(PT_2)^{n-1}x_n - q\| + \gamma_n\|v_n - x_n\| \\ &\leq k_n^{(2)}\|x_n - q\| + \delta_n^{(2)} + \gamma_n C,\end{aligned}$$

and taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - q + \gamma_n(v_n - x_n)\| \leq c. \tag{2.10}$$

Using (1.17) and (2.1), we have

$$
\begin{aligned}
\|x_{n+1}-q\| &\le (1-\alpha_n-\lambda_n)\|T_2(PT_2)^{n-1}y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\| \\
&\quad +\lambda_n\|u_n-q\| \\
&= (1-\alpha_n-\lambda_n)\|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n+T_1(PT_1)^{n-1}y_n-q\| \\
&\quad +\alpha_n\|T_1(PT_1)^{n-1}y_n-q\|+\lambda_n\|u_n-q\| \\
&\le (1-\alpha_n-\lambda_n)\|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\| \\
&\quad +(1-\alpha_n-\lambda_n)\|T_1(PT_1)^{n-1}y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\| \\
&\quad +\lambda_n\|u_n-q\| \\
&= (1-\alpha_n-\lambda_n)\|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\| \\
&\quad +(1-\lambda_n)\|T_1(PT_1)^{n-1}y_n-q\|+\lambda_n\|u_n-q\| \\
&\le \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+(1-\lambda_n)\|T_1(PT_1)^{n-1}y_n-q\| \\
&\quad +\lambda_n\|u_n-y_n\|+\lambda_n\|y_n-q\| \\
&\le \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+(1-\lambda_n)(1+r_n^{(1)})\|y_n-q\|+\delta_n^{(1)} \\
&\quad +\lambda_n\|u_n-y_n\|+\lambda_n\|y_n-q\| \\
&= \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+(1+r_n^{(1)}-\lambda_n-\lambda_n r_n^{(1)})\|y_n-q\| \\
&\quad +\delta_n^{(1)}+\lambda_n\|u_n-y_n\|+\lambda_n\|y_n-q\| \\
&= \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+(1+r_n^{(1)}-\lambda_n r_n^{(1)})\|y_n-q\| \\
&\quad +\delta_n^{(1)}+\lambda_n\|u_n-y_n\| \\
&\le \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+(1+r_n^{(1)})\|y_n-q\| \\
&\quad +\delta_n^{(1)}+\lambda_n\|u_n-y_n\| \\
&\le \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+\|y_n-q\|+r_n^{(1)}\|y_n-q\| \\
&\quad +\delta_n^{(1)}+\lambda_n C \\
&\le \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+\|y_n-q\| \\
&\quad +r_n^{(1)}((1+r_n^{(2)})\|x_n-q\|+\delta_n^{(2)}+\gamma_n M)+\delta_n^{(1)}+\lambda_n C \\
&= \|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}y_n\|+\|y_n-q\| \\
&\quad +(r_n^{(1)}+r_n^{(1)}r_n^{(2)})\|x_n-q\|+r_n^{(1)}\delta_n^{(2)}+r_n^{(1)}\gamma_n M+\delta_n^{(1)}+\lambda_n C. \qquad (2.11)
\end{aligned}
$$

Taking the lim inf on both sides in the inequality (2.11), by (2.9), $\sum_{n=1}^{\infty} r_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} r_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(2)} < \infty$ and $\lim_{n\to\infty} \|x_{n+1}-q\| = c$, we have

$$\liminf_{n\to\infty} \|y_n-q\| \ge c. \qquad (2.12)$$

It follows from (2.3) and (2.12) that $\lim_{n\to\infty}\|y_n - q\| = c$. This implies that

$$\begin{aligned} c = \lim_{n\to\infty}\|y_n - q\| &\le \lim_{n\to\infty}\|(1-\beta_n-\gamma_n)(x_n - q) \\ &\quad + \beta_n(T_2(PT_2)^{n-1}x_n - q) + \gamma_n(v_n - q)\| \\ &\le \lim_{n\to\infty}\|x_n - q\| = c, \end{aligned}$$

and so

$$\lim_{n\to\infty}\|(1-\beta_n-\gamma_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q) + \gamma_n(v_n - q)\| = c. \tag{2.13}$$

From

$$\begin{aligned} &\|(1-\beta_n)(x_n - q + \gamma_n(v_n - x_n)) + \beta_n(T_2(PT_2)^{n-1}x_n - q + \gamma_n(v_n - x_n))\| = \\ &\quad \|(1-\beta_n-\gamma_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q) + \gamma_n(v_n - q)\| \end{aligned}$$

and (2.13), we have

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n - q + \gamma_n(v_n - x_n)) + \beta_n(T_2(PT_2)^{n-1}x_n - q + \gamma_n(v_n - x_n))\| = c. \tag{2.14}$$

Note that $\|x_n - q + \gamma_n(v_n - x_n)\| \le \|x_n - q\| + \gamma_n C$ gives that

$$\limsup_{n\to\infty}\|x_n - q + \gamma_n(v_n - x_n)\| \le c. \tag{2.15}$$

By using (2.10), (2.14), (2.15) and Lemma 1.2 we have

$$\lim_{n\to\infty}\|T_2(PT_2)^{n-1}x_n - x_n\| = 0. \tag{2.16}$$

From $y_n = P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n)$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and (2.16), we have

$$\begin{aligned} \|y_n - x_n\| &= \|P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n) - x_n\| \\ &\le \beta_n\|T_2(PT_2)^{n-1}x_n - x_n\| + \gamma_n\|v_n - x_n\| \\ &\le \|T_2(PT_2)^{n-1}x_n - x_n\| + \gamma_n C \\ &\to 0 \ (\text{as } n\to\infty). \end{aligned} \tag{2.17}$$

Now, since $T_i(i = 1, 2)$ are uniformly L–Lipschitzian for Lipschitz constant $L = \max_{1\le i\le 2}\{L_i\} > 0$. We note that

$$
\begin{aligned}
\|T_1(PT_1)^{n-1}x_n - x_n\| &= \|T_1(PT_1)^{n-1}x_n - y_n + y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - y_n\| + \|y_n - x_n\| \\
&= \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n \\
&\quad + T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le L\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le L\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - T_2(PT_2)^{n-1}y_n\| \\
&\quad + \|T_2(PT_2)^{n-1}y_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}x_n - x_n\| \\
&\quad + \|x_n - y_n\| + \|y_n - x_n\| \\
&\le L\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - T_2(PT_2)^{n-1}y_n\| \\
&\quad + L\|y_n - x_n\| + \|T_2(PT_2)^{n-1}x_n - x_n\| + \|x_n - y_n\| \\
&\quad + \|y_n - x_n\|
\end{aligned}
$$

Thus, it follows from (2.9), (2.16), (2.17) and (2.18) that

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \tag{2.18}$$

Since $T_i(i = 1, 2)$ are uniformly L–Lipschitzian for Lipschitz constant $L = \max_{1\le i\le 2}\{L_i\} > 0$, using (1.17), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le (1 - \alpha_n - \lambda_n)\|T_2(PT_2)^{n-1}y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - x_n\| \\
&\quad + \lambda_n\|u_n - x_n\| \\
&\le (1 - \alpha_n - \lambda_n)\|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + (1 - \alpha_n - \lambda_n)\|T_1(PT_1)^{n-1}y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - x_n\| \\
&\quad + \lambda_n\|u_n - y_n\| + \lambda_n\|y_n - x_n\| \\
&= (1 - \alpha_n - \lambda_n)\|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + (1 - \lambda_n)\|T_1(PT_1)^{n-1}y_n - x_n\| + \lambda_n\|u_n - y_n\| + \lambda_n\|y_n - x_n\| \\
&\le (1 - \alpha_n - \lambda_n)\|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + (1 - \lambda_n)\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\
&\quad + (1 - \lambda_n)\|T_1(PT_1)^{n-1}x_n - x_n\| + \lambda_n\|u_n - y_n\| + \lambda_n\|y_n - x_n\| \\
&\le (1 - \alpha_n - \lambda_n)\|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + (1 - \lambda_n)L\|y_n - x_n\| + (1 - \lambda_n)\|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\quad + \lambda_n\|u_n - y_n\| + \lambda_n\|y_n - x_n\| \\
&\le \|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}y_n\| + L\|y_n - x_n\| \\
&\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| + \lambda_n\|y_n - x_n\| + \lambda_n C
\end{aligned}
$$

It follows from (2.9), (2.17), (2.18), $\sum_{n=1}^{\infty}\lambda_n<\infty$, $\sum_{n=1}^{\infty}r_n^{(1)}<\infty$ and $\sum_{n=1}^{\infty}\delta_n^{(1)}<\infty$ that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.19}$$

Using (2.18) and (2.19), we have

$$\begin{aligned}\|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\| &= \|x_{n+1}-x_n+x_n-T_1(PT_1)^{n-1}x_n\\ &\quad +T_1(PT_1)^{n-1}x_n-T_1(PT_1)^{n-1}x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-1}x_{n+1}-T_1(PT_1)^{n-1}x_n\|\\ &+\|T_1(PT_1)^{n-1}x_n-x_n\|\\ &\le \|x_{n+1}-x_n\|+L\|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-1}x_n-x_n\|,\\ &\to 0 \text{ (as } n\to\infty).\end{aligned} \tag{2.20}$$

In addition, for $n\ge 2$,

$$\begin{aligned}\|x_{n+1}-T_1(PT_1)^{n-2}x_{n+1}\| &= \|x_{n+1}-x_n+x_n-T_1(PT_1)^{n-2}x_n\\ &\quad +T_1(PT_1)^{n-2}x_n-T_1(PT_1)^{n-2}x_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-2}x_n-x_n\|\\ &\quad +\|T_1(PT_1)^{n-2}x_{n+1}-T_1(PT_1)^{n-2}x_n\|\\ &\le \|x_{n+1}-x_n\|+\|T_1(PT_1)^{n-2}x_n-x_n\|\\ &\quad +L\|x_{n+1}-x_n\|.\end{aligned}$$

It follows from (2.19) and (2.20) that

$$\lim_{n\to\infty}\|x_{n+1}-T_1(PT_1)^{n-2}x_{n+1}\|=0. \tag{2.21}$$

We denote as $(PT_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (2.20) and (2.21), we have

$$\begin{aligned}\|x_{n+1}-T_1x_{n+1}\| &= \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}+T_1(PT_1)^{n-1}x_{n+1}-T_1x_{n+1}\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+\|T_1(PT_1)^{n-1}x_{n+1}-T_1x_{n+1}\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|(PT_1)^{n-1}x_{n+1}-x_{n+1}\|\\ &= \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|(PT_1)(PT_1)^{n-2}x_{n+1}-P(x_{n+1})\|\\ &\le \|x_{n+1}-T_1(PT_1)^{n-1}x_{n+1}\|+L\|T_1(PT_1)^{n-2}x_{n+1}-x_{n+1}\|\\ &\to 0 \text{ (as } n\to\infty)\ ,\end{aligned}$$

which implies that $\lim_{n\to\infty}\|x_n-T_1x_n\|=0$. Similary, we may show that

$$\lim_{n\to\infty}\|x_n-T_2x_n\|=0.$$

The proof is completed. ■

We prove the strong convergence of the scheme (1.17) under condition A' which is weaker than the compactness of the domain of the mappings.

Theorem 2.3. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two uniformly L–Lipschitzian, nonself generalized asymptotically quasi–nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\}, \{\delta_n^{(i)}\} \subset [1, \infty)(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty}\delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$, $\{\gamma_n\}, \{\lambda_n\} \subset [0, 1)$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.17) converge strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 2.2 we have $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0$. It follows from condition A' that

$$\lim_{n\to\infty} f(d(x_n,\ F)) \leq \lim_{n\to\infty}\|x_n - T_1x_n\| = 0 \text{ or}$$
$$\lim_{n\to\infty} f(d(x_n,\ F)) \leq \lim_{n\to\infty}\|x_n - T_2x_n\| = 0.$$

In the both case, $\lim_{n\to\infty} f(d(x_n,\ F)) = 0$. Since $f : [0,\ \infty) \to [0,\ \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\ \infty)$, we obtain that $\lim_{n\to\infty} d(x_n,\ F) = 0$. That is

$$\liminf_{n\to\infty}\inf_{y^*\in F}\|x_n - y^*\| = \lim_{n\to\infty} d(x_n,\ F) = 0.$$

It implies that

$$\inf_{y^*\in F}\lim_{n\to\infty}\|x_n - y^*\| = 0.$$

So, for any given $\epsilon > 0$, there exists $p \in F$ and $N > 0$ such that for all $n \geq N$, $\|x_n - p\| < \frac{\epsilon}{2}$. This shows that

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for all $n \geq N$ and $m \geq 1$. Hence, $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. From $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0$ and the continuity of T_1 and T_2, we have $\|T_1u - u\| = \|T_2u - u\| = 0$. Thus $u \in F$. From (2.17), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0,$$

it follows that $\lim_{n\to\infty}\|y_n - u\| = 0$. This completes the proof. ∎

The following result follows from Theorem 2.3.

Theorem 2.4. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself asymptotically nonexpansive mappings of C condition A' with sequences $\{k_n^{(i)}\} \subset [1,\ \infty)$ $(i = 1, 2)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$, $\{\gamma_n\}, \{\lambda_n\} \subset$*

$[0,1)$ *such that* $\sum_{n=1}^{\infty}\gamma_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$ *and* $\{u_n\},\{v_n\}$ *are bounded sequences in* C. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by the iterative scheme* (1.17) *converge strongly to a common fixed point of* T_1 *and* T_2.

In the remainder of this section, we deal with the strong convergence of the new iterative scheme (1.17) to a common fixed point of nonself generalized asymptotically quasi–nonexpansive mappings in a real Banach space.

Theorem 2.5. *Let* X *be a real Banach space and* C *a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T_1, T_2 : C \to X$ *be two nonself generalized asymptotically quasi–nonexpansive mappings of* C *with sequences* $\{k_n^{(i)}\},\{\delta_n^{(i)}\}\subset[1,\infty)$ $(i=1,2)$, *respectively such that* $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty, \sum_{n=1}^{\infty}\delta_n^{(i)}<\infty$ *and* $F=F(T_1)\cap F(T_2)\neq\emptyset$ *is closed. Suppose that* $\{\alpha_n\},\{\beta_n\}$ *are real sequences in* $[\epsilon,1-\epsilon]$ *for some* $\epsilon\in(0,1),\{\gamma_n\},\{\lambda_n\}\subset[0,1)$ *such that* $\sum_{n=1}^{\infty}\gamma_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$ *and* $\{u_n\},\{v_n\}$ *are bounded sequences in* C. *Then the sequence* $\{x_n\}$ *defined by the iterative scheme* (1.17) *converges strongly to a common fixed point of* T_1 *and* T_2 *if and only if* $\liminf_{n\to\infty} d(x_n,\ F)=0$, *where* $d(x_n,\ F)=\inf_{y\in F}\|x_n-y_n\|$, $n\geq 1$.

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. As in the proof of Lemma 2.1 by the arbitrariness of $q\in F$, we have

$$\|x_{n+1}-q\|\leq(1+r_n^{(2)}+r_n^{(2)}++r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)})\|x_n-q\|+\epsilon_n^{(1)},$$

and so

$$d(x_{n+1},\ F)\leq(1+r_n^{(2)}+r_n^{(2)}++r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)})d(x_n,\ F)+\epsilon_n^{(1)},$$

where $\epsilon_n^{(1)}=(1+r_n^{(2)}+r_n^{(1)})\delta_n^{(2)}+(1+r_n^{(2)}+r_n^{(1)})\gamma_nM+\delta_n^{(2)}+\delta_n^{(1)}+\lambda_nM$. Since $\sum_{n=1}^{\infty}(r_n^{(2)}+r_n^{(2)}++r_n^{(2)}r_n^{(2)}+r_n^{(1)}+r_n^{(1)}r_n^{(2)})<\infty$ and $\sum_{n=1}^{\infty}\epsilon_n^{(1)}<\infty$, we obtained by Lemma 1.1 that $\lim_{n\to\infty}d(x_n,F)$ exists. Then, by hypothesis $\liminf_{n\to\infty}d(x_n,F)=0$, we have $\lim_{n\to\infty}d(x_n,F)=0$. From Theorem 2.3 it obtain that $\{x_n\}$ defined by (1.17) is a Cauchy sequence in C. Let $\lim_{n\to\infty}x_n=u$. Now $\lim_{n\to\infty}d(x_n,\ F)=0$ gives that $d(u,\ F)=0$. F is closed; therefore $u\in F$. This completes the proof of Theorem 2.5. ∎

Let $\{a_n\}$ be a sequence that converges to a, with $a_n\neq a$ for all n. If positive constants λ and ϑ exist with $\lim_{n\to\infty}\frac{|a_{n+1}-a|}{|a_n-a|^{\vartheta}}=\lambda$, then $\{a_n\}$ converges to a of order ϑ, with asymptotic error constant λ. If $\vartheta=1$ (and $\lambda<1$), the sequence is linearly convergent and if $\vartheta=2$, the sequence is quadratically convergent (see [2]).

The following examples show that generalized asymptotically quasi–nonexpansive mapping is not nonexpansive mapping and hence asymptotically nonexpansive mapping.

Example 2.6. Let $E = [-\pi, \pi]$ and let T be defined by $Tx = xcosx$ for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi–nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |T_1x - 0| = |x||cosx| \leq |x| = |x - z|,$$

and hence T is generalized asymptotically quasi–nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{\delta_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$|Tx - Ty| = |\frac{\pi}{2}cos\frac{\pi}{2} - \pi cos\pi| = \pi,$$

whereas

$$|x - y| = |\frac{\pi}{2} - \pi| = \frac{\pi}{2}.$$

■

Example 2.7. Let $E = \mathbb{R}$ and let S be defined by

$$S(x) = \begin{cases} \frac{x}{2}cos\frac{1}{x}, & if\ x \neq 0, \\ 0, & if\ x = 0. \end{cases}$$

If $x \neq 0$ and $Sx = x$, then $x = \frac{x}{2}cos\frac{1}{x}$. Thus $2 = cos\frac{1}{x}$. This is not hold. S is a quasi–nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Sx - z| = |Sx - 0| = |\frac{x}{2}||cos\frac{1}{x}| \leq \frac{|x|}{2} < |x| = |x - z|,$$

and hence S is generalized asymptotically quasi–nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{\delta_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$|Tx - Ty| = |\frac{1}{3\pi}cos\frac{3\pi}{2} - \frac{1}{2\pi}cos\pi| = \frac{1}{2\pi},$$

whereas

$$|x - y| = |\frac{2}{3\pi} - \frac{1}{\pi}| = \frac{1}{3\pi}.$$

■

Additionally, Let $\alpha_n = \frac{n}{2n+5}$, $\beta_n = \frac{n}{2n+3}$, $\gamma_n = \frac{n}{3n+3}$, $\lambda_n = \frac{n}{4n+3}$, $v_n = \frac{1}{5n^4}$ and $u_n = \frac{1}{3n^4}$. Suppose that $T_1 = T$ and $T_1 = S$ given in Example 2.6 and Example 2.7, respectively. So, the convergence of the sequence $\{x_n\}$ generated by (1.17) to a point $0 \in F(T) \cap F(S)$ can be received.

We choose $x_1 = 1$ and run our process within 50 iteration. All code were written in Matlab2019b. We obtain the iteration steps and its amplification factor of the proposed algorithms as shown in Table 1. For convenience, we call the iteration (1.17) the proposed iteration process.

Table 1 shows that the proposed method converges to solution of Example 2.6 and Example 2.7. It can be concluded that the proposed method is linearly convergent and its amplification factor less than 0.763.

Table 1. Numerical experiment of the proposed method for Example

The Proposed Iteration Process		
Iteration Number (n)	$\lvert x_n\rvert$	$\frac{\lvert x_{n+1}\rvert}{\lvert x_n\rvert}$
1	1.7187e-01	1.7187e-01
2	3.4087e-03	1.9832e-02
3	6.0751e-04	1.7821e-01
4	3.1363e-04	5.1626e-01
5	1.4623e-04	4.6624e-01
⋮	⋮	⋮
10	9.8621e-06	4.8094e-01
⋮	⋮	⋮
20	6.4370e-07	6.1130e-01
⋮	⋮	⋮
30	1.3558e-07	8.0878e-01
⋮	⋮	⋮
40	4.1139e-08	7.3432e-01
⋮	⋮	⋮
50	1.6900e-08	7.6262e-01

3. Conclusions

Authors constructed a new projection type of two-step iterative procedure with perturbations to approximate a common fixed point for two nonself generalized asymptotically quasinonexpansive mappings in Banach spaces. The authors proved strong convergence results of such mappings in real uniformly convex Banach and real Banach spaces. Our results extend the corresponding results of Thianwan [27] to the more general class of asymptotically quasi-nonexpansive mappings considered in this paper. Our results also improve the related results of Deng and Liu [8] to the case of the more general class of asymptotically quasi-nonexpansive mappings and two-step iteration process with errors considered in this paper. Illustrative examples are also provided as Example 2.6 and Example 2.7. Our results extend, improve and generalize some known results from the existing literature.

Acknowledgements

This research project is supported by the University of Phayao and Thailand Science Research and Innovation under the project FF66-UoE015. C. Chairatsiripong would like to thank the revenue budget for 2020 (PBTSC63021), School of Science, University of Phayao.

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