Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 69–75

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

On Regular Matrix Semirings

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Abstract : A ring R is called a (*von Neumann*) *regular ring* if for every $x \in R$, $x = xyx$ for some $y \in R$. It is well-known that for any ring R and any positive integer n, the full matrix ring $M_n(R)$ is regular if and only if R is a regular ring. This paper examines this property on any additively commutative semiring S with zero. The regularity of S is defined analogously. We show that for a positive integer n, if $M_n(S)$ is a regular semiring, then S is a regular semiring but the converse need not be true for $n = 2$. And for $n \geq 3$, $M_n(S)$ is a regular semiring if and only if S is a regular ring.

Keywords : Matrix ring; Matrix semiring; Regular ring; Regular semiring.
2000 Mathematics Subject Classification : 16Y60, 16S50.

1 Introduction

A triple $(S, +, \cdot)$ is called a *semiring* if $(S, +)$ and $(S, \cdot)$ are semigroups and $\cdot$ is distributive over $+$. An element $0 \in S$ is called a *zero* of the semiring $(S, +, \cdot)$ if $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. Note that every semiring contains at most one zero. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ [$x \cdot y = y \cdot x$] for all $x, y \in S$. We say that $(S, +, \cdot)$ is *commutative* if it is both additively and multiplicatively commutative.

For a positive integer n and an additively commutative semiring S with zero, let $M_n(S)$ be the set of all $n \times n$ matrices over S. Then under the usual addition and multiplication of matrices, $M_n(S)$ is also an additively commutative semiring with zero and the $n \times n$ zero matrix over S is the zero of the matrix semiring $M_n(S)$. For $A \in M_n(S)$ and $i, j \in \{1, \ldots, n\}$, let A_{ij} be the entry of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column.

Denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the set of integers, the set of rational numbers and the set of real numbers, respectively. Let $\mathbb{Z}_0^+ = \{x \in \mathbb{Z} \,|\, x \geq 0\}$, $\mathbb{Q}_0^+ = \{x \in \mathbb{Q} \,|\, x \geq 0\}$ and $\mathbb{R}_0^+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$.

Example 1.1. (1) Under the usual addition and multiplication of real numbers, $\mathbb{Z}_0^+, \mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are commutative semirings with zero 0 which are not rings. If n is a positive integer greater than 1, then $M_n(\mathbb{Z}_0^+), M_n(\mathbb{Q}_0^+)$ and $M_n(\mathbb{R}_0^+)$ are additively commutative semirings with zero which are not multiplicatively commutative. Also, none of them are rings.

(2) If $S = \{0, 1\}$ with $0+0 = 0, 0+1 = 1+0 = 1+1 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$, then $(S, +, \cdot)$ is a commutative semiring with zero 0 which is not a ring.

(3) Let S be a nonempty subset of $\mathbb{R}$ such that $\min S$ exists. Define

$$x \oplus y = \max\{x, y\} \quad \text{and} \quad x \odot y = \min\{x, y\} \quad \text{for all } x, y \in S.$$

Then $(S, \oplus, \odot)$ is a commutative semiring having $\min S$ as its zero. Also, if S contains more than one element, then $(S, \oplus, \odot)$ is not a ring.

A ring R is called a (*von Neumann*) *regular ring* if for every $x \in R, x = xyx$ for some $y \in R$. Regular rings was originally introduced by von Neumann in order to clarify certain aspects of operator algebras. Regular rings have also been widely studied for their own sake. See [1] for various interesting properties of regular rings. An interesting known result of regularity of rings relating to matrix rings is as follows:

Theorem 1.2. ([2] page 114–115) *For a ring R and a positive integer n, the matrix ring $M_n(R)$ is regular if and only if R is a regular ring.*

By making use of Theorem 1.2, Fang Li [3] extended this result to bounded $I \times \Lambda$ matrices over a ring where I and Λ are any index sets. As a consequence of Theorem 1.2, we have that for every positive integer n and every field F, $M_n(F)$ is a regular ring. In particular, $M_n(\mathbb{Q})$ and $M_n(\mathbb{R})$ are regular rings. Also, by Theorem 1.2, $M_n(\mathbb{Z})$ is not a regular ring for every positive integer n.

We define regular semirings analogously. That is, a semiring S is said to be *regular* if for every $x \in S$, $x = xyx$ for some $y \in S$. We can see that the semirings $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are regular but $\mathbb{Z}_0^+$ is not regular. Also, the semirings in Example 1.1 (2) and (3) are regular semirings.

The purpose of this paper is to investigate the analogous results of Theorem 1.2 for additively commutative semirings with zero. We show that

(1) for a positive integer n, if $M_n(S)$ is a regular semiring, then S is a regular semiring and the converse is not generally true for $n = 2$,

(2) for $n \geq 3$, $M_n(S)$ is a regular semiring if and only of S is regular and S is a ring.

2 Regularity of Matrix Semirings

If S is an additively commutative semiring with zero 0, n is a positive integer and $A \in M_n(S)$ is such that $A_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ with $(i, j) \neq (1, 1)$, then for $B \in M_n(S)$, $A = ABA$ implies $A_{11} = (ABA)_{11} = \sum_{k=1}^{n} (AB)_{1k} A_{k1} = (AB)_{11} A_{11} = \left(\sum_{k=1}^{n} A_{1k} B_{k1} \right) A_{11} = A_{11} B_{11} A_{11}$. Hence we have

Proposition 2.1. *Let S be an additively commutative semiring with zero 0 and n a positive integer. If $M_n(S)$ is a regular semiring, then so is S.*

Remark 2.2. The converse of Proposition 2.1 need not be true for $n = 2$, as the following example shows. Let $S = \{0, 1, 2, 3\}$ and $(S, \oplus, \odot)$ the semiring defined in Example 1.1 (3), that is,

$$x \oplus y = \max\{x, y\} \quad \text{and} \quad x \odot y = \min\{x, y\} \quad \text{for all } x, y \in S.$$

Then $(S, \oplus, \odot)$ is regular. Suppose that $M_2(S)$ is a regular semiring. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \in M_2(S)$. Then $A = ABA$ for some $B \in M_2(S)$. Then by the definition of $\oplus$ and $\odot$,

$$\begin{aligned} AB &= \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 \odot B_{21} & 1 \odot B_{22} \\ 2 \odot B_{11} \oplus B_{21} & 2 \odot B_{12} \oplus B_{22} \end{bmatrix} \end{aligned}$$

and thus

$$\begin{aligned} ABA &= (AB)A \\ &= \begin{bmatrix} 1 \odot B_{21} & 1 \odot B_{22} \\ 2 \odot B_{11} \oplus B_{21} & 2 \odot B_{12} \oplus B_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \odot B_{22} & 1 \odot B_{21} \oplus 1 \odot B_{22} \\ (2 \odot B_{12} \oplus B_{22}) \odot 2 & (2 \odot B_{11} \oplus B_{21}) \odot 1 \oplus (2 \odot B_{12} \oplus B_{22}) \end{bmatrix} \end{aligned}$$

Since $A = ABA$, it follows that

$$1 \odot B_{22} = 0 \tag{1}$$

$$(2 \odot B_{11} \oplus B_{21}) \odot 1 \oplus (2 \odot B_{12} \oplus B_{22}) = 3 \tag{2}$$

From (1), $B_{22} = 0$. Since $(2 \odot B_{11} \oplus B_{21}) \odot 1 \leq 1$, by (2), we have $2 \odot B_{12} \oplus B_{22} = 3$. But $B_{22} = 0$, so

$$3 = 2 \odot B_{12} \oplus B_{22} = 2 \odot B_{12} \leq 2$$

which is a contradiction. This show that $M_2(S)$ is not regular, as desired.

Theorem 2.3. *Let S be an additively commutative semiring with zero 0 and n a positive integer with $n \geq 3$. Then the semiring $M_n(S)$ is regular if and only if S is a regular ring.*

Proof. Assume that $M_n(S)$ is a regular semiring. By Proposition 2.1, S is a regular semiring. To show that S is a ring, let $a \in S$. Define $A \in M_n(S)$ by

$$A_{ij} = \begin{cases} a & \text{if } \ (i,j) \in \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,3)\}, \\ 0 & \text{otherwise,} \end{cases}$$

that is,

$$A = \begin{bmatrix} a & a & 0 & 0 & \cdots & 0 \\ 0 & a & a & 0 & \cdots & 0 \\ a & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{1}$$

Since $M_n(S)$ is regular, $A = ABA$ for some $B \in M_n(S)$. Then from (1), we have

$$\begin{aligned} a = A_{11} = (ABA)_{11} &= \sum_{k=1}^{n} A_{1k}(BA)_{k1} \\ &= a(BA)_{11} + a(BA)_{21} \\ &= a(\sum_{k=1}^{n} B_{1k}A_{k1} + \sum_{k=1}^{n} B_{2k}A_{k1}) \\ &= a(B_{11}a + B_{13}a + B_{21}a + B_{23}a) \\ &= a(B_{11} + B_{13} + B_{21} + B_{23})a, \end{aligned} \tag{2}$$

$$\begin{aligned} 0 = A_{13} = (ABA)_{13} &= \sum_{k=1}^{n} A_{1k}(BA)_{k3} \\ &= a(BA)_{13} + a(BA)_{23} \\ &= a(\sum_{k=1}^{n} B_{1k}A_{k3} + \sum_{k=1}^{n} B_{2k}A_{k3}) \\ &= a(B_{12}a + B_{13}a + B_{22}a + B_{23}a) \\ &= a(B_{12} + B_{13} + B_{22} + B_{23})a, \end{aligned} \tag{3}$$

$$\begin{aligned}
0 = A_{21} = (ABA)_{21} &= \sum_{k=1}^{n} A_{2k}(BA)_{k1} \\
&= a(BA)_{21} + a(BA)_{31} \\
&= a(\sum_{k=1}^{n} B_{2k}A_{k1} + \sum_{k=1}^{n} B_{3k}A_{k1}) \\
&= a(B_{21}a + B_{23}a + B_{31}a + B_{33}a) \\
&= a(B_{21} + B_{23} + B_{31} + B_{33})a, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
0 = A_{32} = (ABA)_{32} &= \sum_{k=1}^{n} A_{3k}(BA)_{k2} \\
&= a(BA)_{12} + a(BA)_{32} \\
&= a(\sum_{k=1}^{n} B_{1k}A_{k2} + \sum_{k=1}^{n} B_{3k}A_{k2}) \\
&= a(B_{11}a + B_{12}a + B_{31}a + B_{32}a) \\
&= a(B_{11} + B_{12} + B_{31} + B_{32})a. \qquad (5)
\end{aligned}$$

Then (3) + (4) + (5) yields

$$\begin{aligned}
0 &= a(B_{12} + B_{13} + B_{22} + B_{23} + B_{21} + B_{23} + B_{31} + B_{33} + B_{11} + B_{12} + B_{31} + B_{32})a \\
&= a(B_{11} + B_{13} + B_{21} + B_{23})a + a(2B_{12} + B_{22} + B_{23} + B_{31} + B_{33} + B_{31} + B_{32})a \\
&= a + a(2B_{12} + B_{22} + B_{23} + B_{31} + B_{33} + B_{31} + B_{32})a \qquad \text{from}(2).
\end{aligned}$$

This show that for every $a \in S$,$a + x = 0$ for some $x \in S$. It follows that S is a ring, as desired.

The converse is obtained directly from Theorem 1.2. $\square$

Remark 2.4. Since $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are not rings, it follows from Theorem 2.3 that $M_n(\mathbb{Q}_0^+)$ and $M_n(\mathbb{R}_0^+)$ are not regular for all $n \geq 3$. Hence $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are regular semirings but for all $n \geq 3$, $M_n(\mathbb{Q}_0^+)$ and $M_n(\mathbb{R}_0^+)$ are not regular. The semirings $M_1(\mathbb{Q}_0^+)$ and $M_1(\mathbb{R}_0^+)$ are regular since both $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are regular. It is natural to ask whether $M_2(\mathbb{Q}_0^+)$ and $M_2(\mathbb{R}_0^+)$ are regular semirings. As shown below, neither $M_2(\mathbb{Q}_0^+)$ nor $M_2(\mathbb{R}_0^+)$ is regular. Suppose that $M_2(\mathbb{Q}_0^+)$ is regular. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{Q}_0^+).$$

Then $A = ABA$ for some $B \in M_2(\mathbb{Q}_0^+)$. Since A is invertible over $\mathbb{Q}$, it follows that

$$I_2 = AA^{-1} = ABAA^{-1} = AB$$

where A^{-1} is the inverse of A over $\mathbb{Q}$ and I_2 is the identity 2×2 matrix over $\mathbb{Q}$. This implies that $B = A^{-1}$ over $\mathbb{Q}$, so $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \notin M_2(\mathbb{Q}_0^+)$, a contradiction.

Hence $M_2(\mathbb{Q}_0^+)$ is not regular. By replacing $\mathbb{Q}$ by $\mathbb{R}$ in the above proof, we have that $M_2(\mathbb{R}_0^+)$ is also not regular. Hence for all $n \geq 2$, the semirings $M_n(\mathbb{Q}_0^+)$ and $M_n(\mathbb{R}_0^+)$ are not regular.

Remark 2.5. In the proof of Theorem 2.3, to show that S is a ring, $n \geq 3$ is required. It is reasonable to ask that for an additively commutative semiring S with zero, if $M_2(S)$ is regular, must S be a ring ? A negative answer is given by the following example. Let $(S = \{0, 1\}, +, \cdot)$ be the commutative semiring in Example 1.1 (2), that is, $0+0 = 0, 0+1 = 1+0 = 1+1 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Then S is not a ring. We claim that $M_2(S)$ is a regular semiring. We have that $M_2(S)$ contains exactly 16 elements which are

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix},$$

$$\begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

It is easy to check that for $A \in M_2(S), A^2 = A$ if and only if A is one of the following 11 matrices :

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix},$$

$$\begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

The remaining 5 matrices are

$$A = \begin{bmatrix}0&1\\0&0\end{bmatrix}, B = \begin{bmatrix}0&0\\1&0\end{bmatrix}, C = \begin{bmatrix}0&1\\1&0\end{bmatrix}, D = \begin{bmatrix}0&1\\1&1\end{bmatrix}, E = \begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

Also, $A = ABA$, $B = BAB$, $C = C^3$, $D = DED$ and $E = EDE$. Hence $M_2(S)$ is a regular semiring, as desired.

Remark. During the referee process, it was brought to our attention of the paper of S.N. Ilin on regularity criterion for complete matrix semirings. The results there are deeper and contain ours. However, the proof given here is quite elementary and essentially self-contained.

References

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(Received 13 May 2009)

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