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Completely Invariant Fatou, Julia, and Escaping Sets of Transcendental Semigroups

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Abstract In holomorphic semigroup dynamics, Fatou and escaping sets are, in general, forward invariant and Julia sets are backward invariant. Therefore, some fundamental results of the holomorphic dynamics of a single holomorphic function cannot be generalized and preserved for holomorphic semigroup dynamics. In this paper, we define completely invariant Julia, Fatou, and escaping sets of transcendental semigroups, and we see how far the results of the holomorphic dynamics of a single transcendental entire function can be preserved and generalized for transcendental semigroup dynamics.

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1. Introduction

A *transcendental semigroup* S is a semigroup of transcendental entire functions defined on the complex plane $\mathbb{C}$, with the semigroup operation being functional composition. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Delta\}$ be a set of transcendental entire functions, where the index set Δ is allowed to be infinite unless otherwise stated. When a semigroup S is generated by $\mathscr{F}$, we write $S = \langle f_\alpha \rangle_{\alpha \in \Delta}$ or simply $S = \langle f_\alpha \rangle$. A semigroup generated by finitely many transcendental entire functions $f_i, i = 1, 2, \ldots, n$, is called a *finitely generated transcendental semigroup*, and we write $S = \langle f_1, f_2, \ldots, f_n \rangle$. If S is generated by only one transcendental entire function f, then S is called a *cyclic transcendental semigroup*, and we write $S = \langle f \rangle$. In this case, each $g \in S$ can be written as $g = f^n$, where f^n is the nth iterate of f.

We say that $\mathscr{F}$ is a *normal family* in $\mathbb{C}$ if every sequence $(f_\alpha) \subseteq \mathscr{F}$ has a subsequence (f_{α_k}) which either is uniformly convergent on all compact subsets of $\mathbb{C}$ or diverges to ∞. If there is a neighborhood U of a point $z \in \mathbb{C}$ such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. If $\mathscr{F}$ is a semigroup S such that it is normal family at z, we say that S is normal at z. We say that a function f is *iteratively divergent* at $z \in \mathbb{C}$

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if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup S is *iteratively divergent* at z if every $f \in S$ is iteratively divergent at z.

As in the iteration theory of a single transcendental entire function, the Fatou, Julia, and escaping sets in the settings of transcendental semigroups, are defined as follows:

Definition 1.1 (**Fatou, Julia, and escaping sets**)**.** Let S be a transcendental semigroup. The *Fatou set* of S is defined by

$$F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\}.$$

The *Julia set* $J(S)$ of S is the complement of $F(S)$. The *escaping set* of S is defined by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}.$$

We call each point of the set $I(S)$ an *escaping point*.

If $S = \langle f \rangle$, then the Fatou, Julia, and escaping sets are respectively denoted by $F(f), J(f)$ and $I(f)$. Thus Definition 1.1 generalizes the definition of Fatou, Julia, and escaping sets of a single transcendental entire function. Any maximal connected subset U of the Fatou set $F(S)$ is called a *Fatou component*. Note that for any transcendental semigroup S, we have

(1) $F(S) \subset F(f)$ for all $f \in S$, and hence $F(S) \subset \bigcap_{f \in S} F(f)$.
(2) $J(f) \subset J(S)$ for all $f \in S$, and $J(S) = \overline{\bigcup_{f \in S} J(f)}$.
(3) $I(S) \subset I(f)$ for all $f \in S$, and hence $I(S) \subset \bigcap_{f \in S} I(f)$

Definition 1.2 (**Forward, backward and completely invariant set**)**.** Let f be a map of a set X into itself. A subset $U \subset X$ is said to be

(1) *forward invariant* under f if $f(U) \subset U$;
(2) *backward invariant* under f if $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\} \subset U$;
(3) *completely invariant* under f if it is both forward and backward invariant.

It is well known in classical transcendental dynamics that the Fatou set $F(f)$ is a largest completely invariant open set, and the Julia set $J(f)$ is a smallest completely invariant closed set. The escaping set $I(f)$ is completely invariant but it is neither open nor closed. However, in transcendental semigroup dynamics, the following results hold.

Proposition 1.3. *Let S be a transcendental semigroup. Then $F(S)$ is a forward invariant open set and $J(S)$ is a backward invariant closed set for each element of S.*

Proposition 1.3 was proved by Poon [1, Theorem 2.1].

Proposition 1.4. *Let S be a transcendental semigroup. Then $I(S)$ is a forward invariant set for each element of S.*

This Proposition 1.4 was proved by Kumar and Kumar [2, Theorem 4.1]. We also proved the same in [3, Theorem 3.1] based on Definition 1.1 of escaping set.

From Propositions 1.3 and 1.4, we can say that the sets $F(S)$, $J(S)$ and $I(S)$ are not neccessirly completely invariant under each element of S. In this paper, we generalize the completely invariant notion of the Fatou, Julia, and escaping sets of a single transcendental entire functions to the completely invariant notion of these sets in transcendental semigroup dynamics. Note that Stankewitz [4, 5] studied completely invariant Julia and Fatou sets of rational semigroups. In this paper, we define a completely invariant Julia, Fatou and escaping sets of a transcendental semigoup S as follows.

Definition 1.5. Let S be a transcendental semigroup. We define a *completely invariant Julia set* of S by

$$J_1(S) = \bigcap \{G : G \text{ is a closed, completely invariant set under each } f \in S\}$$

The *completely invariant Fatou set* $F_1(S)$ is defined as the complement of $J_1(S)$ in $\mathbb{C}$.

Definition 1.6. Let S be a transcendental semigroup. We define a *completely invariant escaping set* of S by

$$I_1(S) = \bigcap_{i\in\mathbb{N}} \{G_i : G_i \text{ is a completely invariant set under each } f \in S, \text{ and each } G_i \text{ contains points } z \in \mathbb{C} \text{ such that } f^n(z) \to \infty \text{ as } n \to \infty \text{ for every } f \in S\}$$

Note that in a transcendental semigroup S, $J_1(S)$ is closed and completely invariant under each $f \in S$, and it contains the Julia sets of each element of S. The corresponding Fatou set $F_1(S)$ is open, completely invariant and contained in the Fatou set of each element of S and the set $I_1(S)$ is neither an open nor a closed set and is contained in the escaping set of each element of S. We prove the following results.

Theorem 1.7. *Let S be a transcendental semigroup. If $J_1(S)$ has non-empty interior, then $J_1(S) = \mathbb{C}$.*

Theorem 1.8. *Let S be a transcendental semigroup which contains functions f and g such that $J(f) \neq J(g)$. Then $J_1(S) = \mathbb{C}$.*

Theorem 1.9. *Let $F_1(S)$ be a completely invariant Fatou set of a transcendental semigroup S. Then number of components of $F_1(S)$ is either 0, 1 or ∞.*

Theorem 1.10. *Let S be a transcendental semigroup. Then $E = I_1(S)$, where E is a set defined by $E = \cap_{n\in\mathbb{N}\cup\{0\}} E_n$, where $E_0 = \bigcap_{h\in S} I(h)$, $E_1 = \cup_{h\in S} h^{-1}(E_0) \cup \bigcup_{h\in S} h(E_0)$, $\ldots$, and $E_n = \bigcup_{h\in S} h^{-1}(E_{n-1}) \cup \bigcup_{h\in S} h(E_{n-1})$.*

The organization of this paper is as follows. In section 2, we study completely invariant Julia and Fatou sets of transcendental semigroups and we prove Theorems 1.7, 1.8 and 1.9. In section 3, we study completely invariant escaping sets of transcendental semigroups and we prove Theorem 1.10.

2. Completely Invariant Julia and Fatou Sets of Transcendental Semigroups

In rational semigroups and, in particular, in polynomial semigroups, there are a few studies of such completely invariant Fatou and Julia sets (see for instance [4–6] for more detail) but there are no studies of such sets in transcendental semigroups. In this section, we concentrate on completely invariant Julia and Fatou sets of transcendental semigroups.

The sets $J_1(S)$ and $F_1(S)$ of Definition 1.5 may or may not coincide with the sets $J(S)$ and $F(S)$ respectively. The following examples can help to compare the sets $J_1(S)$ and $J(S)$, and also $F_1(S)$ and $F(S)$.

Example 2.1. Let $S = \langle f, g\rangle$ be a transcendental semigroup generated by $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$, where $0 < |\lambda| < 1$. Then $J_1(S) = J(f) = J(g)$. One can also verify that $J(S) = J(f) = J(g)$. In this case, $J_1(S) = J(S)$, and so $F_1(S) = F(S)$.

There are other examples (see for instance [7, Example 3.2], and [1, Example 2.1]) of transcendental semigroups similar to Example 2.1.

Example 2.2. Let $S = \langle f, g\rangle$ be a transcendental semigroup generated by $f(z) = \lambda e^z, (0 < \lambda < \frac{1}{e})$ and $g(z) = \lambda e^z, (\lambda > \frac{1}{e})$. Then, by Devaney [8], $J(f)$ is a Cantor set (bouquet), and $J(g) = \mathbb{C}$. In this case, $F(g) = \emptyset$. Thus we have $F(S) = \emptyset$ and $J(S) = \mathbb{C}$. It is easy to verify that $J_1(S) = J(S) = \mathbb{C}$ and $F_1(S) = F(S) = \emptyset$.

Example 2.3. Let $S = \langle f, g\rangle$ be a transcendental semigroup generated by $f(z) = \lambda \sin z$, where $\lambda \in \mathbb{C}$ is chosen in such a way that there are two attracting cycles, and $|\Re(\lambda)| \geq \pi/2$, where $\Re(\lambda)$ represents the real part of λ; and $g(z) = \mu e^z$, where $\mu \in (0, 1/e)$. Then, by Osborne [9, Example 6.4], $J(f)$ is a spider web, and by Devaney [8], $J(g)$ is a Cantor bouquet. In this case, it is easy to verify that $J_1(S) = J(S) = \mathbb{C}$ and $F_1(S) = F(S) = \emptyset$.

Note that the Cantor's bouquet and the spider's web are structurally different sets. Cantor's bouquet is closed and has uncountably many components with a single unbounded complementary component; whereas the spider web is connected with infinitely many complementary components, each of which is bounded. Therefore, the Julia set $J(S)$ that contains both $J(f)$ and $J(g)$ of Example 2.3 must be the entire complex plane $\mathbb{C}$. In all of these three examples, we have $J_1(S) = J(S)$ and $F_1(S) = F(S)$. However, in Example 2.1, we have $J(f) = J(g)$, but in Examples 2.2 and 2.3, we have $J(f) \neq J(g)$. In the following example, an entire (polynomial) semigroup, we have $J(f) \neq J(g)$ as well as $F_1(S) \neq F(S)$ and $J_1(S) \neq J(S)$.

Example 2.4. [4, Example-2] Let $S = \langle z^2, z^2/a\rangle$, where $a \in \mathbb{C}, |a| > 1$. Then the Julia set $J(S) = \{z : 1 \leq |z| \leq |a|\}$, which is not forward invariant. Hence $J_1(S) \neq J(S)$. In this case, $J_1(S) = \mathbb{C}_\infty$. Note that $J(f) = \{z : |z| = 1\}$ and $J(g) = \{z : |z| = |a|\}$. The Fatou set $F(S) = \{z : |z| < 1 \text{ or } |z| > |a|\}$ is not backward invariant, and so $F_1(S) \neq F(S)$. In this case it is obvious that $F_1(S) = \emptyset$.

One of the main result in classical complex dynamics is that if a Julia set has non-empty interior, then the Julia set explodes and it becomes whole complex plane $\mathbb{C}$. This result is generalized and preserved to completely invariant Julia set $J_1(S)$. For this, we workout some constructions for the comparison of sets $J_1(S)$ and $J(S)$. Let $S = \langle f_1, f_2, \dots f_n\rangle$ be a finitely generated transcendental semigroup. Note that $J(h) \subset J_1(S)$ for all $h \in S$ and so $\bigcup_{h\in S} J(h) \subset J_1(S)$. Let us define the following countable collections of sets:

$$\mathscr{E}_0 = \{J(h)\} \text{ for all } h \in S$$

$$\mathscr{E}_1 = \bigcup_{h\in S} h^{-1}(\mathscr{E}_0) \cup \bigcup_{h\in S} h(\mathscr{E}_0)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\mathscr{E}_{n+1} = \bigcup_{h\in S} h^{-1}(\mathscr{E}_n) \cup \bigcup_{h\in S} h(\mathscr{E}_n)$$

and

$$\mathscr{E} = \bigcup_{n=0}^{\infty} \mathscr{E}_n$$

where $h^{-1}(\mathscr{E}_i) = \{h^{-1}(E) : E \in \mathscr{E}_i\}$ and $h(\mathscr{E}_i) = \{h(E) : E \in \mathscr{E}_i\}$ for any collection of sets $\mathscr{E}_i, (i = 1, 2, \dots)$ and a function $h \in S$. The following result gives an alternate description of the set $J_1(S)$ of a transcendental semigroup S.

Theorem 2.5. *For a transcendental semigroup* $\langle f_1, f_2, \dots f_n \rangle$, *we have* $J_1(S) = \overline{\bigcup_{E\in\mathscr{E}} E}$.

Proof. By Definition 1.5, $J_1(S)$ is closed, is completely invariant under each $h \in S$, and contains $J(h)$ for all $h \in S$. Therefore we can write

$$J_1(S) \supset \overline{\bigcup_{E\in\mathscr{E}} E}$$

Since the set $\overline{\bigcup_{E\in\mathscr{E}} E}$ is closed and contains $J(h)$ for all $h \in S$, it remains to show that it is also completely invariant under each $h \in S$. Since h is a continuous closed map, $h(\overline{\bigcup_{E\in\mathscr{E}} E})$ and $h^{-1}(\overline{\bigcup_{E\in\mathscr{E}} E})$ are closed sets for each $h \in S$. This proves our claim. ■

Theorem 2.6. *The set* $J_1(S)$ *is a perfect set.*

Proof. By definition, $J(h) \subset J_1(S)$ for all $h \in S$ and $J(h)$ is perfect, unbounded and contains an infinite number of points for each $h \in S$. The assertion will be proved if we show $J_1(S)$ has no isolated points. Suppose $\alpha \in J_1(S)$ is an isolated point. Then it an isolated point of some $E \in \mathscr{E}$. Choose a neighborhood U of α so that $U - \{\alpha\} \subset F_1(S)$ where $F_1(S)$ is a completely invariant Fatou set of S. Since $h^{-1}(F_1(S)) \subset F_1(S)$ and $h(F_1(S)) \subset F_1(S)$ for all $h \in S$. Each $h \in S$ omits $J_1(S)$ on $U - \{\alpha\}$, and hence every element in S is normal on U. This is a contradiction. ■

Proof of Theorem 1.7. Let $\text{int.}(J_1(S)) \neq \emptyset$, where $\text{int.}J_1(S)$ denotes the interior of $J_1(S)$. Then there exists a disk $D = \{|z - z_0| < r\} \subset J_1(S)$ such that D intersects $J(h)$ for some $h \in S$. Then by [10, Theorem 3.9], for each finite value a, there is sequence $z_k \to z_0 \in J(h)$ and a sequence of positive integers $n_k \to \infty$ such that $f^{n_k}(z_k) = a, (k = 1, 2, 3, \dots)$ except at most for a finite value. Then by backward invariance of $J(h)$, $z_k \in J(h)$, and by forward invariance of $J(h)$, $a \in J(h)$. This shows that every finite value is in $J(h)$, except at most a single value. Since $h \in S$ is arbitrary, so we must have $J_1(S) = \mathbb{C}$. ■

Corollary 2.7. *If* $J_1(S) \neq \mathbb{C}$, *then* $F_1(S)$ *is unbounded.*

Proof. Suppose by the way of contradiction that $F_1(S)$ is bounded. Then $J_1(S)$ has interior points. By Theorem 1.7, $J_1(S) = \mathbb{C}$, which is a contradiction. ■

Similar to the description of the sets $J_1(S)$ in Theorem 2.5, we can give analogous description of the Julia set $J(S)$ of transcendental semigroup S. Let us define the following countable collections of sets:

$$\mathscr{F}_0 = \{J(h)\} \text{ for all } h \in S$$

$$\mathscr{F}_1 = \bigcup_{h\in S} h^{-1}(\mathscr{F}_0)$$

$$\dots \quad \dots \quad \dots$$

$$\mathscr{F}_{n+1} = \bigcup_{h\in S} h^{-1}(\mathscr{F}_n)$$

and

$$\mathscr{F} = \bigcup_{n=0}^{\infty} \mathscr{F}_n$$

where $h^{-1}(\mathscr{F}_i) = \{h^{-1}(F) : F \in \mathscr{F}_i\}$ for any collection of sets $\mathscr{F}_i, (i = 1, 2, \ldots)$ and any function $h \in S$. The following result will give a convenient description of the set $J(S)$ of transcendental semigroup S.

Theorem 2.8. *Let S be a finitely generated transcendental semigroup. Then $J(S) = \overline{\bigcup_{F \in \mathscr{F}} F}$.*

Proof. By Proposition 1.3, $J(S)$ is closed, backward invariant under each $h \in S$ and contains $J(h)$ for all $h \in S$. Therefore we can write

$$J(S) \supset \overline{\bigcup_{F \in \mathscr{F}} F}$$

Since the set $\overline{\bigcup_{F \in \mathscr{F}} F}$ is closed and contains $J(h)$ for all $h \in S$, it remains to show that it is also backward invariant under each $h \in S$. Since h is a continuous closed map, $h^{-1}(\overline{\bigcup_{F \in \mathscr{F}} F})$ is a closed set for each $h \in S$. This proves our claim. ∎

Corollary 2.9. *Let S be a finitely generated transcendental semigroup. Then $J(S) \subset J_1(S)$.*

Proof. By construction, $\mathscr{F} \subset \mathscr{E}$, and hence the assertion follows from Theorems 2.5 and 2.8. ∎

Corollary 2.10. *If $J(S)$ has non-empty interior, then $J_1(S) = \mathbb{C}$.*

Proof. This corollary follows from Theorem 1.7 and Corollary 2.9. ∎

Before proving Theorem 1.8, we prove the following lemma.

Lemma 2.11. *Let $S = \langle f, g \rangle$ be a transcendental semigroup such that $J(f) \neq J(g)$. Then $J_1(S) = \mathbb{C}$.*

Proof. Let U be a completely invariant component of $F(f)$. Then by [10, Theorem 4.36], U is unbounded and simply connected, and $\partial U = J(f)$. Likewise, a completely invariant component V of $F(g)$ is unbounded and simply connected, and $\partial V = J(g)$. $J(f) \neq J(g)$ implies that $\partial U \neq \partial V$. By [10, Theorem 3.8], $J(f)$ and $J(g)$ are unbounded, so $U \cap J(g) \neq \emptyset$ and $V \cap J(f) \neq \emptyset$. The fact that $\partial U \neq \partial V$ implies that $J(f)$ must intersect the interior of V and $J(g)$ must intersect the interior of U.

Let $z \in J(f) \cap \text{int.}(V)$, where $\text{int.}(V)$ is the interior of V. Then, by the forward invariance of $J_1(S)$ and $\text{int.}(V)$ under the map g, one has $g^n(z) \in J_1(S)$ and $g^n(z) \in \text{int.}(V)$ for all $n \in \mathbb{N}$. Likewise, $f^n(z) \in J_1(S)$ and $J_1(S)$ intersects open sets $\text{int.}(U)$ and $\text{int.}(V)$. Thus $J_1(S)$ intersects $\text{int.}(U) \cap \text{int.}(V)$. Since $J_1(S)$ is perfect and completely invariant, it contains all limits of the sequences (f^n) and (g^n). This proves that $\text{int.}(J_1(S)) \neq \emptyset$, and hence, by Theorem 1.7, $J_1(S) = \mathbb{C}$. ∎

Proof of Theorem 1.8. The proof follows by using Lemma 2.11 and Theorem 1.7. ∎

$F_1(S)$ may also have a completely invariant component, as in classical transcendental dynamics.

Lemma 2.12. *If S is a transcendental semigroup, then $F_1(S)$ has at most one completely invariant component.*

Proof. For a transcendental entire function f, $F(f)$ has at most one completely invariant component [10, Theorem 4.38]. By Definition 1.5, $F_1(S) \subset F(h)$ for all $h \in S$. The assertion follows. ∎

$F_1(S)$ can have either 0 or infinitely many multiply connected components as in classical transcendental dynamics.

Lemma 2.13. *Let $F_1(S)$ be a completely invariant Fatou set of a transcendental semigroup S. Then the number of multiply connected components of $F_1(S)$ is either 0 or ∞.*

Proof. For a transcendental entire function f, the number of multiply connected components of $F(f)$ is either 0 or ∞ [10, Theorem 4.43]. By Definition 1.5, $F_1(S) \subset F(h)$ for all $h \in S$. The assertion follows. ∎

Proof of Theorem 1.9. The proof follows from Lemmas 2.12 and 2.13. ∎

3. Completely Invariant Escaping Sets

In [2, Theorem 4.1] and [3, Theorem 2.3], it was proved that the escaping set $I(S)$ of a transcendental semigroup S is forward invariant for each $f \in S$. There are several classes of transcendental semigroups from which we get backward invariant escaping sets. In [11, Theorem 2.1] and [3, Theorem 3.3], it was proved that the escaping set $I(S)$ of an abelian transcendental semigroup S is backward invariant for each $f \in S$. This is a condition for a completely invariant escaping set of a transcendental semigroup. It is a generalization of the completely invariant property of classical escaping sets of a single function to the more general settings of semigroups. In this section, we generalize the classical completely invariant notion of escaping sets of a single function to the completely invariant notion of escaping sets of transcendental semigroups. By Definition 1.6, the set $I_1(S)$ is completely invariant under each element of S, and this set coincides with the escaping set $I(S)$ if and only if S is an abelian semigroup. There are non-abelian transcendental semigroups from which one can get completely invariant escaping sets. The following assertion will be a good source of several examples.

Theorem 3.1. *If $S = \langle f, g \rangle$ and $I(f) = I(g)$, then $I_1(S) = I(S)$.*

Proof. We know that $I(f)$ is completely invariant under f and $I(g)$ is completely invariant under g. If $I(f) = I(g)$, then $I(h) = I(f) = I(g) = I(S)$ for all $h \in S$. In this case, $I_1(S) = I(S)$. ∎

As an example, the semigroup $S = \langle f, g \rangle$ generated by the functions $f(z) = e^{\lambda z}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $g(z) = f^k + \frac{2\pi i}{\lambda}$, $k \in \mathbb{N}$, is completely invariant. Here we find that $I(h) = I(f) = I(g) = I(S)$ for all $h \in S$. Another example of the same kind is the semigroup $S = \langle f, g \rangle$ generated by the functions $f(z) = \lambda \sin z$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $g(z) = f^k + 2\pi$, $k \in \mathbb{N}$. Note that in both of examples, the semigroup S is not abelian. From this discussion, we can conclude that the escaping set $I(S)$ may be completely invariant even if the semigroup S is not abelian. In such a case, the escaping set $I(S)$ is nothing other than the set $I_1(S)$. There are transcendental semigroups where escaping sets and completely invariant escaping sets are empty.

Example 3.2. Suppose that $S = \langle f, g\rangle$, where $f(z) = e^z$ and $g(z) = e^{-z}$. Then both $I_1(S)$ and $I(S)$ are empty. If $z \in I(f)$, then $g(f^n(z)) = 1/e^{f^n(z)} = 1/f(f^n(z)) = 1/f^{n+1}(z) \to 0$ as $n \to \infty$.

Theorem 3.3. *Let S be a transcendental semigroup. Then $I_1(S) \subset I(f)$ for every $f \in S$.*

Proof. Let $z \in I_1(S)$. Then, by Definition 1.6, $z \in G_i$ for all i and $f^n(z) \to \infty$ as $n \to \infty$ for every $f \in S$. This proves that $z \in I(S)$. ∎

Let S be a transcendental semigroup such that $I_1(S) \neq \emptyset$. Then by Theorem 3.3, we can write $I_1(S) \subset I(h)$ for every $h \in S$ since each $I(h)$ is completely invariant, their intersection $\bigcap_{h\in S} I(h)$ is also completely invariant. Define

$$E_0 = \bigcap_{h\in S} I(h)$$
$$E_1 = \bigcup_{h\in S} h^{-1}(E_0) \cup \bigcup_{h\in S} h(E_0)$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$E_{n+1} = \bigcup_{h\in S} h^{-1}(E_n) \cup \bigcup_{h\in S} h(E_n)$$

and

$$E = \bigcap_{n\in\mathbb{N}\cup\{0\}} E_n \tag{3.1}$$

Theorem 3.4. *The set $E = \bigcap_{n\in\mathbb{N}\cup\{0\}} E_n$ is non-empty.*

Proof. We show that $I_1(S) \subset E_n$ for every $n \in \mathbb{N}\cup\{0\}$ by induction. That $I_1(S) \subset E_0$ is obvious. By the completely invariant property of $I_1(S)$ under each $h \in S$, $I_1(S)$ is subset of each set $h^{-1}(E_0)$ and $h(E_0)$ for all $h \in S$. This shows $I_1(S) \subset E_1$. Let us suppose $I_1(S) \subset E_n$. By construction, $E_{n+1} = h^{-1}(E_n) \cup h(E_n)$ for all $h \in S$. By the similar argument, we can show that $I_1(S)$ is subset of each of the sets $h^{-1}(E_n)$ and $h(E_n)$ for all $h \in S$. This shows that $I_1(S) \subset E_{n+1}$ for each $n \in \mathbb{N}\cup\{0\}$. This proves that $E \neq \emptyset$. ∎

Theorem 1.10 can give a convenient description of the completely invariant escaping set of a transcendental semigroup. First, we prove the following lemma.

Lemma 3.5. *Let f be a transcendental entire function and let $E \subset \mathbb{C}$. The closure $\overline{E}$ of E is completely invariant under f if and only if the set E itself is completely invariant under f.*

Proof. Let $\overline{E}$ be completely invariant under f. Then $f(\overline{E}) \subset \overline{E}$ and $f^{-1}(\overline{E}) \subset \overline{E}$. Let $z \in \overline{E}$. Then $f(z) \in f(\overline{E})$ and so $f(z) \in \overline{E}$. Also, $z \in \overline{E}$ implies that there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in E such that $z_n \to z$ as $n \to \infty$. From the continuity of the function f we can write $f(z_n) \to f(z)$ as $n \to \infty$. As $f(z) \in \overline{E}$, we must have $f(z_n) \in E$. Note that $f(z_n) \in f(E)$ as $z_n \in E$. Thus we must have $f(E) \subset E$.

Next, let $z \in \overline{E}$, then $f^{-1}(z) \in f^{-1}(\overline{E}) \subset \overline{E}$. So there exists $f^{-1}(z_n) \in E$ such that $f^{-1}(z_n) \to f^{-1}(z)$ as $n \to \infty$. However, it is obvious that $f^{-1}(z_n) \in f^{-1}(E)$. Thus we must have $f^{-1}(E) \subset E$.

The converse part of this lemma follows from [12, Theorem 3.2.3]. ∎

Note that under the assumption of Lemma 3.5, not only is the closure of completely invariant set completely invariant, but also its complement, interior and boundary are completely invariant (see for instance [12, Theorem 3.2.3]).

Proof of Theorem 1.10. By Definition 1.6, $I_1(S)$ is completely invariant under each $h \in S$ and is contained in $I(h)$ for each $h \in S$. Hence, by Theorem 3.4, $I_1(S)$ is contained in E_n for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $I_1(S) \subset E$. On the other hand, E is contained in $I(h)$ where each of $I(h)$ is completely invariant. We need to show that E is completely invariant under each $f \in S$, and for every $z \in E$, $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$. $f \in S$ is continuous in $\mathbb{C}$ and $E \subset \mathbb{C}$. So by the usual topological argument, $f(\overline{E}) \subset \overline{f(E)} \Rightarrow f^{-1}(\overline{f(E)})$ is closed in $\mathbb{C}$ for every $f \in S \Rightarrow f$ is a continuous closed map. This shows that $f(\overline{E})$ and $f^{-1}(\overline{E})$ are both closed sets in $\mathbb{C}$. Each $f \in S$ is a continuous closed map and $f(\overline{E_n}) \subset \overline{E_n}$ and $f^{-1}(\overline{E_n}) \subset \overline{E_n}$ for all n. This shows that $f(\overline{E}) \subset \overline{E}$ and $f^{-1}\overline{(E)} \subset \overline{E}$. By Lemma 3.5, it proves that E is completely invariant under each $f \in S$.

Finally, every $z \in E$ belongs to E_n for all n. Again, E_n is a union of all images and pre-images of E_{n-1} under the each map $f \in S$. In this way, the point z belongs to the image or pre-image of E_0 under each map $f \in S$. Since E_0 is contained in $I(f)$ for all $f \in S$, so this point z goes to infinity under the iteration of any function in S. Hence $E \subset I_1(S)$. ∎

Note that $I_1(S) = E$ by Theorem 1.10, and $E \subset I(S)$ by construction of the set E (that is, E is completely invariant and $I(S)$ is not) in (3.1). This fact is equivalent to $I_1(S) \subset I(S)$.

We also can construct $I(S)$ by similar fashion as in $I_1(S)$. Note that $I(S) \subset \bigcap_{h \in S} I(h)$. Suppose $I(S) \neq \emptyset$. Define

$$F_0 = \bigcap_{h \in S} I(h),$$
$$F_1 = \bigcup_{h \in S} (h(F_0))$$
$$\cdots \quad \cdots \quad \cdots$$
$$F_{n+1} = \bigcup_{h \in S} (h(F_n))$$

and

$$F = \bigcap_{n \in \mathbb{N}} F_n \tag{3.2}$$

We can show that $F \neq \emptyset$ by a process similar to the one used in the proof of Theorem 3.4. The fundamental difference between this set F and the set E of (3.1) is that F is constructed from only the forward invariant property under each $h \in S$., whereas the set E was constructed from the completely invariant property under each $h \in S$. The following theorem provides an alternative definition of the escaping set of transcendental semigroup.

Theorem 3.6. *Let S be a transcendental semigroup. Then $F = I(S)$, where $F = \bigcap_{n \in \mathbb{N}} F_n$.*

Proof. Since $I(S)$ is forward invariant under each $h \in S$ and is contained in each $I(h)$, it is contained in F. On the other hand, the set F is contained in $I(S)$ by the construction above. ∎

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