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Inertial Proximal Gradient Method Using Adaptive Stepsize for Convex Minimization Problems

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Abstract This work aims to propose an inertial proximal gradient method using the adaptive stepsize to solve unconstrained minimization problems. We prove that our algorithm weakly converges to a solution of the problems. Finally, we give numerical experiments on image restoration problem. It shows that the proposed algorithms outrun known algorithms introduced by many authors.

MSC: 65K05; 90C25; 90C30
Keywords: inertial technique; proximal gradient method; self-adaptive algorithm; weak convergence

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1. Introduction and Preliminaries

In various fields of applied sciences, economics and engineering, such as signal recovery, image restoration, and machine learning, can be formulated as the unconstrained minimization problem, which is described as follows:

$$\min_{x\in H}(f(x)+g(x)), \tag{1.1}$$

where H is a real Hilbert space and $f, g : H \to \mathbb{R}\cup\{+\infty\}$ are proper, lower semi-continuous and convex functions that f is differentiable on H. The proximal operator $\text{prox}_g : H \to \text{dom}g$ is defined by $\text{prox}_g(z) = (Id + \partial g)^{-1}(z), z \in H$ where Id denotes the identity operator on H. It is well-known that the proximal operator is single-valued with full domain. It is also known that for all $z \in H$

$$\frac{z-\text{prox}_{\alpha g}(z)}{\alpha} \in \partial g(\text{prox}_{\alpha g}(z)) \tag{1.2}$$

where $\alpha > 0$.

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The minimization problem (1.1) which is equivalent to the following fixed point equation:

$$x = \text{prox}_{\alpha g}(x - \alpha\nabla f(x)),$$

where $\alpha > 0$, ∇f denotes the gradient of f and $\text{prox}_{\alpha g}$ stands for the proximal operator of g. For solving (1.1), we can construct a simple iteration: let $x^0 \in H$ and

$$x^{n+1} = \text{prox}_{\alpha g}(x^n - \alpha\nabla f(x^n)), \tag{1.3}$$

where $\alpha > 0$. By this point of view, we know that (1.3) is called a classical forward-backward algorithm. As a consequence, it has been studied by many authors (see [1–8]). The set $\text{argmin}(f + g)$ is the solution set of (1.1).

To accelerate the convergence of sequence, Moudafi and Oliny [9] introduced the inertial proximal gradient method. In 2009, Beck and Teboulle [10] introduce a fast iterative shrinkage-thresholding algorithm (FISTA) as follows:

Algorithm 1.1. Let $t_1 = 1$ and $x^0 = x^1 \in H$. Calculate

$$y^n \;=\; x^n + \theta_n(x^n - x^{n-1})$$

where $\theta_n = \dfrac{t_n - 1}{t_{n+1}}$ and $t_{n+1} = \dfrac{1 + \sqrt{1 + 4t_n^2}}{2}$. Next, calculate

$$x^{n+1} \;=\; \text{prox}_{\frac{1}{L}g}(y^n - \frac{1}{L}\nabla f(x^n)).$$

where L is the Lipschitz constant of ∇f.

Recently, Verma and Shukla [11] proposed a new accelerated gradient algorithm (NAGA). It is defined by:

Algorithm 1.2. Let $\theta_n \in [0, 1]$ and $x^0 = x^1 \in H$. Calculate

$$\begin{aligned} y^n &= x^n + \theta_n(x^n - x^{n-1}) \\ x^{n+1} &= T_n[(1 - \alpha_n)y^n + \alpha_n T_n y^n] \end{aligned}$$

where $T_n = \text{prox}_{\alpha_n g}(Id - \alpha_n\nabla f)$ and $\alpha_n \in (0, 2/L)$.

In 2016, Cruz and Nghia [1] proposed a fast forward-backward method (FFB) based on the linesearch method for solving (1.1). The main advantage is that the Lipschitz condition on the gradient of functions is dropped in computing. It is defined as follows:

Algorithm 1.3. Given $\sigma, \theta \in (0, 1)$ and $\delta \in (0, \frac{1}{2})$. Let $\Omega := \text{dom}g$ is closed, $x^0 = x^1 \in \text{dom}g$ and $t_1 = 1$. Calculate

$$y^n \;=\; P_\Omega\left(x^n + \left(\frac{t_n - 1}{t_{n+1}}\right)\right)$$

where $t_{n+1} = \dfrac{1 + \sqrt{1 + 4t_n^2}}{2}$ and calculate the next step via

$$x^{n+1} = \text{prox}_{\alpha_n g}(x^n - \alpha_n\nabla f(x^n))$$

where the stepsize α_n generated by Linesearch 1 in [1].
Then (x^n) converges weakly to a minimizer of (1.1).

This work introduces inertial proximal gradient methods using the adaptive stepsize for convex minimization problems. Our algorithms do not require the Lipschitz condition of the gradient. Under some conditions, we derive a weak convergent method for unconstrained minimization problem (1.1). Our content is organized as follows: In Section 2, we construct our main theorems. In Section 3, we provide numerical experiments in image restoration. Finally, we give the conclusion of this work in Section 4.

2. Main Results

Now, we propose a new inertial forward-backward algorithm and prove the weak convergence. Assume that $f : H \to \mathbb{R} \cup \{+\infty\}$ and $g : H \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semi-continuous that ∇f is L-Lipschitz continuous on H.

Algorithm 2.1. Given $\delta \in (0, 1)$, $\alpha_1 \geq 0$ and $\theta_n \geq 0$. Let $x^0 = x^1 \in H$ be arbitrary and calculate:

$$z^n = x^n + \theta_n(x^n - x^{n-1}) \tag{2.1}$$

and

$$x^{n+1} = \mathrm{prox}_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n)) \tag{2.2}$$

where

$$\alpha_{n+1} = \begin{cases} \min\{\frac{\delta\|z^n - x^{n+1}\|}{\|\nabla f(z^n) - \nabla f(x^{n+1})\|}, \alpha_n\} & \text{if } \nabla f(z^n) - \nabla f(x^{n+1}) \neq 0 \\ \alpha_n & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.2. *Let (x^n) be defined by Algorithm 2.1. Assume that $\theta_n \geq 0$ and $\sum_{n=1}^{\infty} \theta_n < +\infty$. Then, we have*

(1) for each $x_* \in \mathrm{argmin}(f + g)$, $\|x^{n+1} - x_*\| \leq K \cdot \prod_{j=1}^{n}(1 + 2\theta_j)$ where $K = \max\{\|x^1 - x_*\|, \|x^2 - x_*\|\}$.
(2) The sequence (x^n) weakly converges to a point in $\mathrm{argmin}(f + g)$.

Proof. Using (1.2) and (2.2), we see that

$$\frac{z^n - x^{n+1}}{\alpha_n} - \nabla f(z^n) = \frac{z^n - \mathrm{prox}_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))}{\alpha_n} - \nabla f(z^n) \in \partial g(x^{n+1}).$$

From the convexity of g, we have

$$g(x) - g(x^{n+1}) \geq \left\langle \frac{z^n - x^{n+1}}{\alpha_n} - \nabla f(z^n), x - x^{n+1} \right\rangle \tag{2.4}$$

for all $x \in H$. Also the convexity of f gives

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle \tag{2.5}$$

for all $x, y \in H$. Combining (2.4) and (2.5) with $y = z^n$, we obtain

$$\begin{aligned} g(x) - g(x^{n+1}) + f(x) - f(z^k) &\geq \langle \frac{z^n - x^{n+1}}{\alpha_n} - \nabla f(z^n), x - x^{n+1}\rangle + \langle \nabla f(z^n), x - z^n\rangle \\ &= \frac{1}{\alpha_n}\langle z^n - x^{n+1}, x - x^{n+1}\rangle + \langle \nabla f(z^n), x^{n+1} - z^n\rangle \\ &= \frac{1}{\alpha_n}\langle z^n - x^{n+1}, x - x^{n+1}\rangle + \langle \nabla f(x^{n+1}), x^{n+1} - z^n\rangle \\ &\quad + \langle \nabla f(z^n) - \nabla f(x^{n+1}), x^{n+1} - z^n\rangle \\ &\geq \frac{1}{\alpha_n}\langle z^n - x^{n+1}, x - x^{n+1}\rangle + \langle \nabla f(x^{n+1}), x^{n+1} - z^n\rangle \\ &\quad - \|\nabla f(z^n) - \nabla f(x^{n+1})\|\|x^{n+1} - z^n\| \end{aligned} \tag{2.6}$$

By definition of (α_n), we have

$$\|\nabla f(z^n) - \nabla f(x^{n+1})\| \leq \frac{\delta}{\alpha_{n+1}}\|z^n - x^{n+1}\| \tag{2.7}$$

Indeed, if $\nabla f(z^n) = \nabla f(x^{n+1})$, then the inequality (2.7) hold. Otherwise, from (2.3), we have

$$\alpha_{n+1} = \min\{\frac{\delta\|z^n - x^{n+1}\|}{\|\nabla f(z^n) - \nabla f(x^{n+1})\|}, \alpha_n\} \leq \frac{\delta\|z^n - x^{n+1}\|}{\|\nabla f(z^n) - \nabla f(x^{n+1})\|}.$$

This implies that

$$\|\nabla f(z^n) - \nabla f(x^{n+1})\| \leq \frac{\delta}{\alpha_{n+1}}\|z^n - x^{n+1}\|.$$

Therefore, the inequality follows from (2.6) and (2.7). It then follows that

$$\begin{aligned} \langle z^n - x^{n+1}, x^{n+1} - x\rangle &\geq \alpha_n[f(z^n) + g(x^{n+1}) - (f+g)(x) + \langle \nabla f(x^{n+1}), x^{n+1} - z^n\rangle] \\ &\quad - \frac{\delta\alpha_n}{\alpha_{n+1}}\|z^n - x^{n+1}\|^2. \end{aligned}$$

Using $2\langle z^n - x^{n+1}, x^{n+1} - x\rangle = \|z^n - x\|^2 - \|z^n - x^{n+1}\|^2 - \|x^{n+1} - x\|^2$, we get

$$\begin{aligned} \|x^{n+1} - x\|^2 &\leq \|z^n - x\|^2 - 2\alpha_n[(f+g)(x^{n+1}) - (f+g)(x)] \\ &\quad - (1 - \frac{2\delta\alpha_n}{\alpha_{n+1}})\|z^n - x^{n+1}\|^2. \end{aligned} \tag{2.8}$$

Now, we let $x_* \in \operatorname{argmin}(f+g)$ and we will show that (x^n) is bounded. By (2.1) and (2.8), we see that

$$\begin{aligned} \|x^{n+1} - x_*\| &\leq \|z^n - x_*\| \\ &= \|x^n + \theta_n(x^n - x^{n-1}) - x_*\| \\ &\leq \|x^n - x_*\| + \theta_n(\|x^n - x_*\| + \|x^{n-1} - x_*\|). \end{aligned}$$

It implies that

$$\|x^{n+1} - x_*\| \leq (1+\theta_n)\|x^n - x_*\| + \theta_n\|x^{n-1} - x_*\|.$$

By Lemma 5 in [12], we have

$$\|x^{n+1} - x_*\| \leq K \cdot \prod_{j=1}^{n}(1 + 2\theta_j)$$

where $K = \max\{\|x^1 - x_*\|, \|x^2 - x_*\|\}$. Since $\sum_{n=1}^{\infty} \theta_n < +\infty$, we have (x^n) is bounded. From (2.1) and (2.8), we see that

$$\begin{aligned}
\|x^{n+1} - x_*\|^2 &\leq \|x^n + \theta_n(x^n - x^{n-1}) - x_*\|^2 - 2\alpha_n[(f+g)(x^{n+1}) - (f+g)(x_*)] \\
&\quad -(1 - \frac{2\delta\alpha_n}{\alpha_{n+1}})\|z^n - x^{n+1}\|^2 \\
&\leq \|x^n - x_*\|^2 + 2\theta_n\|x^n - x_*\|\|x^n - x^{n-1}\| + \theta_n^2\|x^n - x^{n-1}\|^2 \\
&\quad -2\alpha_n[(f+g)(x^{n+1}) - (f+g)(x_*)] - (1 - \frac{2\delta\alpha_n}{\alpha_{n+1}})\|z^n - x^{n+1}\|^2.
\end{aligned} \tag{2.9}$$

From Remark 3.1 in [13], we know that α_n is bounded from below by $\min\left\{\alpha_1, \frac{\delta}{L}\right\}$ and $\lim_{n\to\infty} \alpha_n = \alpha > 0$. By Lemma 1 in [14] and (2.9) we obtain that $\lim_{n\to\infty} \|x^n - x_*\|^2$ exists. Since $\lim_{n\to\infty}(1 - \frac{\delta\alpha_n}{\alpha_{n+1}}) = 1 - \delta > 0$, we have

$$\lim_{n\to\infty} \|z^n - x^{n+1}\| = 0.$$

From definition of z^n, it is easily seen that $\lim_{n\to\infty} \|z^n - x^n\| = 0$ and implies that $\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0$. By the boundedness of (x^n), we know that the set of its weak accumulation points is nonempty. Let x^∞ be a weak accumulation point of (x^n). So there is a subsequence (x^{n_i}) of (x^n) such that (x^n) converges weakly to x^∞. Next, we show that $x^\infty \in \operatorname{argmin}(f+g)$. Let $(v,u) \in \operatorname{Graph}(\nabla(f) + \partial(g))$, that is $u - \nabla f(v) \in \partial g(v)$. Since $x^{n_i+1} = \operatorname{prox}_{\alpha_{n_i} g}(I - \alpha_{n_i}\nabla f)z^{n_i}$, we obtain

$$(I - \alpha_{n_i}\nabla f)z^{n_i} \in (I + \alpha_{n_i}\partial g)x^{n_i+1},$$

which yields

$$\frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1} - \alpha_{n_i}\nabla f(z^{n_i})) \in \partial g(x^{n_i+1}).$$

Since ∂g is maximal monotone, we have

$$\langle v - x^{n_i+1}, u - \nabla f(v) - \frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1} - \alpha_{n_i}\nabla f(z^{n_i}))\rangle \geq 0.$$

This shows that

$$\begin{aligned}
\langle v - x^{n_i+1}, u\rangle &\geq \langle v - x^{n_i+1}, \nabla f(v) + \frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1} - \alpha_{n_i}\nabla f(z^{n_i}))\rangle \\
&= \langle v - x^{n_i+1}, \nabla f(v) - \nabla f(x^{n_i})\rangle + \langle v - x^{n_i+1}, \frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1})\rangle \\
&= \langle v - x^{n_i+1}, \nabla f(v) - \nabla f(z^{n_i+1})\rangle + \langle v - x^{n_i+1}, \nabla f(x^{n_i+1}) - \nabla f(z^{n_i})\rangle \\
&\quad +\langle v - x^{n_i+1}, \frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1})\rangle \\
&\geq \langle v - x^{n_i+1}, \nabla f(x^{n_i+1}) - \nabla f(z^{n_i})\rangle + \langle v - x^{n_i+1}, \frac{1}{\alpha_{n_i}}(z^{n_i} - x^{n_i+1})\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty}\|z^n - x^{n+1}\| = 0$ and by assumptions, we have $\lim_{n\to\infty}\|\nabla f(z^n) - \nabla f(x^{n+1})\| = 0$. Hence we obtain

$$\langle v - x^\infty, u\rangle = \lim_{i\to\infty}\langle v - x^{n_i+1}, u\rangle \geq 0.$$

Hence, $0 \in (\nabla f + \partial g)x^\infty$, and consequently $x^\infty \in \operatorname{argmin}(f+g)$. This gives that (x^n) converges weakly to a point in $\operatorname{argmin}(f+g)$ by applying Theorem 4.1 in [18]. We thus complete the proof. ∎

3. Numerical Experiments

In this section, we provide the numerical examples and compare the proposed algorithm with some existing algorithms in the literature. All computational experiment were written in Matlab 2020b and preformed on a 64-bit MacBook Pro Chip Apple M1 and 8 GB of RAM.

In this experiments, firstly, we apply Algorithm 2.1 to solve image restoration problem, we consider the following linear equation system:

$$b = Aa + \varepsilon, \tag{3.1}$$

where $a \in \mathbb{R}^{N\times 1}$ is the original image, $b \in \mathbb{R}^{N\times 1}$ is the observed image, $\varepsilon \in \mathbb{R}^{N\times 1}$ is the additive noise and $A \in \mathbb{R}^{N\times N}$ is the blurring operation. It is known that to solve (3.1) can be seen as solving the LASSO problem:

$$\min_{a\in\mathbb{R}^N} \frac{1}{2}\|b - Aa\|_2^2 + \lambda\|a\|_1,$$

where $\lambda > 0$. To measure the quality of restored images, we use the peak signal-to-noise ratio (PSNR) and the structural similarity index measure (SSIM) [19], which is defined by

$$\text{PSNR} = 20\log\left(\frac{255^2}{\|a_r - a\|_2^2}\right)$$

and

$$\text{SSIM} = \frac{(2u_a u_{a_r} + c_1)(2\sigma_{aa_r} + c_2)}{(u_a^2 + u_{a_r}^2 + c_1)(\sigma_a^2 + \sigma_{a_r}^2 + c_2)}$$

where a is the original image, a_r is the restored image, u_a and u_{a_r} are the mean values of the original image a and restored image a_r, respectively, σ_a^2 and $\sigma_{a_r}^2$ are the variances, $\sigma_{aa_r}^2$ is the covariance of two images, $c_1 = (0.01L)^2$ and $c_2 = (0.03L)^2$ and L is the dynamic range of pixel values. SSIM ranges from 0 to 1, where 1 means flawless recovery.

Next, we consider the original images that are corrupted by the following blur types:

(BM1) Gaussian blur of the filter size 5×5 with standard deviation $\sigma = 5$.

(BM2) Out of focus blur (disk) with radius $r = 7$.

(BM3) Motion blur specified with the motion length of 45 pixels and motion orientation $\theta = 45$.

The original image and three different types of original image degraded by the blurring matrices are shown in Figure 1.

FIGURE 1. The original image (size 448×332) and the blurred image by the blurred matrices (BM1), (BM2) and (BM3), respectively.

All parameters are chosen as in Table 1. The initial point a^0 and a^1 are zero vectors with the stopping criterion 900^{th} Iter. The inertial parameter θ_n of FISTA, FFA and NAGA is defined as in Algorithm 1.1. For IFBAS, we set θ_n as follows:

$$\theta_n = \begin{cases} \dfrac{1}{n^2}, & \text{if } n < 50 \\ \dfrac{t_n - 1}{t_{n+1}}, & \text{if } n \geq 50 \end{cases}$$

where $t_{n+1} = \dfrac{(1/10) + \sqrt{(1/50) + 4t_n^2}}{2}$.

The results of deblurred image for each algorithm are shown in Table 2 and

TABLE 1. Chosen parameters of each algorithm.

Algorithms	Parameters				
	$\gamma = 1/\|A\|$	$\theta = 0.2$	$\delta = 0.4$	$\sigma = 0.2$	$t_1 = 1$
FISTA	√	-	-	-	√
FFA	-	√	√	√	√
NAGA	√	-	-	-	√
IFBAS	-	-	√	-	√

TABLE 2. The results of deblurred image for each algorithm.

Blurred matrices	Measurement	Algorithms			
		FISTA	FFB	NAGA	IFBAS
(BM1)	PSNR	38.7126	39.4788	39.2057	40.1099
	SSIM	0.9734	0.9774	0.9760	0.9803
(BM2)	PSNR	33.8526	34.4556	34.2232	34.8416
	SSIM	0.9256	0.9324	0.9296	0.9349
(BM3)	PSNR	32.2596	33.9459	32.9459	34.2778
	SSIM	0.9102	0.9187	0.9169	0.9271

Figure 2. The restored images by (MB1) for FISTA (PSNR:38.7126, SSIM:0.9734), FFB (PSNR:39.4788, SSIM:0.9774), NAGA (PSNR:39.2057 SSIM:0.9760) and IFBAS (PSNR:40.1099, SSIM:0.9803), respectively.

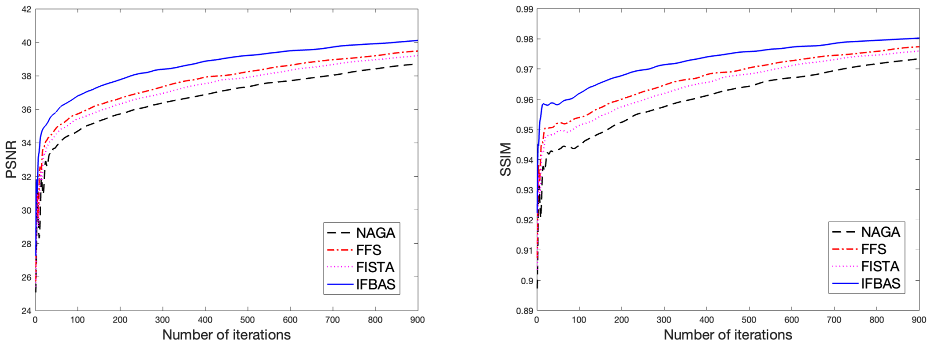

Figure 3. Graph of PSNR and SSIM of Figure 2.

Figure 4. The restored images by (MB2) for FISTA (PSNR:33.8526, SSIM:0.9256), FFB (PSNR:34.4556, SSIM:0.9324), NAGA (PSNR:34.2232 SSIM:0.9296) and IFBAS (PSNR:34.8416, SSIM:0.9349), respectively.

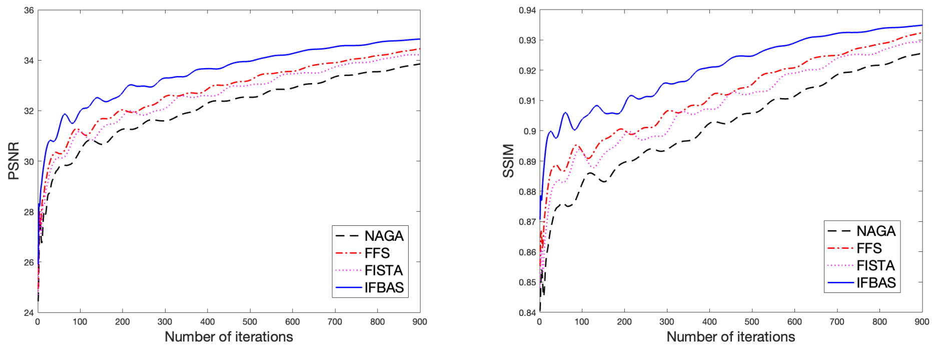

FIGURE 5. Graph of PSNR and SSIM of Figure 4.

FIGURE 6. The restored images by (MB3) for FISTA (PSNR:32.2596, SSIM:0.9102), FFB (PSNR:33.2292, SSIM:0.9187), NAGA (PSNR:32.9459 SSIM:0.9169) and IFBAS (PSNR:34.2778, SSIM:0.9271), respectively.

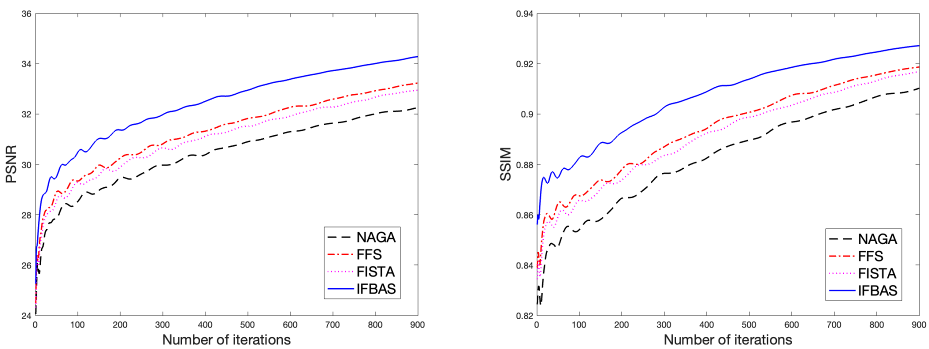

FIGURE 7. Graph of PSNR and SSIM of Figure 6.

From Tables 2 and Figures 3, 5 and 7, we see that our proposed method has a better convergence behavior than FISTA, FFB and NAGA in terms of PSNR and SSIM.

4. Conclusion

In this paper, we proposed a modified proximal gradient method with adaptive stepsize to solve minimization problems in real Hilbert spaces. We obtained the weak convergence theorems under some conditions that do not depend on the Lipschitz condition on the gradient. We gave numerical results to image deblurring by using our algorithms. It was shown that the proposed proposed methods outperform other methods in comparison.

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