Thai Journal of Mathematics
Volume 21 Number 2 (2023)
Pages 265–276

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Fixed Point Theorems for L-Fuzzy Monotone Multifunctions

Abdelghani Derardja[1], Lemnaouar Zedam[1,*] and Soheyb Milles[2]

[1] *Laboratory of Pure and Applied Mathematics, Department of Mathematics, University of M'sila, Algeria*
e-mail : abdelghani.derardja@univ-msila.dz (A. Derardja); lemnaouar.zedam@univ-msila.dz (L. Zedam)

[2] *Departement of Mathematics, Institute of Science, University Center of Barika, Algeria*
e-mail : soheyb.milles@cu-barika.dz (S. Milles)

Abstract In this paper, we introduce three types of monotonicity of an L-fuzzy multifunction on a given L-fuzzy complete lattice and investigate their various properties. Furthermore, we show that any L-fuzzy multifunction with respect to these types of monotonicity has the fixed point property. Specific attention is paid to show for one type of these monotonicities that the set of fixed points of an L-fuzzy monotone multifunction is an L-fuzzy complete lattice.

MSC: 03E72; 06B05; 37C25

Keywords: fuzzy set; fuzzy multifunction; fixed point property

Submission date: 09.11.2021 / Acceptance date: 09.04.2023

1. Introduction

It is well known that fixed point theory is among the oldest acquaintances of modern mathematics. This theory plays a prominent role in many mathematical branches (both pure and applied), and it has three approaches in literature. The first one is the metric approach which makes use of the metric properties of the underlying spaces and self-maps (e.g., Banach's contraction mapping theorem [2]). The second approach is the topological one in which one utilizes the topological properties of the underlying spaces and continuity of self-mappings (e.g., Brouwer's fixed point theorem [7]). The third approach is the order-theoretic one for the category of complete lattice, complete Heyting algebras, pseudo-ordered sets (e.g., Abian and Brown fixed point theorem [1], Tarski and Davis fixed point theorems [11, 20], Skala fixed point theorem [18], ... etc).

In fuzzy setting, several authors have discussed the same approaches, they have investigated some fixed point theorems in different ways and on different structures. Shen et al. [17] established several fixed point theorems for a new class of self-maps in M-complete fuzzy metric spaces and compact fuzzy metric spaces, respectively. Lu [16] studied some fixed point theorems for fuzzy mappings in general topological spaces. Recently, Včelař and Patikova [21] have presented a fuzzification of Tarski's fixed point theorem without

*Corresponding author.

Published by The Mathematical Association of Thailand.

the assumption of transitivity. For the fuzzy monotone multifunctions Stouti [19] proved under suitable conditions the existence of fixed points by using iteration method in the unit interval [0.1].

Motivated by recent developments relating to this framework, in this paper, we introduce and study three types of monotonicity of an L-fuzzy multifunction on a given L-fuzzy complete lattice and investigate their various properties. Furthermore, we show that any L-fuzzy multifunction with respect to these types of monotonicity has the fixed point property. Specific attention is paid to show for one type of these monotonicities that the set of fixed points of an L-fuzzy monotone multifunction is an L-fuzzy complete lattice.

The contents of the paper are organized as follows. In Section 2, we recall the necessary basic concepts and properties of residuated lattices, fuzzy relations, and fuzzy complete lattices. In Section 3, we introduce the notion of L-fuzzy monotone multifunction on a given L-fuzzy complete lattice and investigate its various properties. In Section 4, we extend some fixed point theorems for L-fuzzy monotone multifunctions. Also, we show that the set of fixed points of an L-fuzzy monotone multifunction is an L-fuzzy complete lattice. Finally, we present some concluding remarks in Section 5.

2. Basic Concepts

This section serves an introductory purpose. First, we recall some basic definitions and properties of residuated lattice. Second, we recall some notions and results of L-fuzzy relations, as well as L-fuzzy complete lattices.

2.1. Residuated Lattices

A poset $(L, \leq)$ (see, e.g., [10]) is called a lattice if any two elements x and y have a smallest upper bound, denoted $x \vee y$ and called the join (or the supremum) of x and y, as well as a greatest lower bound, denoted $x \wedge y$ and called the meet (or the infimum) of x and y. A lattice can also be defined as an algebraic structure namely a set L equipped with two binary operations $\vee$ and $\wedge$ that are idempotent, commutative and associative, and satisfy the absorption laws ($x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$, for any $x, y \in L$). The order relation, the meet and the join operations are related as follows: $x \leq y$ if and only if $x \vee y = y$; $x \leq y$ if and only if $x \wedge y = x$.
A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$, for any x in L. Often, the notation $(L, \leq, \vee, \wedge, 0, 1)$ is used. A complete lattice is a poset in which every subset has a supremum or join (the least upper bound) and an infimum or meet (the greatest lower bound).

Definition 2.1. [12] A triangular norm (t-norm, for short) $*$ on a bounded lattice L is a binary operation on L that is commutative (i.e., $\alpha * \beta = \beta * \alpha$, for any $\alpha, \beta \in L$), associative (i.e., $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$, for any $\alpha, \beta, \gamma \in L$), has neutral element 1 (i.e., $\alpha * 1 = \alpha$, for any $\alpha \in L$) and is order-preserving (i.e., if $\alpha \leq \beta$, then $\alpha * \gamma \leq \beta * \gamma$, for any $\alpha, \beta, \gamma \in L$).

Example 2.2. The following four operations are the most common t-norms on $L = [0, 1]$:

(1) Minimum: $x * y = min\{x, y\}$.
(2) Product: $x * y = x.y$.
(3) Lukasiewicz: $x * y = max\{x + y - 1, 0\}$.

(4) Drastic product: $x * y = \begin{cases} x \ if \ y = 1\,, \\ y \ if \ x = 1\,, \\ 0 \ if \ x, y < 1. \end{cases}$

Definition 2.3. [3] A residuated lattice is an algebra $(L, \vee, \wedge, *, \rightarrow, 0, 1)$ such that

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(ii) $(*, \rightarrow)$ forms an adjoint couple on L, i.e., for any $a, b, c \in L$:
 (R1) If $a \leq b$ and $c \leq d$, then $a * c \leq b * d$;
 (R2) If $b \leq c$, then $a \rightarrow b \leq a \rightarrow c$;
 (R3) If $a \leq b$, then $b \rightarrow c \leq a \rightarrow c$;
 (R4) $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$ (adjointness condition);
(iii) $(L, *, 1)$ forms a commutative monoïd, i.e., for any $a, b, c \in L$:
 (R5) $(a * b) * c = a * (b * c)$;
 (R6) $a * b = b * a$;
 (R7) $1 * a = a$.

A residuated lattice L is called complete if $(L, \vee, \wedge, 0, 1)$ is a complete lattice. The $*$ and $\rightarrow$ will be called multiplication and residuum, respectively. Multiplication is isotone while residuum is isotone in the first argument and antitone in the second argument (w.r.t. lattice order $\leq$).

Remark 2.4. When the operator $*$ is exactly the min $(\wedge)$ operation of the residuated lattice, such structure is called a Heyting algebra. A complete Heyting algebra is a special case of a complete residuated lattice which is also called a frame.

The following properties of a complete residuated lattices will be used in the paper (see, e.g., Bělohlávek [3, 4], Ćirić [8] and Hájek [15]). For any $a, b, c \in L$, it holds that:

(1) $a \rightarrow b = \bigvee\{c \in L \mid a * c \leq b\}\,,$
(2) $1 \rightarrow a = a,\ a \leq b \Leftrightarrow a \rightarrow b = 1\,,$
(3) $b \leq c$ implies $\begin{cases} a \wedge b \leq a \wedge c\,, \\ a \rightarrow b \leq a \rightarrow c\,, \\ b \rightarrow a \geq c \rightarrow a\,, \end{cases}$
(4) $a \wedge (a \rightarrow b) \leq b,\ \ a \leq (a \rightarrow b) \rightarrow b,\ \ (a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c,$
(5) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c,\ \ (a \rightarrow b) \wedge (c \rightarrow d) \leq a \rightarrow (c \rightarrow (b \wedge d)),$
(6) $a \rightarrow (\bigwedge B) = \bigwedge\{a \rightarrow b \mid b \in B\},\ a \rightarrow (\bigvee B) \geq \bigvee\{a \rightarrow b \mid b \in B\},$
(7) $(\bigvee B) \rightarrow a = \bigwedge\{b \rightarrow a \mid b \in B\},\ (\bigwedge B) \rightarrow a \geq \bigvee\{b \rightarrow a \mid b \in B\}.$

Throughout this paper, $(L, \vee, \wedge, 0, 1)$ is a complete residuated lattice.

2.2. L-Fuzzy Relations

Goguen [14] introduced the notion of L-fuzzy set as a generalization of Zadeh's-fuzzy set with L being a bounded lattice. An L-fuzzy set A on X is a mapping $A : X \longrightarrow L$. The set of all L-fuzzy sets on X is denoted by L^X.
A binary L-fuzzy relation (L-relation, for short) R on X is an L-fuzzy set on X^2, i.e., is a mapping $R : X \times X \longrightarrow L$. If $L = \{0, 1\}$, crisp relations are obtained. For a crisp relation, we use the usual infix notation, xRy, for any $x, y \in X$. An L-relation R_1 is said to be included in an L-relation R_2, denoted $R_1 \subseteq R_2$, if $R_1(x, y) \leq R_2(x, y)$, for any $x, y \in X$. The intersection of two L-relations R_1 and R_2 on X is the L-relation $R_1 \cap R_2$ on X defined by $R_1 \cap R_2(x, y) = R_1(x, y) \wedge R_2(x, y)$, for any $x, y \in X$. Similarly, the

union of two L-relations R_1 and R_2 on X is the L-relation $R_1 \cup R_2$ on X defined by $R_1 \cup R_2(x,y) = R_1(x,y) \vee R_2(x,y)$, for any $x, y \in X$. The transpose R^t of R is defined by $R^t(x,y) = R(y,x)$, for any $x, y \in X$.

Next, we need the fommowin definition of fuzzy order on a nonempty set. For more details on fuzzy order relations, we refer to [4–6, 13, 26].

Definition 2.5. Let X be a nonempty set. An L-fuzzy relation $R : X \times X \longrightarrow L$ is called an L-fuzzy order on X if the following statements hold:

(E1) Reflexivity, $\forall x \in X,\ R(x,x) = 1$,
(E2) Antisymmetry, $\forall x, y \in X,\ R(x,y) = R(y,x) = 1$ implies $x = y$,
(E3) Transitivity, $\forall x, y, z \in X,\ R(x,y) \wedge R(y,z) \leq R(x,z)$.

The pair (X, R) is called a L-fuzzy partially ordered set (an L-fuzzy poset, for short).

Example 2.6. (1) On a residuated lattice L, the L-fuzzy relation $R_L : L \times L \longrightarrow L$ defined by $R_L(x,y) = x \rightarrow y$ is an L-fuzzy ordered relation on L.

(2) Let $X = \{a, b, c, d\}$ and $L = \{0, 0.5, 1\}$. We define $R : X \times X \longrightarrow L$ as follows:

$R(.,.)$	a	b	c	d
a	1	0	0	0
b	1	1	0.5	0
c	1	0.5	1	0
d	1	1	1	1

Then (X, R) is an L-fuzzy poset on X.

2.3. L-Fuzzy Complete Lattices

In this subsection, we discuss the notion of L-fuzzy complete lattice. First, we recall the notion of supremum and infimum of an L-fuzzy set.

Definition 2.7 (Zhang and Fan [24, 25])**.** Let (X, R) be an L-fuzzy poset on a set X and $A \in L^X$.

(1) An element $s \in X$ is called a join (or a supremum) of A (w.r.t. the L-fuzzy order R) if it holds that
 (i) For any $x \in X,\ A(x) \leq R(x,s)$;
 (ii) For any $y \in X,\ \bigwedge_{x \in X} (A(x) \longrightarrow R(x,y)) \leq R(s,y)$.

(2) An element $m \in X$ is called a meet (or an infimum) of A (w.r.t. the L-fuzzy partial order R) if it holds that
 (i) For any $x \in X, A(x) \leq R(m,x)$;
 (ii) For any $y \in X,\ \bigwedge_{x \in X} (A(x) \longrightarrow R(y,x)) \leq R(y,m)$.

Theorem 2.8 (Bělohlvek [4, 5], Xie [22])**.** *Let (X, R) be an L-Fuzzy poset, $A \in L^X$ and $s, m \in X$. The following equivalences hold:*

(i) s is a join of A if and only if $(R(s,y) = \bigwedge_{x \in X} (A(x) \longrightarrow R(x,y)))$, for any $y \in X$;

(ii) m is a meet of A if and only if $(R(y,m) = \bigwedge_{x \in X} (A(x) \longrightarrow R(y,x)))$, for any $y \in X$.

Corollary 2.9 (Zhang [24]). *Let (X, R) be an L-fuzzy poset, $A \in L^X$. If the join and the meet of A exist, then they are unique.*

For a given L-fuzzy set A of X, the $\sqcup A$ (resp. $\sqcap A$) denotes the join (resp. the meet) of A.

Definition 2.10 (Bělohlavek [4, 5]). Let (X, R) be an L-Fuzzy poset and $A \in L^X$.

(i) $A^u \in L^X$ is called the set of upper bounds of A and defined by

$$A^u(x) = \bigwedge_{z \in X} (A(z) \longrightarrow R(z, x)), \quad \text{for any } x \in X.$$

(ii) $A^\ell \in L^X$ is called the set of lower bounds of A and defined by

$$A^\ell(x) = \bigwedge_{z \in X} (A(z) \longrightarrow R(x, z)), \quad \text{for any } x \in X.$$

Combining Definition 2.10 and Theorem 2.8 leads to the following result.

Proposition 2.11. *Let (X, R) be an L- fuzzy poset and $A \in L^X$. If $\sqcup A$ (resp., $\sqcap A$) exists, then $A^u(\sqcup A) = 1$ (resp., $A^l(\sqcap A) = 1$).*

Theorem 2.12 (Yao [23]). *Let (X, R) be an L-fuzzy poset and $A \in L^X$. It holds that*

(i) if $\sqcap A^u$ exists, then so is $\sqcup A$ and $\sqcup A = \sqcap A^u$;
(ii) if $\sqcup A^\ell$ exists, then so is $\sqcap A$ and $\sqcap A = \sqcup A^\ell$.

Definition 2.13 (Bělohlavek [4, 5], Lai and Zhang [25] and Zhang [26]). An L-fuzzy poset (X, R) is called an L-fuzzy complete lattice if $\sqcup A$ and $\sqcap A$ exist for any L-fuzzy subset A of X.

Theorem 2.14 (Bělohlavek [5], Lai and Zhang [25] and Zhang [26]). *Let (X, R) be an L-Fuzzy poset. The following statements are equivalent:*

(i) (X, R) is an L-fuzzy complete lattice;
(ii) For any $A \in L^X$, $\sqcup A$ exists;
(iii) For any $A \in L^X$, $\sqcap A$ exists.

3. L-Fuzzy Multifunctions

In this section, we introduce three types of monotonicity of an L-fuzzy multifunction on a given L-fuzzy complete lattice and investigate their various properties. Fist we recall the notion of an L-fuzzy multifunction on a given set.

Definition 3.1. Let X be a nonempty set. An L-fuzzy multifunction on X is a mapping $\phi : X \longrightarrow L^X$.

For a given L-fuzzy multifunction $\phi : X \longrightarrow L^X$, we denote ϕ_x instead of $\phi(x)$ i.e., $\phi_x : X \longrightarrow L$, for any $x \in X$.
Next, we introduce the notion of L-fuzzy monotone multifunction on a given L-fuzzy complete lattice.

Definition 3.2. Let (X, R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an L-fuzzy multifunction.

(1) ϕ is called $\sqcup$-fuzzy monotone if it holds that
(i) $R(x, y) \leq R(\sqcup\phi_x, \sqcup\phi_y)$, for any $x, y \in X$;

(ii) $\phi_x(\sqcup\phi_x) = 1$, for any $x \in X$.

(2) ϕ is called $\sqcap$-fuzzy monotone if it holds that

(i) $R(x,y) \le R(\sqcap\phi_x, \sqcap\phi_y)$, for any $x, y \in X$;

(ii) $\phi_x(\sqcap\phi_x) = 1$, for any $x \in X$.

(3) ϕ is called $(\sqcup,\sqcap)$-fuzzy monotone (or an L-fuzzy monotone, simply) if it holds that

(i) $R(x,y) \le R(\sqcup\phi_x, \sqcap\phi_y)$, for any $x, y \in X$;

(ii) $\phi_x(\sqcup\phi_x) = \phi_x(\sqcap\phi_x) = 1$, for any $x \in X$.

Remark 3.3. If ϕ is a simple function, then Definition 3.2 coincides with the definition of fuzzy monotone function (see, e.g., Definition 1.4 [26]).

Example 3.4. Let $X = \{x, y, z\}$ and $L = \{0, a, b, 1\}$ be the lattice given by the Hasse diagram in Figure 1. Define the fuzzy relation $R : X \times X \longrightarrow L$ as follows:

$R(.,.)$	x	y	z
x	1	1	1
y	b	1	b
z	b	1	1

It is easy to verify that (X, R) is an L-fuzzy complete lattice.

We define the L-fuzzy multifunctions $\phi, \varphi, \psi : X \longrightarrow L^X$ as follows:

	x	y	z
$\phi_x(.)$	b	1	a
$\phi_y(.)$	b	1	1
$\phi_z(.)$	1	1	a

	x	y	z
$\varphi_x(.)$	1	b	1
$\varphi_y(.)$	1	a	1
$\varphi_z(.)$	1	0	b

	x	y	z
$\psi_x(.)$	1	b	0
$\psi_y(.)$	0	1	0
$\psi_z(.)$	0	b	1

Then we obtain that

	$\sqcup$	$\sqcap$
ϕ_x	y	z
ϕ_y	y	z
ϕ_z	y	x

	$\sqcup$	$\sqcap$
φ_x	z	x
φ_y	y	x
φ_z	x	x

	$\sqcup$	$\sqcap$
ψ_x	x	x
ψ_y	y	y
ψ_z	z	z

Where,

(i) ϕ is $\sqcup$-fuzzy monotone multifunction;

(ii) φ is $\sqcap$-fuzzy monotone multifunction;

(iii) ψ is ($\sqcup$-$\sqcap$) fuzzy monotone multifunction.

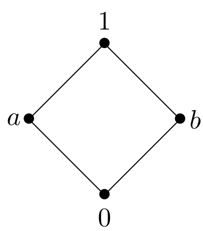

FIGURE 1. Hasse diagram of $L = \{0, a, b, 1\}$.

Notation 3.5. For a given L-fuzzy multifunction $\phi : X \longrightarrow L^X$, we associate two L-fuzzy multifunctions $\phi^u, \phi^\ell : X \longrightarrow L^X$ defined for any $x \in X$ as:

$$\phi^u(x) = \phi^u_x \text{ and } \phi^\ell(x) = \phi^\ell_x.$$

The following proposition shows the interaction between the monotonicity of a given L-fuzzy multifunction ϕ on a set X and that of ϕ^u and ϕ^ℓ.

Proposition 3.6. *Let (X,R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an L-fuzzy multifunction with the corresponding L-fuzzy multifunctions ϕ^ℓ and ϕ^u. The following equivalences hold:*

(i) ϕ is $\sqcup$-fuzzy monotone if and only if ϕ^u is $\sqcap$-fuzzy monotone;
(ii) ϕ is $\sqcap$-fuzzy monotone if and only if ϕ^ℓ is $\sqcup$-fuzzy monotone.

Proof. We only prove (i), as both cases are analogous. For the first implication, we show that $R(x,y) \leq R(\sqcap\phi^u_x, \sqcap\phi^u_y)$ and $\phi^u_x(\sqcap\phi^u_x) = 1$, for any $x, y \in X$.

(a) Since ϕ is $\sqcup$-monotone, it follows that $R(x,y) \leq R(\sqcup\phi_x, \sqcup\phi_y)$, for any $x, y \in X$. From Theorem 2.12, it holds that $\sqcup\phi_x = \sqcap\phi^u_x$, for any $x \in X$. Hence, $R(x,y) \leq R(\sqcap\phi^u_x, \sqcap\phi^u_y)$, for any $x, y \in X$.

b) Since $\sqcap\phi^u_x = \sqcup\phi_x$, it follows that

$$\phi^u_x(\sqcap\phi^u_x) = \phi^u_x(\sqcup\phi_x) = \bigwedge_{y\in X} (\phi_x(y) \longrightarrow R(y, \sqcup\phi_x))\,.$$

Since $\phi_x(y) \leq R(y, \sqcup\phi_x)$, for any $y \in X$, it follows that

$$\phi^u_x(\sqcap\phi^u_x) = \bigwedge_{y\in X} (\phi_x(y) \longrightarrow R(y, \sqcup\phi_x)) = 1\,.$$

Thus, ϕ^u is $\sqcap$-fuzzy monotone. The converse goes in the same way. ■

Next, we need the following lemma.

Lemma 3.7. *Let (X,R) be an L-fuzzy complete lattice and $A \in L^X$ satisfies that there exists x_0 in X such that $A(x_0) = 1$. Then it holds that $R(\sqcap A, \sqcup A) = 1$.*

Proof. Let $m = \sqcap A$ and $s = \sqcup A$, then $A(x) \leq R(m,x)$ and $A(x) \leq R(x,s)$, for any $x \in X$. This implies that $R(x_0, s) = 1$ and $R(m, x_0) = 1$. By transitivity of R, we obtain that $R(m,s) = 1$. ■

The above Lemma 3.7 leads to show the relationship between $(\sqcup,\sqcap)$-fuzzy monotonicity and $\sqcup$-fuzzy (resp. $\sqcap$-fuzzy) monotonicity.

Proposition 3.8. *Let (X,R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an L-fuzzy multifunction. If ϕ is $(\sqcup,\sqcap)$-fuzzy monotone, then ϕ is $\sqcup$-fuzzy monotone and $\sqcap$-fuzzy monotone.*

Proof. Suppose that ϕ is $(\sqcup,\sqcap)$-fuzzy monotone and let $x, y \in X$. Since $\phi_y(\sqcup\phi_y) = 1$, it follows from Lemma 3.7 that $R(\sqcap\phi_y, \sqcup\phi_y) = 1$. Since $R(x,y) \leq R(\sqcup\phi_x, \sqcap\phi_y)$, it holds that

$$R(x,y) \leq R(\sqcup\phi_x, \sqcap\phi_y) \wedge \underbrace{R(\sqcap\phi_y, \sqcup\phi_y)}_{=1},$$

The transitivity of R guarantees that

$$R(x,y) \leq R(\sqcup\phi_x, \sqcup\phi_y)\,.$$

Furthermore, one easily verifies that $\phi_x(\sqcup\phi_x) = 1$. Thus, ϕ is $\sqcup$-fuzzy monotone multifunction. In analogous way, we obtain that ϕ is $\sqcap$-fuzzy monotone multifunction. ■

Remark 3.9. The converse of the above implication does not necessarily hold. Indeed, let ϕ and φ are two L-fuzzy multifunctions given in Example 3.4. We know that ϕ is $\sqcup$-fuzzy monotone and φ is $\sqcap$-fuzzy monotone. Since $R(x,y) \nleq R(\sqcup\phi_x, \sqcap\phi_y) = R(y,z)$ (resp. $R(x,y) \nleq R(\sqcup\varphi_x, \sqcap\varphi_y) = R(z,x)$) it holds that ϕ (resp. φ) is not $(\sqcup,\sqcap)$- fuzzy monotone multifunction.

4. Fixed Point Theorems for L-Fuzzy Monotone Multifunctions

The aim of the present section is to show some fixed point theorems of L-fuzzy multifunction with respect to the three introduced types of monotonicity. Furthermore, we show that the set of fixed points of an L-fuzzy monotone multifunction is an L-fuzzy complete lattice.

4.1. Existence Theorems

In this subsection, we investigate the existence of a fixed point for any L-fuzzy monotone multifunction on an L-fuzzy complete lattice. First, we recall the following definition.

Definition 4.1. Let $\phi : X \longrightarrow L^X$ be an L-fuzzy multifunction. An element $x \in X$ is called a fixed point of ϕ if it holds that $\phi_x(x) = 1$.

We denote by $Fix(\phi)$ the set of all fixed points of ϕ. More details on fixed points of multifunctions can be found in [9, 19].

The following theorems shows that any L-fuzzy monotone multifunction has the fixed point property.

Theorem 4.2. *Let (X, R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an $\sqcup$-fuzzy monotone multifunction. Then it holds that ϕ has at least a fixed point.*

Proof. Let P be an L-fuzzy set on X defined as:

$$P(x) = R(\sqcup\phi_x, x), \text{ for any } x \in X,$$

Since (X, R) is an L-fuzzy complete lattice, it follows that P has an infimum. Let $m = \sqcap P$, i.e., $P(x) \leq R(m, x)$ and $\bigwedge_{x\in X}(p(x) \longrightarrow R(y, x)) \leq R(y, m)$, for any $x, y \in X$. Since ϕ is $\sqcup$-fuzzy monotone multifunction, it follows that $R(m, x) \leq R(\sqcup\phi_m, \sqcup\phi_x)$. Since $P(x) \leq R(m, x) \leq R(\sqcup\phi_m, \sqcup\phi_x)$ and $P(x) = R(\sqcup\phi_x, x)$, it follows that $P(x) \leq R(\sqcup\phi_m, \sqcup\phi_x) \wedge R(\sqcup\phi_x, x)$. From the transitivity of R, it follows that $P(x) \leq R(\sqcup\phi_m, x)$. In the other hand $\bigwedge_{x\in X}(P(x) \longrightarrow R(y, x)) \leq R(y, m)$, for any $y \in X$. Setting $y = \sqcup\phi_m$, then we obtain that $\bigwedge_{x\in X}(P(x) \longrightarrow R(\sqcup\phi_m, x)) \leq R(\sqcup\phi_m, m)$. The fact that $P(x) \leq R(\sqcup\phi_m, x)$, implies that $P(x) \longrightarrow R(\sqcup\phi_m, x) = 1$, for any $x \in X$. Hence, $\bigwedge_{x\in X}(P(x) \longrightarrow R(\sqcup\phi_m, x)) = 1$. Thus,

$$R(\sqcup\phi_m, m) = 1 \tag{4.1}$$

Since ϕ is $\sqcup$-fuzzy monotone, it holds that $\underbrace{R(\sqcup\phi_m, m)}_{=1} \leq R(\sqcup\phi_{\sqcup\phi_m}, \sqcup\phi_m)$. This implies that, $R(\sqcup\phi_{\sqcup\phi_m}, \sqcup\phi_m) = 1$. Since $P(x) = R(\sqcup\phi_x, x)$, it follows that $P(\sqcup\phi_m) = 1$. Since $P(x) \leq R(m, x)$, it follows that $P(\sqcup\phi_m) \leq R(m, \sqcup\phi_m)$. Hence,

$$R(m, \sqcup\phi_m) = 1 \tag{4.2}$$

From the equations (4.1) and (4.2), and by the antisymmetry of R, we obtain that $m = \sqcup\phi_m$. From the second condition of $\sqcup$- monotonicity, we obtain that $\phi_m(\sqcup\phi_m) = 1$, this implies that $\phi_m(m) = 1$. We conclude that, m is a fixed point of ϕ. ∎

In the same way, the following theorem shows that any $\sqcap$-fuzzy monotone multifunction has at least a fixed point.

Theorem 4.3. *Let (X, R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an $\sqcap$-fuzzy monotone multifunction. Then it holds that ϕ has a fixed point.*

Proof. The proof can be obtained by combining Proposition 3.6 and Theorem 4.2. ∎

A combination of Proposition 3.8, Theorem 4.2 and Theorem 4.3 leads to the following fixed point property for $(\sqcup, \sqcap)$-fuzzy monotone multifunctions.

Theorem 4.4. *Let (X, R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be a $(\sqcup, \sqcap)$-fuzzy monotone multifunction. Then ϕ has a fixed point.*

4.2. Structure of the Fixed Points Set of an L-Fuzzy Monotone Multifunction

In this subsection, we show that the set of fixed points of an L-fuzzy monotone multifunction is an L-fuzzy complete lattice.

Theorem 4.5. *Let (X, R) be an L-fuzzy complete lattice and $\phi : X \longrightarrow L^X$ be an L-fuzzy monotone multifunction, then the set of fixed point of ϕ in X is a nonempty L-fuzzy complete lattice.*

Proof. In order to show that $Fix(\phi)$ is an L-fuzzy complete lattice, we prove that $\sqcup A$ exists, for any $A \in L^{Fix(\phi)}$. Let's consider the L-fuzzy subset B of L^X defined for any $x \in X$ by:

$$B(x) = (\bigwedge_{y \in Fix(\phi)} A(y) \longrightarrow R(x, y)) \wedge R(x, \sqcup\phi_x).$$

Assume that $s = \sqcup B$, then it holds that $B(x) \leq R(x, s)$, for any $x \in X$. Since ϕ is $(\sqcup, \sqcap)$-fuzzy monotone multifunction, it follows that $R(x, s) \leq R(\sqcup\phi_x, \sqcap\phi_s)$, for any $x \in X$. This implies that

$$B(x) \leq R(\sqcup\phi_x, \sqcap\phi_s), \text{ for any } x \in X. \tag{4.3}$$

Using the fact that $B(x) \leq R(x, \sqcup\phi_x)$ and equation (4.3), it follows that $B(x) \leq R(x, \sqcup\phi_x)$ $\wedge R(\sqcup\phi_x, \sqcap\phi_s)$, for any $x \in X$. Now, the transitivity of R guarantees that

$$B(x) \leq R(x, \sqcap\phi_s), \text{ for any } x \in X. \tag{4.4}$$

In the same line, Proposition 3.8 guarantees that $R(x, s) \leq R(\sqcup\phi_x, \sqcup\phi_s)$, for any $x \in X$. Hence, $B(x) \leq R(x, \sqcup\phi_x) \wedge R(\sqcup\phi_x, \sqcup\phi_s)$, for any $x \in X$. Thus,

$$B(x) \leq R(x, \sqcup\phi_s), \text{ for any } x \in X. \tag{4.5}$$

We know from Theorem 2.8 that $\bigwedge_{x \in X} (B(x) \longrightarrow R(x, y)) = R(s, y)$, for any $y \in X$. By setting $y = \sqcup\phi_s$, it holds that $\bigwedge_{x \in X} (B(x) \longrightarrow R(x, \sqcup\phi_s)) = R(s, \sqcup\phi_s)$.

Equation (4.5) implies that $B(x) \longrightarrow R(x, \sqcup\phi_s) = 1$, for any $x \in X$. Hence,

$$R(s, \sqcup\phi_s) = 1. \tag{4.6}$$

Equation (4.6) and the fact of $(\sqcup,\sqcap)$-fuzzy monotonicity of ϕ imply that $\underbrace{R(s,\sqcup\phi_s)}_{=1} \leq R(\sqcup\phi_s,\sqcup\phi_{\sqcup\phi_s})$. Thus,

$$R(\sqcup\phi_s,\sqcup\phi_{\sqcup\phi_s}) = 1. \tag{4.7}$$

Similarly, by the definition of B and the fact that $\bigwedge_{x\in X}(B(x) \longrightarrow R(x,y)) = R(s,y)$, for any $y \in Fix(\phi)$, it follows that

$$R(s,y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.8}$$

From equations (4.6) and (4.8), it holds that $R(s,y) \wedge R(s,\sqcup\phi_s) = 1,$ for any $y \in Fix(\phi)$. Thus, $B(s) = 1$.
Also, the fact of $(\sqcup,\sqcap)$-fuzzy monotonicity of ϕ imply that $1 = R(s,y) \leq R(\sqcup\phi_s,\sqcap\phi_y)$. Hence,

$$R(\sqcup\phi_s,\sqcap\phi_y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.9}$$

Since $\phi_y(z) \leq R(\sqcap\phi_y,z)$, for any $z \in X$, it follows that $\phi_y(y) \leq R(\sqcap\phi_y,y)$. The fact that$\phi_y(y) = 1$, then implies that

$$R(\sqcap\phi_y,y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.10}$$

Equations (4.9), (4.10) and the transitivity of R, imply that $1 = R(\sqcup\phi_s,\sqcap\phi_y)\wedge R(\sqcap\phi_y,y) \leq R(\sqcup\phi_s,y)$. Thus,

$$R(\sqcup\phi_s,y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.11}$$

From equations (4.7) and (4.11), it follows that $R(\sqcup\phi_s,\sqcup\phi_{\sqcup\phi_s}) \wedge R(\sqcup\phi_s,y) = 1$, for any $y \in Fix(\phi)$. Hence,

$$B(\sqcup\phi_s) = 1. \tag{4.12}$$

Since $B(x) \leq R(x,s)$ for any $x \in X$, it follows that $B(\sqcup\phi_s) \leq R(\sqcup\phi_s,s)$ Hence,

$$R(\sqcup\phi_s,s) = 1. \tag{4.13}$$

Equations (4.6), (4.13) and the antisymmetry of R imply that

$$s = \sqcup\phi_s. \tag{4.14}$$

Using the second condition of $(\sqcup,\sqcap)$-monotonicity of ϕ, it holds that $\phi_s(s) = \phi_s(\sqcup\phi_s) = 1$. Thus, s is a fixed point of ϕ.

We complete this proof by showing that s is the infimum of A (i.e., $s = \sqcap A$) in $Fix(\phi)$. Let $t \in Fix(\phi)$ such that $A(y) \leq R(t,y)$, for any $y \in Fix(\phi)$. The fact that $1 = \phi_t(t) \leq R(t,\sqcup\phi_t)$ and the transitivity of R imply that $\underbrace{R(t,\sqcup\phi_t)}_{=1} \wedge R(\sqcup\phi_t,y) \leq R(t,y)$. This implies that

$$R(\sqcup\phi_t,y) \leq R(t,y), \text{ for any } y \in Fix(\phi). \tag{4.15}$$

In similar way, we obtain $R(\sqcap\phi_y,y) \leq R(\sqcup\phi_t,y)$, for any $y \in Fix(\phi)$. Now, equation (4.10) implies that

$$R(\sqcup\phi_t,y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.16}$$

Equations (4.15) and (4.16) imply that

$$R(t,y) = 1, \text{ for any } y \in Fix(\phi). \tag{4.17}$$

Furthermore, from the equations (4.8) and (4.17), it holds that $R(s,t) = R(t,s) = 1$. The antisymmetry of R guarantees that $s = t$. Thus, $A(x) \leq R(s,x)$, for any $x \in Fix(\phi)$. Now we show that $\bigwedge_{x\in Fix(\phi)} (A(x) \longrightarrow R(y,x)) \leq R(y,s)$, for any $y \in Fix(\phi)$. Indeed,

$$\bigwedge_{x\in Fix(\phi)} (A(x) \longrightarrow R(y,x)) = \bigwedge_{x\in Fix(\phi)} (A(x) \longrightarrow R(y,x)) \wedge \underbrace{R(y, \sqcup\phi_y)}_{=1} = B(y) \leq R(y,s),$$

for any $y \in Fix(\phi)$. Thus, $s = \sqcap A$. Therefore, Theorem 2.14 guarantees that $Fix(\phi)$ is an L-fuzzy complete lattice. ∎

5. Conclusion

In this work, we have established some fixed point theorems of L-fuzzy monotone multifunctions. Moreover, we have provided that the set of all fixed points of L-fuzzy monotone multifunction has the structure of complete lattice. We anticipate that these results will facilitate the study of fixed point theorems for fuzzy monotone multifunctions on more general fuzzy ordered structures.

References

[1] S. Abian, A.B. Brown, A theorem on partially ordered sets, with applications to fixed point theorems, Canad. J. Math. 13 (1961) 78–82.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. 3 (1922) 133–181.

[3] R. Bělohlávek, Some properties of residuated lattices, Czech. Math. J. 53 (2003) 161–171.

[4] R. Bělohlávek, Concept lattices and order in fuzzy logic, Ann, Pure Appl. Logic. 128 (2004) 277–298.

[5] R. Bělohlávek, Fuzzy Relational Systems, Foundations and Principles, Springer, New York, 2012.

[6] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets and Systems 137 (2003) 113–136.

[7] L.E.J. Brouwer, Über Abbildungen von mannigfaltigkeiten, Mathematische Annalen 71 (1911) 97–115.

[8] M.D. Ćirić, J. Ignjatović, S. Bogdanović, Fuzzy equivalence relations and their equivalence classes, Fuzzy Sets and Systems 158 (2007) 1295–1313.

[9] R. Dacic, Fixed points of monotone multifunctions in partially ordered sets, Publ. Inst. Math.(Beograd) 27 (1980) 41–50.

[10] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002.

[11] A.C. Davis, A characterization of complete lattices, Pacific Journal of Mathematics (1955) 311–319.

[12] B. De Baets, R. Mesiar, Triangular norms on product lattices, Fuzzy Sets and Systems 104 (1999) 61–75.

[13] L. Fan, A new approach to quantitative domain theory, Electronic Notes in Theoretical Computer Science 45 (2001) 77–87.

[14] J.A. Goguen, L-Fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.

[15] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic, Kluwer Academic Publishers, Dordrecht, 1998.

[16] H. Lu, Continuous selection and fixed point theorems for fuzzy mappings in general topological spaces, Second International Conference on Intelligent Computation Technology and Automation, Changsha, Hunan (2009) 681–684.

[17] Y. Shen, D. Qiu, W. Chen, Fixed point theorems in fuzzy metric spaces, Applied Mathematics Letters 25 (2012) 138–141.

[18] H.L. Skala, Trellis theory, Algebra Universalis 1 (1971) 218–233.

[19] A. Stouti, Fixed points of fuzzy monotone multifunctions, Archivum Mathematicum 39 (2003) 209–212.

[20] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics (1955) 285–309.

[21] F. Včelař, Z. Patikova, On fuzzification of Tarski's fixed point theorem without transitivity, Fuzzy Sets and Systems 320 (2017) 93–113.

[22] W. Xie, Q. Zhang, L. Fan, The Dedekind-MacNeille completions for fuzzy posets, Fuzzy Sets and Systems 160 (2009) 2292–2316.

[23] W. Yao, L.X. Lu, Fuzzy Galois connections on fuzzy posets, Mathematical Logic Quarterly 55 (2009) 105–112.

[24] Q.Y. Zhang, L. Fan, Continuity in quantitative domains, Fuzzy Sets and Systems 154 (2005) 118–131.

[25] Q.Y. Zhang, L. Fan, A kind of L-fuzzy complete lattices and adjoint functor theorem for LF-posets. In Report on the Fourth International Symposium on Domain Theory, Hunan University, Changsha, China (2006).

[26] Q. Zhang, W. Xie, L. Fan, Fuzzy complete lattices, Fuzzy Sets and Systems 160 (2009) 2275–2291.