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Common Fixed Point Theorem of Family of Contraction Maps and Its Applications in Integral Equations

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Abstract The aim of this article is to prove existence of a unique common fixed point for a family of contractive type self maps on a complete metric space. Finally we apply this theorem to study the existence and uniqueness of a common solution for functional integral equations.

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1. Introduction and Preliminaries

Fixed point theory constitutes an important part of the subject of nonlinear functional analysis and it is useful for proving the existence theorems for nonlinear differential and integral equations. The Banach contraction principle is the simplest and one of the most versatile elementary results in fixed point theory, which is a very popular tool for solving existence problems in many branches of mathematical analysis. Several authors have extended Banach's fixed point theorem in various ways. The family of contraction mappings was introduced and studied by Ćirić [5] and Tasković [13]. Also in the process, the study of existence of common fixed point for finite and infinite families of self-mappings has been carried out by many authors. For example, one may refer [1–4, 6, 8, 10–12, 14, 15].

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The aim of this paper is to define some new conditions of common contractivity for an infinite family of mappings and give some new results on the existence and uniqueness of common fixed points in the setting of complete metric space. The following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set and $\{T_n\}$ a family of self-mappings on X. A point $x_0 \in X$ is called a common fixed point for this family if and only if $T_n(x_0) = x_0$, for each $n \in \mathbb{N}$.

The following interesting theorem was given by Ćirić [5] for a family of generalized contractions.

Theorem 1.2. *Let (X, d) be a complete metric space and $\{T_\alpha\}_{\alpha\in J}$ a family of self-mappings of X. If there exists fixed $\beta \in J$ such that for each $\alpha \in J$:*

$$d(T_\alpha x, T_\beta y) \leq \lambda \max\left\{d(x,y), d(x,T_\alpha x), d(y,T_\beta y), \frac{1}{2}[d(x,T_\beta y) + d(y,T_\alpha x)]\right\},$$

for some $\lambda = \lambda(\alpha) \in (0,1)$ and all x, $y \in X$, then all T_α's have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha \in J$.

Definition 1.3. Let $\{T_n\}$ be a sequence of mappings and g a self-mapping on X. If $y = gx = T_n x$ for all $n \in \mathbb{N}$ and for some $x \in X$, then x is called a coincidence point of $\{T_n\}$ and g, where y is called a point of coincidence of $\{T_n\}$ and g.

Definition 1.4. [7] Let f and g be two self-mappings defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Definition 1.5. [9] Let S denote the class of those functions $\beta : \mathbb{R}_+ \to [0,1)$ which satisfy the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

The following generalization of Banach contraction principle is due to Geraghty [9].

Theorem 1.6. *Let (X, d) be a complete metric space and $T : X \to X$. Suppose there exists $\beta \in S$ such that for each $x, y \in X$,*

$$d(Tx,Ty) \leq \beta(d(x,y))\ d(x,y).$$

Then T has a unique fixed point $z \in X$, and $\{T^n(x)\}$ converges to z, for each $x \in X$.

In Section 2, we prove a version of Theorem 1.6, for infinite families of self mappings of a complete metric space. In Section 3, existence of a unique common solution for the functional integral equation (3.1) is obtained under suitable conditions.

2. Common Fixed Point Theorems

In this section, we prove existence of a unique common fixed point for a family of contractive type self maps on a complete metric space.

Theorem 2.1. *Let (X, d) be a complete metric space and $\{T_n\}$ a sequence of self-mappings on X. Suppose that there exists $\beta \in S$ such that*

$$d(T_i x, T_j y) \leq \beta(M_{i,j}(x,y)) \max\{d(x,y), d(x,T_i x), d(y,T_j y)\}, \tag{2.1}$$

where

$$M_{i,j}(x,y) = \max\left\{d(x,y), d(x,T_i x), d(y,T_j y), \frac{d(x,T_j y) + d(y,T_i x)}{2}\right\},$$

for all $x, y \in X$, $i, j = 1, 2, \ldots$ with $x \neq y$, $i \neq j$, then all T_n's have a unique common fixed point in X.

Proof. For any $x_0 \in X$, let $x_n = T_n(x_{n-1})$, $n = 1, 2, \ldots$, then using (2.1) we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T_n(x_{n-1}), T_{n+1}(x_n)) \\ &\leq \beta(M_{n,n+1}(x_{n-1}, x_n)) \max\{d(x_{n-1}, x_n), d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1} x_n)\} \\ &= \beta(M_{n,n+1}(x_{n-1}, x_n)) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Since $\beta : [0, \infty) \longrightarrow [0, 1)$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(M_{n,n+1}(x_{n-1}, x_n)) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &< \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

This shows that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n),$$

and so

$$d(x_n, x_{n+1}) \leq \beta(M_{n,n+1}(x_{n-1}, x_n)) d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{2.2}$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers and, consequently, there exists $\delta \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = \delta. \tag{2.3}$$

We show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T_n and the existence part of the proof is finished. Suppose that $x_{n-1} \neq x_n$ for every $n \in \mathbb{N}$. Then, from (2.2)

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(M_{n,n+1}(x_{n-1}, x_n)) = \beta(d(x_{n-1}, x_n)) < 1. \tag{2.4}$$

Indeed, since

$$\frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

then,

$$\begin{aligned} &M_{n,n+1}(x_{n-1}, x_n) \\ &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1} x_n), \frac{d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})}{2}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n). \end{aligned}$$

Taking the limit as $n \to \infty$ in (2.4), we obtain

$$1 = \frac{\delta}{\delta} = \lim_{n \to \infty} \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \lim_{n \to \infty} \beta(d(x_{n-1}, x_n)) \leq 1.$$

Consequently, $\lim_{n \to \infty} \beta(d(x_{n-1}, x_n)) = 1$ and, since $\beta \in S$, $\delta = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0$. This contradicts that $\delta > 0$. So

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Now, we claim that for any positive integers m and n with $m \geq n$, we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Assume, on the contrary that

$$\limsup_{n,m\to\infty} d(x_n, x_m) > 0. \tag{2.6}$$

Using the triangle inequality, we obtain

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\
&\leq d(x_n, x_{n+1}) + d(x_{m+1}, x_m) + d(T_{n+1}x_n, T_{m+1}x_m) \\
&\leq d(x_n, x_{n+1}) + d(x_{m+1}, x_m) \\
&\quad + \beta(M_{n+1,m+1}(x_n, x_m)) \max\{d(x_n, x_m), d(x_n, T_{n+1}x_n), d(x_m, T_{m+1}x_m)\} \\
&= d(x_n, x_{n+1}) + d(x_{m+1}, x_m) \\
&\quad + \beta(M_{n+1,m+1}(x_n, x_m)) \max\{d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1})\} \\
&\leq d(x_n, x_{n+1}) + d(x_{m+1}, x_m) + \beta(M_{n+1,m+1}(x_n, x_m)) d(x_n, x_m) \\
&\quad + \max\{d(x_n, x_{n+1}), d(x_m, x_{m+1})\} \\
&\leq \beta(M_{n+1,m+1}(x_n, x_m)) d(x_n, x_m) + 3\max\{d(x_n, x_{n+1}), d(x_m, x_{m+1})\},
\end{aligned}$$

since $\beta \in S$, then $1 - \beta(M_{n+1,m+1}(x_n, x_m)) > 0$. Therefore, from the last inequality, it follows

$$d(x_n, x_m) \leq 3[1 - \beta(M_{n+1,m+1}(x_n, x_m))]^{-1} \max\{d(x_n, x_{n+1}), d(x_m, x_{m+1})\}.$$

From the above inequality and (2.5), we have $\lim_{n\to\infty} [1 - \beta(M_{n+1,m+1}(x_n, x_m))]^{-1} = \infty$ or $\lim_{n\to\infty} \beta(M_{n+1,m+1}(x_n, x_m)) = 1$ or $\lim_{n\to\infty} M_{n+1,m+1}(x_n, x_m) = 0$. Since

$$\begin{aligned}
&d(x_n, x_m) \\
&\leq M_{n+1,m+1}(x_n, x_m) \\
&= \max\left\{d(x_n, x_m), d(x_n, T_{n+1}x_n), d(x_m, T_{m+1}x_m), \frac{d(x_n, T_{m+1}x_m) + d(x_m, T_{n+1}x_n)}{2}\right\} \\
&= \max\left\{d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1}), \frac{d(x_n, x_{m+1}) + d(x_m, x_{n+1})}{2}\right\},
\end{aligned}$$

we get $\limsup_{n,m\to\infty} d(x_n, x_m) = 0$, a contradiction to (2.6). Therefore (2.5) holds and we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence and by completeness of X, $\{x_n\}$ converges to x (say) in X, that is,

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{2.7}$$

Now, we will prove that for any positive integer m, x is a common fixed point of $\{T_m\}$. Observe that

$$\begin{aligned}
d(x, T_m x) &\leq d(x, x_n) + d(x_n, T_m x) = d(x, x_n) + d(T_n x_{n-1}, T_m x) \\
&\leq d(x, x_n) + \beta(M_{n,m}(x_{n-1}, x)) \max\{d(x_{n-1}, x), d(x_{n-1}, T_n x_{n-1}), d(x, T_m x)\} \\
&\leq d(x, x_n) + \beta(M_{n,m}(x_{n-1}, x)) \max\{d(x_{n-1}, x), d(x_{n-1}, x_n)\} \\
&\quad + \beta(M_{n,m}(x_{n-1}, x)) d(x, T_m x) \\
&< d(x, x_n) + \max\{d(x_{n-1}, x), d(x_{n-1}, x_n)\} + \beta(M_{n,m}(x_{n-1}, x)) d(x, T_m x).
\end{aligned}$$

Suppose that $T_m x \neq x$. Now, since

$$\begin{aligned} &M_{n,m}(x_{n-1}, x) \\ &= \max\left\{d(x_{n-1}, x), d(x_{n-1}, T_n x_{n-1}), d(x, T_m x), \frac{1}{2}[d(x_{n-1}, T_m x) + d(x, T_n x_{n-1})\right\} \\ &= \max\left\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, T_m x), \frac{1}{2}[d(x_{n-1}, T_m x) + d(x, x_n)\right\}, \end{aligned}$$

then

$$\lim_{n\to\infty} M_{n,m}(x_{n-1}, x) = d(x, T_m x). \tag{2.8}$$

Therefore

$$d(x, T_m x) \leq [1 - \beta(M_{n,m}(x_{n-1}, x))]^{-1}[d(x, x_n) + \max\{d(x_{n-1}, x), d(x_{n-1}, x_n)\}].$$

Taking the limit as $n \to \infty$ and using (2.5), (2.8) and $\beta \in S$, we obtain $d(x, T_m x) = 0$ a contradiction to $T_m x \neq x$, so that $T_m x = x$. Let y be another fixed point of $\{T_n\}$, then

$$d(x, y) = d(T_n x, T_m y) \leq \beta(M_{n,m}(x, y)) \max\{d(x, y), d(x, T_n x), d(y, T_m y)\} < d(x, y).$$

a contradiction. Hence, x is the unique common fixed point of $\{T_n\}$. ∎

If $\beta(t) = k$, where $0 \leq k < 1$, then we have the following result.

Corollary 2.2. *Let (X, d) be a complete metric space and $\{T_n\}$ a sequence of self-mappings on X. Suppose that there exists $0 \leq k < 1$ such that*

$$d(T_i x, T_j y) \leq k \ \max\{d(x, y), d(x, T_i x), d(y, T_j y)\}, \tag{2.9}$$

for all $x, y \in X$, $i, j = 1, 2, \ldots$ with $x \neq y$, $i \neq j$. Then all T_n's have a unique common fixed point in X.

From Theorem 2.1, we deduce the following corollary.

Corollary 2.3. *Let (X, d) be a complete metric space and $\{T_n\}$ a sequence of self-mappings on X. Suppose that there exists $\beta \in S$ such that*

$$d(T_i x, T_j y) \leq \beta(M_{i,j}(x, y)) d(x, y), \tag{2.10}$$

where

$$M_{i,j}(x, y) = \max\left\{d(x, y), d(x, T_i x), d(y, T_j y), \frac{d(x, T_j y) + d(y, T_i x)}{2}\right\},$$

for all $x, y \in X$, $i, j = 1, 2, \ldots$ with $x \neq y$, $i \neq j$. Then all T_n's have a unique common fixed point in X.

From Corollary 2.3, we deduce the following corollary.

Corollary 2.4. *Let (X, d) be a complete metric space and $\{T_n\}$ a sequence of self-mappings on X. Suppose that there exists $\beta \in S$ such that*

$$d(T_i x, T_j y) \leq \beta(d(x, y)) d(x, y), \tag{2.11}$$

for all $x, y \in X$, $i, j = 1, 2, \ldots$ with $x \neq y$, $i \neq j$. Then all T_n's have a unique common fixed point in X.

Now we introduce the following theorem for discussing the existence of unique common fixed point of the sequence (T_n) of self mappings and g on a complete metric space (X, d).

Theorem 2.5. *Let (X,d) be a complete metric space. Let $0 \leq a_{i,j}+b_{i,j} < 1$ $(i,j=1,2,...)$ satisfy*

(i) for each i, $\overline{\lim_{j\to\infty}} a_{i,j} < 1$ and $\overline{\lim_{j\to\infty}} b_{i,j} < 1$;

(ii) $\sum_{n=1}^{\infty} A_n < \infty$, where $A_n = \prod_{i=1}^{n} \frac{b_{i,i+1}}{1-a_{i,i+1}}$.

Also, let $\{T_n : X \longrightarrow X : n \in \mathbb{N}\}$ be a sequence of self-mappings and $g : X \longrightarrow X$. Assume that there exists $n_0 \in \mathbb{N}$ such that $T_{n_0}(X) \subseteq g(X)$, $g(X)$ is closed subset of X and T_{n_0} is weakly compatible with g on X such that

$$d(T_{n_0}x, T_n y) \leq a_{n_0,n} d(gy, T_n y)\varphi(d(gx, T_{n_0}x), d(gx, gy)) + b_{n_0,n} d(gx, gy), \quad (2.12)$$

for all $x, y \in X$, $n_0, n \in \mathbb{N}$ with $n_0 \neq n$ where $\varphi : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that $\varphi(t,t) = 1$ for all $t \in [0,\infty)$. Then g and $\{T_n\}$ have a unique common fixed point in X.

Proof. Suppose that g and $\{T_n\}$ have two common fixed points $x, y \in X$. Then, for all $n \in \mathbb{N}$, we have

$$x = gx = T_n x \text{ and } y = gy = T_n y. \quad (2.13)$$

From (2.12) and (2.13), we have

$$\begin{aligned} d(x,y) &= d(T_{n_0}x, T_n y) \\ &\leq a_{n_0,n} d(gy, T_n y)\varphi(d(gx, T_{n_0}x), d(gx, gy)) + b_{n_0,n} d(gx, gy) \\ &= b_{n_0,n} d(x,y), \end{aligned}$$

since $\overline{\lim_{n\to\infty}} b_{n_0,n} < 1$, it follows from the above inequality that $d(x,y) = 0$, that is, $x = y$. Hence, the common fixed point of g and $\{T_n\}$, if it exists, is unique. Now suppose that x is a common fixed point of g and T_{n_0}. Then

$$x = gx = T_{n_0}x. \quad (2.14)$$

For any $n \in N$, from (2.12) using (2.14), we have

$$\begin{aligned} d(x, T_n x) &= d(T_{n_0}x, T_n x) \\ &\leq a_{n_0,n} d(gx, T_n x)\varphi(d(gx, T_{n_0}x), d(gx, gx)) + b_{n_0,n} d(gx, gx) \\ &= a_{n_0,n} d(x, T_n x) \end{aligned}$$

since $\overline{\lim_{n\to\infty}} a_{n_0,n} < 1$, it follows from the above inequality that $d(x, T_n x) = 0$, that is, $x = T_n x$. Then $x = gx = T_n x$ for all $n \in \mathbb{N}$, that is, x is common fixed point of g and $\{T_n\}$. Hence, any common fixed point of g and T_{n_0} is a common fixed point of g and $\{T_n\}$. The converse part is trivial. Next we claim that T has a fixed point. To substantiate our claim we assume that $x_0 \in X$. As $T_{n_0}(X) \subseteq g(X)$, we can define a sequence $\{x_n\}$ in X as follows (for all $n \in \mathbb{N}$) $gx_n = T_n(x_{n-1})$. Then from (2.12), we obtain

$$\begin{aligned} d(gx_1, gx_2) &= d(T_1(x_0), T_2(x_1)) \\ &\leq a_{1,2} d(gx_1, T_2 x_1)\varphi(d(gx_0, T_1 x_0), d(gx_0, gx_1)) + b_{1,2} d(gx_0, gx_1) \\ &\leq a_{1,2} d(gx_1, gx_2)\varphi(d(gx_0, gx_1), d(gx_0, gx_1)) + b_{1,2} d(gx_0, gx_1) \end{aligned}$$

$$\leq a_{1,2}d(gx_1, gx_2) + b_{1,2}d(gx_0, gx_1),$$

implies

$$(1 - a_{1,2})d(gx_1, gx_2) \leq b_{1,2}d(gx_0, gx_1).$$

Hence, we have

$$d(gx_1, gx_2) \leq \frac{b_{1,2}}{1 - a_{1,2}}d(gx_0, gx_1).$$

Also,

$$\begin{aligned} d(gx_2, gx_3) &= d(T_2(x_1), T_3(x_2)) \\ &\leq a_{2,3}d(gx_2, T_3x_2)\varphi(d(gx_1, T_2x_1), d(gx_1, gx_2)) + b_{2,3}d(gx_1, gx_2) \\ &\leq a_{2,3}d(gx_2, gx_3)\varphi(d(gx_1, gx_2), d(gx_1, gx_2)) + b_{2,3}d(gx_1, gx_2) \\ &\leq a_{2,3}d(gx_2, gx_3) + b_{2,3}d(gx_1, gx_2). \end{aligned}$$

Then

$$\begin{aligned} d(gx_2, gx_3) &\leq \frac{b_{2,3}}{1 - a_{2,3}}d(gx_1, gx_2) \\ &\leq \frac{b_{1,2}}{1 - a_{1,2}} \times \frac{b_{2,3}}{1 - a_{2,3}}d(gx_0, gx_1). \end{aligned}$$

Generally, we conclude that

$$d(gx_n, gx_{n+1}) \leq \prod_{i=1}^{n} \frac{b_{i,i+1}}{1 - a_{i,i+1}}d(gx_0, gx_1) = A_n d(gx_0, gx_1). \tag{2.15}$$

Therefore, for $m, n \in \mathbb{N}$, $m \geq n$, and using (2.15), we get

$$\begin{aligned} d(gx_n, gx_m) &\leq \sum_{k=n}^{m-1} d(gx_k, gx_{k+1}) \\ &\leq \sum_{k=n}^{m-1} A_k d(gx_0, gx_1). \end{aligned}$$

Now, passing limit $n, m \to \infty$, we get $d(gx_n, gx_m) \to 0$. Thus $\{gx_n\}$ is a Cauchy sequence and by completeness of X, $\{gx_n\}$ converges to y in X, that is, $\lim_{n\to\infty} gx_n = y \in X$. Since $T_{n_0}(X) \subseteq g(X)$ and $g(X)$ is closed subset of X, therefore there exists $x \in X$ such that $y = gx$. Assume that $T_{n_0}x \neq gx$. From the condition (2.12), we have

$$\begin{aligned} d(T_{n_0}x, gx) &\leq d(gx, gx_n) + d(T_{n_0}x, gx_n) = d(gx, gx_n) + d(T_{n_0}x, T_n x_{n-1}) \\ &\leq d(gx, gx_n) + a_{n_0,n}d(gx, T_{n_0}x)\varphi(d(gx_{n-1}, T_n x_{n-1}), d(gx_{n-1}, gx)) \\ &\quad + b_{n_0,n}d(gx_{n-1}, gx) \\ &\leq d(gx, gx_n) + a_{n_0,n}d(gx, T_{n_0}x)\varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n)) \\ &\quad + b_{n_0,n}d(gx_{n-1}, gx) \\ &\leq d(gx, gx_n) + a_{n_0,n}d(gx, T_{n_0}x) + b_{n_0,n}d(gx_{n-1}, gx). \end{aligned}$$

Taking $\overline{\lim}$ as $n \to \infty$, we get

$$d(T_{n_0}x, gx) \leq \lim a_{n_0,n}d(T_{n_0}x, gx) < d(T_{n_0}x, gx),$$

a contradiction. Therefore, it must be the case $T_{n_0}x = gx$. Therefore, we have $y = gx = T_{n_0}x$, it follows that x is a coincidence point of g and T_{n_0}. Since T_{n_0} weakly compatible with g on X, we have $gy = g(T_{n_0}x) = T_{n_0}(gx) = T_{n_0}y$. So,

$$\begin{aligned} d(T_{n_0}y, y) &= d(T_{n_0}y, T_{n_0}x) \\ &\leq a_{n_0,n_0} d(gx, T_{n_0}x)\varphi(d(gy, T_{n_0}y), d(gy, gx)) + b_{n_0,n_0} d(gy, gy), \end{aligned}$$

it follows from the above inequality that $d(T_{n_0}y, y) = 0$, that is, $T_{n_0}y = y$. Therefore $y = gy = T_{n_0}y$. So, y is a common fixed point of g and T_{n_0}. By what we have already proved, y is the unique common fixed point of g and $\{T_n\}$. ■

Corollary 2.6. *Let (X, d) be a complete metric space and $T, G, g : X \longrightarrow X$ be three mappings. Assume that $T(X) \subseteq g(X)$, $g(X)$ is closed subset of X and T commutes with g on X such that*

$$d(Tx, Gy) \leq a\ d(gy, Gy)\varphi(d(gx, Tx), d(gx, gy)) + b\ d(gx, gy),$$

for all $x, y \in X$ where a, b non-negative real numbers with $a + b < 1$ and $\varphi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0, \infty)$. Then g, G and T have a unique common fixed point in X.

Corollary 2.7. *Let (X, d) be a complete metric space and $T, G : X \longrightarrow X$ be two mappings. Assume that*

$$d(Tx, Gy) \leq a\ d(y, Gy)\varphi(d(x, Tx), d(x, y)) + b\ d(x, y),$$

for all $x, y \in X$ where a, b non-negative real numbers with $a + b < 1$ and $\varphi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0, \infty)$. Then T and G have a unique common fixed point in X.

Corollary 2.8. *Let (X, d) be a complete metric space. Let $0 \leq a_{i,j} + b_{i,j} < 1$ $(i, j = 1, 2, ...)$, satisfy*

(i) for each i, $\overline{\lim_{j\to\infty}} a_{i,j} < 1$ and $\overline{\lim_{j\to\infty}} b_{i,j} < 1$;

(ii) $\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{b_{i,i+1}}{1 - a_{i,i+1}}$.

Also, $\{T_n : X \longrightarrow X : n \in \mathbb{N}\}$ be a sequence of mappings and $g : X \longrightarrow X$ be a self mapping and there exists $n_0 \in \mathbb{N}$ such that $T_{n_0}(X) \subseteq g(X)$ and $g(X)$ closed subset of X, and T_{n_0} commutes with g on X such that

$$d(T_{n_0}x, T_n y)\ \leq a_{n_0,n} d(gy, T_n y)\frac{1 + d(gx, T_{n_0}x)}{1 + d(gx, gy)} + b_{n_0,n} d(gx, gy),$$

for all $x, y \in X$, $n_0, n \in \mathbb{N}$ with $x \neq y$ and $n_0 \neq n$. Then g and $\{T_n\}$ have a unique common fixed point in X.

Corollary 2.9. *Let (X, d) be a complete metric space. Let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty)\ \leq a\ d(y, Ty)\frac{d(x, Tx)}{d(x, y)} + b\ d(x, y),$$

for all $x, y \in X$ with $x \neq y$ and a, b non-negative real numbers with $a + b < 1$. Then T has a unique fixed point in X.

3. Application to Integral Equations

In this section, we present the application of Corollary 2.4 from the theory of integral equations, which are of theoretical interest.
We denote by Φ the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ verifying the following conditions:

(i) φ is increasing;
(ii) for each $t > 0$, $\varphi(t) < t$;
(iii) $\beta(t) = \dfrac{\varphi(t)}{t} \in S$.

For example, $\varphi(t) = kt$, where $0 \le k < 1$, $\varphi(t) = \dfrac{t}{t+1}$ are in Φ.
Throughout this section, we assume that $X = C[0,1]$ is the set of all continuous functions defined on $I = [0,1]$. As an application of our results, we study the existence and uniqueness of a common solution for the following system of functional integral equations:

$$x(t) = \int_0^1 k(t,s)\ f_i(s, x(s))ds,\ t \in I,\ i \in \mathbb{N}. \tag{3.1}$$

Also, consider the following assumptions:

(a_1) the function $k : [0,1] \times [0,1] \to [0, \infty)$ is continuous and bounded with
$$K = \sup\{k(t,s) : t, s \in [0,1]\} \le 1;$$
(a_2) the functions $f_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ $(i \in \mathbb{N})$ are continuous and there exists a function $\varphi \in \Phi$ such that
$$|f_i(t,x) - f_j(t,y)| \le \varphi(|x - y|);$$

Theorem 3.1. *Under conditions (a_1) and (a_2), the system of integral equations given in (3.1) has a unique common solution in $C(I)$.*

Proof. Define $T_i : X \to X$ by
$$(T_i(x))(t) = \int_0^1 k(t,s)\ f_i(s, x(s))ds,\ t \in I,\ i \in \mathbb{N}.$$
Now, we check that hypotheses in Corollary 2.4 are satisfied. Indeed, (X, d) is a complete metric space, if we choose
$$d(x,y) = \sup_{t \in I} |x(t) - y(t)|,\ x, y \in X.$$
Also, by virtue of our assumptions, T_i is well defined (this means that for $x, y \in X$ then $T_i(x) \in X, i \in \mathbb{N}$). Besides, for $x, y \in X$ and $i, j \in \mathbb{N}$ with $i \neq j$, we can get
$$\begin{aligned} d(T_i x, T_j y) &= \sup_{t \in I} |(T_i x)(t) - (T_j y)(t)| \\ &= \sup_{t \in I} \Big| \int_0^1 k(t,s)\ f_i(s, x(s))ds - \int_0^1 k(t,s)\ f_j(s, y(s))ds \Big| \\ &\le \sup_{t \in I} \int_0^1 k(t,s) |f_i(s, x(s)) - f_j(s, y(s))| ds \\ &\le \sup_{t \in I} \int_0^1 k(t,s) \varphi(|x(s) - y(s)| ds. \end{aligned}$$

As the function φ is increasing, then $\varphi(|x(t)-y(t)| \leq \varphi(d(x,y))$ we obtain

$$\begin{aligned} d(T_ix, T_jy) &\leq \sup_{t\in I} \int_0^1 k(t,s)\varphi(|x(s)-y(s)|ds \\ &\leq \varphi(d(x,y)) \sup_{t\in I} \int_0^1 k(t,s)ds \\ &\leq K\varphi(d(x,y)) \\ &\leq \frac{\varphi(d(x,y))}{d(x,y)}\, d(x,y) = \beta(d(x,y))\, d(x,y). \end{aligned}$$

Then for $x, y \in X$ and $i, j \in \mathbb{N}$ with $i \neq j$

$$d(T_ix, T_jy) \leq \beta(d(x,y))\, d(x,y).$$

Finally, Corollary 2.4 gives that T_n's have a unique common fixed point in X. ■

Example 3.2. Consider the following equation

$$x(t) = \int_0^1 \frac{tsx(s)\cos s}{2i} ds \tag{3.2}$$

for $t \in I = [0,1],\ i \in \mathbb{N}$.

Here we have

$$\varphi(t) = \frac{t}{2},\ k(t,s) = ts \text{ and } f_i(t,x) = \frac{x\cos t}{2i}.$$

In this example, we have k is continuous and bounded on $[0,1]\times[0,1]$. Also,

$$K = \sup\{ts : t, s \in [0,1]\} \leq 1.$$

Again, the function f_i is continuous on $[0,1]\times\mathbb{R}$ for all $i \in \mathbb{N}$ and

$$\begin{aligned} \left| f_i(t,x) - f_j(t,y) \right| &= \left| \frac{x\cos t}{2i} - \frac{y\cos t}{2j} \right| \\ &\leq \frac{1}{2}\,|x-y| \\ &= \varphi\left(|x-y|\right). \end{aligned}$$

Hence conditions from $(a_1) - (a_2)$ are satisfied. Thus by Theorem 3.1 the equation (3.2) has at least one solution in $C(I)$.

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