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The Existence of Best Proximity Points of Generalized p-Cyclic Weak (F, ψ, φ) Contractions in p-Cyclic Metric Spaces

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Abstract In this manuscript, we discuss some property of p-cyclic maps which belong to the class Ω and we extend the notion of a generalized cyclic weak (F, ψ, φ)-contractin to a generalized p-cyclic weak (F, ψ, φ)-contraction, where $p \geq 2$. We prove best proximity point results of such map in p-cyclic complete metric spaces. Our results extend and generalize the related result in the literature. We also give an example in support of our main result.

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1. Introduction

Fixed point theory is a hot area of research and is one of the most important topics in mathematics, especially in analysis. Many researchers took interest in fixed point theory and its applications in diverse fields ranging from different branches of mathematics to engineering, and from economics to biology. For example, optimization problems, variational inequalitiey problems, equilibrium problems including minimization problems, and variational inclusion problems, among others, are known to be very useful in diverse fields such as economics, computer science, and engineering, and they also find applications in machine learning. Many problems arising from these fields can be modeled as optimization problems. The Banach contraction principle is a fundamental result in fixed point theory [6]. For one century, due to its importance and simplicity, several authors have

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obtained many interesting extensions and generalizations of the Banach contraction principle in several direction. One possible direction is the notion of best proximity point results. In this line of research, [1, 2, 27] obtained interesting best proximity point results and derived fixed point results as consequences of their works in which Banach fixed point is one of them. Also, taking the key role of the notion of the metric in mathematics and hence in quantitative sciences, it has been extended and generalized in several distinct directions by many authors.

In 1968, Bryant [8] constructed a remarkable result in fixed point theory and proved that, in a complete metric space, if, for some positive integer $n \geq 2$, the nth iteration of the given mapping forms a contraction, then it possess a unique fixed point. Another outstanding approach was proposed by Kirk, Srinivasan and Veeramani [13] by introducing the notion of cyclic contraction. More precisely, every cyclic contraction in a complete metric space possess a unique fixed point. Later, the concept of the cyclic contractions has been investigated immensely by a considerable large number of authors who brought several brilliant notions and derived a number of interesting results (see, e.g., [3, 10, 11, 14–20, 23, 25, 26, 28] and the references therein). Let T be a self-mapping on a metric space (X, η). Suppose that A and B are nonempty subsets of X such that $X = A \cup B$. A self-mapping $T : A \cup B \to A \cup B$ is called a cyclic contraction [13] if

1). $T(A) \subseteq B$ and $T(B) \subseteq A$.

2). If there is a $k \in (0, 1)$ such that the following inequality is satisfied

$$\eta(Tx, Ty) \leq k\eta(x, y), \quad \text{for all } x \in A, y \in B.$$

After this initial investigation, several extensions of cyclic mappings and cyclic contractions have been introduced. In this paper, we mainly follow the notations defined in [17, 22]. In [17], a notion of p-cyclic map is introduced. Let $D_1, D_2, \ldots, D_p (p \geq 2)$ be nonempty sets. A p-cyclic map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is defined such that $T(D_i) \subseteq D_{i+1}, \forall i \in \{1, 2, \ldots, p\}$, $x = x_0 \in D_i$ defines a sequence $\{x_n\} \subset \cup_{i=1}^{p} D_i$ as $x_n = Tx_{n-1}$. Then, $\{x_{pn}\}$ is a subsequence in D_i, $\{x_{pn+1}\}$ is a subsequence in D_{i+1} and so on. From the arrangement of such a sequence formed by a p-cyclic map, Karapinar et al. in [22] introduced a notion of p-cyclic sequence (Definition 2.1(1)). If $D_i s$ are subsets of a metric space (X, η), then, to obtain a best proximity point of T under various contractive conditions (some of them given in the literature), it is enough to prove that: given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\eta(x_{pn}, x_{pm+1}) < dist(D_i, D_{i+1}) + \varepsilon, \forall n, m \geq N_0$$

where $dist(D_i, D_{i+1}) = \inf\{\eta(x, y) : x \in D_i, y \in D_{i+1}\}$.

The authors [22] introduced a concept of p-cyclic Cauchy sequence and p-cyclic complete metric space ([22], Definition 2.1). They investigated the behavior of such p-cyclic maps, and found that, if $\eta(x, y) > dist(D_i, D_{i+1})$, then $\eta(Tx, Ty) < \eta(x, y)$ and, if $\eta(x, y) = dist(D_i, D_{i+1})$, then $\eta(Tx, Ty) = \eta(x, y), x \in D_i, y \in D_{i+1}$. A p-cyclic map with this property is called to be a *p-cyclic strict contraction map* (Definition 3.1). Note that, if the distances between the adjacent sets are zero, then a p-cyclic strict contraction map is a strict contraction map in the usual sense. All such maps invariably satisfy the condition: $x, y \in D_i, \eta(T^{pn}x, T^{pn+1}y) \to dist(D_i, D_{i+1})$ as $n \to \infty$. In the paper [22], all p-cyclic maps which satisfy the above two properties are said to belong to class Ω (Definition 3.4). Finally, the authors proved the existence and convergence of best proximity points of mappings which belong to the class Ω in a p-cyclic complete metric space.

Now we recollect some essential definitions.

Definition 1.1 ([4])**.** A continuous function $F:[0,\infty)^2 \to \mathbb{R}$ is called a C-class function, if for any $s,t \in [0,\infty)$, the following conditions hold:
(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s=0$ or $t=0$.

Remark 1.2. We denote the class of all C-class functions as $\mathbb{C}$.

Example 1.3 ([4])**.** Following examples show that the class $\mathbb{C}$ of C-class functions is nonempty:

(1) $F(s,t) = s - t$.
(2) $F(s,t) = ms, 0 < m < 1$
(3) $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
(4) $F(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.
(5) $F(s,t) = \ln(1+a^s)/2$, for $a > e$. Indeed $F(s,1) = s$ implies that $s = 0$.
(6) $F(s,t) = s\log_{t+a} a$, for $a > 1$.
(7) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
(8) $F(s,t) = s\beta(s)$, where $\beta:[0,\infty) \to [0,1)$.
(9) $F(s,t) = s - \frac{t}{k+t}$.
(10) $F(s,t) = s - (\frac{2+t}{1+t})t$.

More examples of C-class functions can be founf in [4].

Definition 1.4 ([21])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied:
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.5 ([4])**.** An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0 \ \ , t > 0$ and $\varphi(0) \geq 0$

Remark 1.6. We let Ψ denote the class of altering distance functions and Ψ_u denote the class of ultra altering distance functions. Let $\mathbb{R}^+ = [0,\infty)$.

Motivated by the above mentioned definitions, we introduce a new generalized cyclic weak (F,ψ,φ)-contraction based on the generalized weak φ-contraction which is proposed in [5]. Moreover, we obtain corresponding best proximity point theorems for these cyclic mappings under certain conditions. Our results extend and improve the results obtained in [5] as well as some other related results in the literature.

In what follows, we recollect some definitions and fundamental results which are crucial in the sequel.

Definition 1.7. ([17], Definitions 3.1). For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Define $D_{p+i} := D_i$, for all $i \in \{1,2,\ldots,p\}$. A map $T : \cup_{i=1}^{p} D \to \cup_{i=1}^{p} D$ is called a p-cyclic map, if $T(D_i) \subseteq D_{i+1}, \forall i \in \{1,2,\ldots,p\}$. If $p = 2$, then T is called a cyclic map. A point $x \in D_i$ is said to be a best proximity point of T in D_i, if $\eta(x,Tx) = dist(D_i, D_{i+1})$, where $dist(D_i, D_{i+1}) := \inf\{\eta(x,y) : x \in D_i, y \in D_{i+1}\}$.

In [22], the authors introduced the conditions for the underlying space and for the subsets of the space, to have a unique best proximity point under a p-cyclic map, if it exists, irrespective of the contraction condition imposed on the map.

Proposition 1.8 ([22]). *Let $D_1, D_2, \ldots, D_p, (p \geq 2)$ be nonempty convex subsets of a strictly convex norm linear space X such that $dist(D_i, D_{i+1}) > 0, i \in \{1, 2, \ldots, p\}$. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic map. Then T has at most one best proximity point in $D_i, 1 \leq i \leq p$.*

Let T be a p-cyclic map as given in Definition 1.7. T is said to be a p-cyclic nonexpansive map if for all $x \in D_i, y \in D_{i+1}$, the following holds:

$$\eta(Tx, Ty) \leq \eta(x, y), \forall i \in \{1, 2, \ldots, p\}.$$

The following lemma naturally follows for a p-cyclic nonexpansive map.

Lemma 1.9. ([17], Lemma 3.3). *For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. If $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is a p-cyclic nonexpansive map, then*

$$dist(D_i, D_{i+1}) = dist(D_{i+1}, D_{i+2}) = dist(D_1, D_2), \forall i \in \{1, 2, \ldots, p\}. \tag{1.1}$$

In addition, if $\nu \in D_i \cap \mathbf{D}(T)_i \neq \emptyset$, then $T^j \nu \in D_{i+1} \cap \mathbf{D}(T)_{i+j} \neq \emptyset$, for all j = 1, 2, ..., (p-1), where $\mathbf{D}(T)_k$ is the set of best proximity point of the mapping T in D_k.

The following lemma (see [11, 22]) is crucial to prove that a given sequence is Cauchy.

Lemma 1.10. ([11], Lemma 3.7). *For a uniformly convex Banach space $(X, \|.\|)$, we suppose that D_1, D_2 are nonempty closed subsets of X and $\{a_n\}, \{b_n\} \subset D_1$ and $\{d_n\} \subset D_2$. If D_1 is convex such that*

(i) $\|b_n - d_n\| \to dist(D_1, D_2)$; and

(ii) for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n > N$,

$$\|a_m - d_n\| \leq dist(D_1, D_2) + \varepsilon,$$

then for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$, $\|a_m - b_n\| \leq \varepsilon$.

Next, we recall few p-cyclic maps with some contraction conditions imposed on them, which are defined in [3, 9, 12, 17, 18].

Definition 1.11. ([18], Definition 3.1). For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic map, T is said to be a p-cyclic contraction, if there exists $k \in (0, 1)$ such that for all $x \in D_i$ and $y \in D_{i+1}$, we have

$$\eta(Tx, Ty) \leq k\eta(x, y) + (1 - k)dist(D_i, D_{i+1}), \forall i \in \{1, 2, \ldots, p\}.$$

Definition 1.12. ([3], Definition 2.1). For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and D_1 and D_2 are nonempty subsets of X. A cyclic map $T : D_1 \cup D_2 \to D_1 \cup D_2$ is said to be a cyclic φ-contraction if

$$\eta(Tx, Ty) \leq \eta(x, y) - \varphi(\rho(x, y)) + \varphi(dist(D_1, D_2)), \forall x \in D_1, y \in D_2,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map.

Definition 1.13. ([9], Definition 2.1). Let D_1 and D_2 be nonempty subsets of a metric space (X, η). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A cyclic map $T : D_1 \cup D_2 \to D_1 \cup D_2$ is said to be a generalized cyclic weak φ-contraction, if for any $x \in D_1, y \in D_2$

$$\eta(Tx, Ty) \leq m(x, y) - \varphi(m(x, y)) + \varphi(dist(D_1, D_2)) \tag{1.2}$$

where $m(x,y) = \max\{\eta(x,y), \eta(x,Tx), \eta(y,Ty), \frac{1}{2}[\eta(x,Ty)+\eta(y,Tx)]\}$.

Definition 1.14. ([5], Definition 2.1). Let D_1 and D_2 be nonempty subsets of a metric space(X,η). Suppose that $\varphi, \psi : [0,\infty) \to [0,\infty)$ and φ is a strictly increasing map. A cyclic map $T : D_1 \bigcup D_2 \to D_1 \bigcup D_2$ is called a generalized cyclic weak (F,ψ,φ)-contraction, if for any $x \in D_1$ and $y \in D_2$,

$$\begin{aligned}\psi(\eta(Tx,Ty)) \leq F\Big(&\psi(m(x,y)) - \psi(dist(D_1,D_2)),\\ &\varphi(m(x,y)) - \varphi(dist(D_1,D_2))\Big) + \psi(dist(D_1,D_2)),\end{aligned} \tag{1.3}$$

where $F \in \mathbb{C}, \psi \in \Psi$ with $\psi(s+t) \leq \psi(s) + \psi(t)$, $\varphi \in \Psi_u$ and

$$m(x,y) = \max\{\eta(x,y), \eta(x,Tx), \rho(y,Ty), \frac{1}{2}[\eta(x,Ty)+\eta(y,Tx)]\}.$$

Remark 1.15. If we take $F(s,t) = s-t$ and $\psi(t) = t$ in Definition 1.14, the we obtain Definition 1.13 that mentioned above.

2. p-Cyclic Sequences and p-Cyclic Complete Metric Spaces

Throughout this article, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In [22], Karapinar et al. introduced the notion of p-cyclic sequence as follows:

Definition 2.1 ([22])**.** For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X.

1 . A sequence $\{x_n\}_{n=1}^{\infty} \subset \cup_{i=1}^{p} D_i$ is called a p-cyclic sequence if $x_{pn+i} \in D_i$, for all $n \in \mathbb{N}_0$ and $i = 1,2,\ldots,p$.

2 . We say that $\{x_n\}_{n=1}^{\infty}$ is a p-cyclic Cauchy sequence, if for given $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that for some $i \in \{1,2,\ldots,p\}$, we have

$$\eta(x_{pn+i}, x_{pm+i+1}) < dist(D_i, D_{i+1}) + \varepsilon, \forall m,n \geq N_0. \tag{2.1}$$

3 . A p-cyclic sequence $\{x_n\}_{n=1}^{\infty}$ in $\cup_{i=1}^{p} D_i$ is said to be p-cyclic bounded, if $\{x_{pn+i}\}_{n=1}^{\infty}$ is bounded in D_i for some $i \in \{1,2,...,p\}$.

4 . Let $\{x_n\}_{n=1}^{\infty}$ be a p-cyclic sequence in $\cup_{i=1}^{p} D_i$. If for some $j \in \{1,2,...,p\}$ the subsequence $\{x_{pn+j}\}$ of $\{x_n\}_{n=1}^{\infty}$ converges in D_j, then we say that $\{x_n\}_{n=1}^{\infty}$ is p-cyclic convergent.

5 . Under the assumption that $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty closed subsets of a metric space (X,η), we say that $\cup_{i=1}^{p} D_i$ is p-cyclic complete if every p-cyclic Cauchy sequence in $\cup_{i=1}^{p} D_i$ is p-cyclic convergent.

6 . If there are subsets $D_1, D_2, \ldots, D_p, (p \geq 2)$ of (X,η) such that $X = \cup_{i=1}^{p} D_i$ and $\cup_{i=1}^{p} D_i$ is p-cyclic complete, then we call (X,η) to be p-cyclic complete.

Remark 2.2. Note that a p-cyclic sequence which is a Cauchy sequence in the usual sense is a p-cyclic Cauchy sequence. On the other hand, p-cyclic Cauchy sequences need not be Cauchy sequences in the usual sense, even if $dist(D_i, D_{i+1}) = 0,\ \forall i \in \{1,2,...,p\}$.

An example which illustrates the notion of p-cyclic sequence and p-cyclic Cauchy sequence can be found in ([22], Example 1 and 2). And a complete metric space need not be p-cyclic complete, (see [22], Remark 2, for example).

The following proposition shows that a p-cyclic Cauchy sequence is p-cyclic bounded.

Proposition 2.3 ([22]). *For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Then, every p-cyclic Cauchy sequence in $\cup_{i=1}^{p} D_i$ is p-cyclic bounded.*

The following proposition is an example of two-cyclic complete metric space.

Proposition 2.4 ([22]). *Let D_1 and D_2 be subsets of a uniformly convex Banach space X, which are nonempty and closed. If either D_1 or D_2 is convex, then $D_1 \cup D_2$ is two-cyclic complete.*

3. p-Cyclic Strict Contraction Maps

In [22], Karapinar et. al. introduced a notion of p-cyclic strict contraction, which is a generalization of strict contraction in the usual sense.

Definition 3.1 ([22]). For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. A p-cyclic map T is said to be p-cyclic strict contraction if, for all $x \in D_i, y \in D_{i+1}, 1 \leq i \leq p$:

(i) $\eta(x, y) > dist(D_i, D_{i+1}) \Rightarrow \eta(Tx, Ty) < \eta(x, y)$; and
(ii) $\eta(x, y) = dist(D_i, D_{i+1}) \Rightarrow \eta(Tx, Ty) = \eta(x, y)$.

Remark 3.2. Note that, if $D_i = A$, for all $i = 1, 2, \ldots, p$, then p-cyclic strict contraction is a strict contraction in the usual sense. It is clear that the p-cyclic strict contraction also forms a p-cyclic nonexpansive map.

The following proposition proves an important property of p-cyclic strict contraction map

Proposition 3.3 ([22]). *For a nonempty set M, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Let $x \in D_i (1 \leq i \leq p)$. Suppose that $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is a p-cyclic strict contraction map and if for all $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that*

$$\eta(T^{pn}x, T^{pm+1}x) < dist(D_i, D_{i+1}) + \varepsilon, n, m \geq n_0, \tag{3.1}$$

then for a given $\varepsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\eta(T^{pn+k}x, T^{pm+k+1}x) < dist(D_{i+k}, D_{i+k+1}) + \varepsilon, n, m \geq n_1, k \in \{1, 2, \ldots, p\}.$$

In [22], Karapinar et al. introduced the notion of p-cyclic maps with various contractive conditions and possed some common properties. They also introduced a notion of class Ω, a certain class of mappings.

Definition 3.4 ([22]). For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. A p-cyclic map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is said to belong to the class Ω if

(1) T is p-cyclic strict contraction.
(2) If $x, y \in D_i$, then $\lim\limits_{n\to\infty} \eta(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1}), 1 \leq i \leq p$.

In this manuscript, we list some p-cyclic maps different from those given in [22] which belong to the class Ω. First, we prove that a p-cyclic contraction map, which is defined via the notion of C-class functions, belongs to the class Ω. We give the following new definition via C-class functions.

Definition 3.5. Let $D_1, D_2, \ldots, D_p$ be nonempty subsets of a metric space (X, η). Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is called a p-cyclic (F, ψ, φ)-contraction map, if it satisfies

$$\psi\big(\eta(Tx, Ty)\big) \leq F\big(\psi(\eta(x, y)) - \psi(dist(D_i, D_{i+1})), \varphi(\eta(x, y)) - \varphi(dist(D_i, D_{i+1}))\big), \\ + \psi(dist(D_i, D_{i+1})),$$

for all $i \in \{1, 2, \ldots, p\}$, where $F \in \mathbb{C}$, $\psi \in \Psi$ and $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map.

We note that such a p-cyclic (F, ψ, φ)-contraction map T belongs to the class Ω (See [24]).

Remark 3.6. If we take $F(s, t) = s - t$, $\psi(t) = t$ and $p = 2$ in Definion 3.5, then we obtain Definition 1.12 (Definition 2.1, defined in [3]).

Remark 3.7. Karapinar et al.[22] showed that the p-cyclic Meir-Keeler map (p-cyclic MK-map) introduced in [17] belongs to the class Ω. See Example 4 in [22].

Next, we give a result on p-cyclic map satisfying a contraction condition of Geraghtys type [7] and show that it belongs to the class Ω. Here, we use the notion of C-class functions introduced in [4] combining with a class of functions S introduced by Geraghty [7], where, if S is the class of all functions $\vartheta : [0, \infty) \to [0, 1)$ that satisfies $\vartheta(t_n) \to 1$, then $t_n \to 0, t_n \in [0, \infty)$ for $n \in \mathbb{N}$.

Theorem 3.8. *For a nonempty set X, suppose $\eta : X \times X \to \mathbb{R}^+$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic $(F, \psi, \varphi, \vartheta)$-map such that*

$$\eta(Tx, Ty) \leq F\Big(\psi(\vartheta(\eta(x, y)))\eta(x, y) - \psi(\vartheta(\eta(x, y)))dist(D_i, D_{i+1}), \\ \varphi\big(\vartheta(\eta(x, y))\eta(x, y)\big) - \varphi\big(\vartheta(\eta(x, y))dist(D_i, D_{i+1})\big)\Big) \\ +\psi\big(\vartheta(\eta(x, y))\big)dist(D_i, D_{i+1}),$$

for all $i \in \{1, 2, \ldots, p\}$, where $F \in \mathbb{C}$, $\psi \in \Psi$ where $\psi(t) < t$ and $\vartheta \in S$. Then

(a) T is a p-cyclic strict contraction.

(b) $\lim_{n\to\infty} \eta(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1}), x \in D_i, y \in D_{i+1}$.

Proof. (a) Let $x \in D_i, y \in D_{i+1}$.

Case (1): If $\eta(x, y) > dist(D_i, D_{i+1})$, we have

$$\eta(Tx, Ty) \leq F\big(\psi(\vartheta(\eta(x, y)))\rho(x, y) - \psi(\vartheta(\eta(x, y)))dist(D_i, D_{i+1}), \\ \varphi\big(\vartheta(\eta(x, y))\rho(x, y)\big) - \varphi(\vartheta(\eta(x, y))dist(D_i, D_{i+1}))\big), \\ + \psi(\vartheta(\eta(x, y)))dist(D_i, D_{i+1}) \\ \leq \psi(\vartheta(\eta(x, y))\big[\eta(x, y) - dist(D_i, D_{i+1}) + dist(D_i, D_{i+1})\big](*) \\ \leq \psi(\vartheta(\eta(x, y))\eta(x, y).$$

Therefore

$$\eta(Tx, Ty) < \eta(x, y).$$

Case (2): If $\eta(x, y) = dist(D_i, D_{i+1})$, then from (*), we have $\eta(Tx, Ty) \leq \eta(x, y)$. By equation (1.1),

$$\eta(x, y) = dist(D_i, D_{i+1}) = dist(D_{i+1}, D_{i+2}) \leq \eta(Tx, Ty) \leq \eta(x, y),$$

therefore

$$\eta(Tx, Ty) = \eta(x, y).$$

Hence, T is p-cyclic strict contraction.

(b) Let $x, y \in D_i$. Since T is p-cyclic nonexpansive, $\{\eta(T^{pn}x, T^{pn+1}y)\}$ is a decreasing sequence and is bounded below by $dist(D_i, D_{i+1})$. Therefore,

$$\eta(T^{pn}x, T^{pn+1}y) \to r \text{ as } n \to \infty \text{ and } r \geq dist(D_i, D_{i+1}),$$

where $r = \inf_{n\geq 1} \eta(T^{pn}x, T^{pn+1}y)$.

Claim: $r = dist(D_i, D_{i+1})$.

If $\eta(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1})$ for some n, then by the p-cyclic non-expansiveness of T,

$$\eta(T^{pn+k}x, T^{pn+k+1}y) = \eta(T^{pn}x, T^{pn+1}y), k = 1, 2, \ldots.$$

Hence, we have

$$\eta(T^{pn}x, T^{pn+1}y) \to dist(D_i, D_{i+1}) \text{ as } n \to \infty.$$

Let us assume that $\eta(T^{pn}x, T^{pn+1}y) > dist(D_i, D_{i+1})$, $n \in \mathbb{N}$. Suppose that $r > dist(D_i, D_{i+1})$. Since T is p-cyclic non expansive,

$$\begin{aligned}
&\eta(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \eta(T^{pn+1}x, T^{pn+2}y) \\
&\leq F\Big(\psi(\vartheta(\eta(T^{pn}x, T^{pn+1}y)))\eta(T^{pn}x, T^{pn+1}y) - \psi(\vartheta(\eta(T^{pn}x, T^{pn+1}y)))dist(D_i, D_{i+1})\big), \\
&\quad \varphi\big(\vartheta(\eta(T^{pn}x, T^{pn+1}y))\big)\eta(T^{pn}x, T^{pn+1}y) - \varphi\big(\vartheta(\eta(T^{pn}x, T^{pn+1}y)))dist(D_i, D_{i+1})\big)\Big) \\
&\quad + \psi(\vartheta(\eta(T^{pn}x, T^{pn+1}y)))dist(D_i, D_{i+1}) \\
&\leq \psi(\vartheta(\eta(T^{pn}x, T^{pn+1}y))\big[\eta(T^{pn}x, T^{pn+1}y) - dist(D_i, D_{i+1}) + dist(D_i, D_{i+1})\big].
\end{aligned}$$

Then

$$\begin{aligned}
\eta(T^{p(n+1)}x, &T^{p(n+1)+1}y) \\
&\leq \psi(\vartheta(\eta(T^{pn}x, T^{pn+1}y))\big[\eta(T^{pn}x, T^{pn+1}y)\big].
\end{aligned}$$

Since $\vartheta \in S$ and $\psi(t) < t$,

$$\frac{\eta(T^{p(n+1)}x, T^{p(n+1)+1}y)}{\eta(T^{pn}x, T^{pn+1}y)} \leq \vartheta(\eta(T^{pn}x, T^{pn+1}y)) < 1. \tag{3.2}$$

Since $r = \lim_{n\to\infty} \eta((T^{p(n+1)}x, T^{p(n+1)+1}y)) > dist(D_i, D_{i+1})$ by our assumption, letting $n \to \infty$ in equation (3.2), we get

$$1 \leq \lim_{n\to} \vartheta(\eta(T^{pn}x, T^{pn+1}y)) \leq 1,$$

that is,

$$\lim_{n\to\infty} \vartheta(\eta(T^{pn}x, T^{pn+1}y)) = 1.$$

However, $\lim_{n\to\infty} \eta(T^{pn}x, T^{pn+1}y) = r > 0$, which contradicts $\vartheta \in S$. Hence,

$$r = dist(D_i, D_{i+1}).$$

This proves (b). From (a) and (b) we conclude that T belongs to Ω. ∎

Remark 3.9. For some more interesting results on best proximity point theorems in p-cyclic metric spaces, we refer readers to [24].

The purpose of this manuscript is to extend the generalized **cyclic weak** (F,ψ,φ)-**contraction** introduced in [5] to a generalized p**-cyclic weak** (F,ψ,φ)**-contraction**, where $p \geq 2$. Our results extends and generalizes the results in [5] and some other related results in the literature.

4. Best Proximity Point Results for Generalized p-Cyclic Weak (F,ψ,φ)-Contractins

We introduce the following definition via C-class functions.

Definition 4.1. Let $D_1, D_2, \ldots, D_p, (p \geq 2)$ be nonempty subsets of a p-cyclic metric space(X,η). Suppose that $\varphi,\psi : [0,\infty) \to [0,\infty)$ and φ is a strictly increasing map. A cyclic map $T : \cup_i^p D_i \to \cup_i^p D_i$ is called a **generalized** p**-cyclic weak** (F,ψ,φ)**-contraction**, if for any $x \in D_i$ and $y \in D_{i+1}$,

$$\begin{aligned}\psi(\eta(Tx,Ty)) \leq &F\Big(\psi(\mathcal{M}(x,y)) - \psi(dist(D_i,D_{i+1})),\\ &\varphi(\mathcal{M}(x,y)) - \varphi(dist(D_i,D_{i+1}))\Big) + \psi(dist(D_i,D_{i+1})),\end{aligned} \tag{4.1}$$

where $F \in \mathcal{C}, \psi \in \Psi$ with $\psi(s+t) \leq \psi(s) + \psi(t)$, $\varphi \in \Psi_u$ and

$$\mathcal{M}(x,y) = \max\{\eta(x,y), d(x,Tx), \eta(y,Ty), \frac{1}{2}[\eta(x,Ty) + \eta(y,Tx)]\}.$$

We prove the following result.

Lemma 4.2. *Let $D_1, D_2, \ldots, D_p$ be nonempty subsets of a metric space (X,η). Let $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ is a p-cyclic (F,ψ,φ)-contraction map (4.1). Then, $T \in \Omega$.*

Proof. We first show that p-cyclic strict contraction. Because the the map T is a p-cyclic (F,ψ,φ)- contraction, we have

$$\begin{aligned}\psi\big(\eta(Tx,Ty)\big) \leq &F\Big(\psi(\mathcal{M}(x,y)) - \psi(dist(D_i,D_{i+1})),\\ &\varphi(\mathcal{M}(x,y)) - \varphi(dist(D_i,D_{i+1}))\Big) + \psi(dist(D_i,D_{i+1})),\end{aligned}$$

for all $i \in \{1,2,\ldots,p\}$, where $F \in \mathcal{C}$, $\psi \in \Psi$, $\varphi \in \Psi_u$. Taking $F(s,t) = s - t$, we have

$$\psi\big(\eta(Tx,Ty)\big) \leq \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) + \varphi(dist(D_i,D_{i+1})).$$

If $\eta(x,y) = \mathcal{M}(x,y) = dist(D_i,D_{i+1})$, we have

$$\eta(Tx,Ty) \leq \eta(x,y).$$

Since $\eta(x,y) = dist(D_i,D_{i+1}) \leq \eta(Tx,Ty)$, we then have

$$\eta(Tx,Ty) = \eta(x,y).$$

In addition, if $\eta(x,y) = \mathcal{M}(x,y) > dist(D_i,D_{i+1})$, then

$$\begin{aligned}\psi\big(\eta(Tx,Ty)\big) \leq\ &F\Big(\psi(\mathcal{M}(x,y)) - \psi(dist(D_i,D_{i+1})),\\ &\varphi(\mathcal{M}(x,y)) - \varphi(dist(D_i,D_{i+1}))\Big) + \psi(dist(D_i,D_{i+1}))\\ &\leq \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) + \varphi(dist(D_i,D_{i+1}))\\ &< \psi(\eta(x,y)) - \varphi(\eta(x,y)) + \varphi(\eta(x,y)).\end{aligned}$$

Therefore

$$\eta(Tx, Ty) < \eta(x, y).$$

Therefore, T is a p-cyclic strict contraction. The second condition of Definition 3.4 follows from Lemma 3.3 in [18]. Hence, $T \in \Omega$. ∎

Theorem 4.3. *For a nonempty set X, suppose $\eta : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic maps satisfying a* **generalized p-cyclic weak** (F, ψ, φ)**-contraction** (4.1). *Assume for some $k \in \mathbb{N}$ and $x \in D_i, (1 \leq i, k \leq p), \{T^{pn+k}x\}$ converges to $\nu \in D_{i+k}$. Then, ν is a best proximity point of T in D_{i+k}.*

Proof. Let $x \in D_i$. By equation (1.1), for each $n \in N$, we have,

$$\begin{aligned} dist(D_{i+k}, D_{i+k+1}) &= dist(D_{i+k-1}, D_{i+k}) \\ &\leq \eta(T^{pn+k-1}x, \nu) \\ &\leq \eta(T^{pn+k-1}x, T^{pn+k}x) + \eta(T^{pn+k}x, \nu). \end{aligned}$$

By Lemma 4.2, $T \in \Omega$, so

$$\lim_{n\to\infty} (\eta(T^{pn+k-1}x, T^{pn+k}x) + \eta(T^{pn+k}x, \nu)) = dist(D_{i+k-1}, D_{i+k}).$$

Therefore,

$$\lim_{n\to\infty} \eta(T^{pn+k-1}x, \nu) = dist(D_{i+k-1}, B_{i+k}) = dist(D_{i+k}, D_{i+k+1}). \tag{4.2}$$

Now,

$$\begin{aligned} dist(D_{i+k}, D_{i+k+1}) &\leq \eta(\nu, T\nu) \\ &= \lim_{n\to\infty} \eta(T^{pn+k}x, T\nu) \\ &\leq \lim_{n\to\infty} \eta(T^{pn+k-1}x, \nu) \\ &= dist(D_{i+k}, D_{i+k+1}), \text{ (by equation(4.2))}. \end{aligned}$$

Hence, $\eta(\nu, T\nu) = dist(D_{i+k}, D_{i+k+1})$. ∎

Theorem 4.4. *For a nonempty set X, suppose $\eta : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are nonempty subsets of X. Suppose that $X = \cup_{i=1}^{p} D_i$ and $\cup_{i=1}^{p} D_i$ is p-cyclic complete. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic mapping which satisties a* **generalized p-cyclic weak** (F, ψ, φ)**-contraction** (4.1). *Then, there exists a best proximity point of T in D_j for some $j \in \{1, 2, \ldots, p\}$.*

Proof. Let $x \in D_i, 1 \leq i \leq p$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in (X, η) by

$$x_n := T^n x \text{ for } n \in \mathbb{N}.$$

Claim: $\{T^n x\}_{n=1}^{\infty}$ is a p-cyclic Cauchy sequence.
Let $m, n \in \mathbb{N}$ be such that $m > n$,

$$\begin{aligned} \eta(T^{pm}x, T^{pn+1}x) &= \eta(T^{p(n+r)}x, T^{pn+1}x), \text{ where } m = n + r, r \in \mathbb{N} \\ &= \eta(T^{pn}y, T^{pn+1}x), \text{ where } y = T^{pr}x \in D_i \\ &\to dist(D_i, D_{i+1}), \text{as } n \to \infty \text{ (because } T \in \Omega). \end{aligned}$$

This implies that, for all $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\eta(T^{pm}x, T^{pn+1}x) < \varepsilon + dist(D_i, D_{i+1}), m, n \geq n_0.$$

By Proposition 3.3, for a given $\varepsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\eta(T^{pm+k}x, T^{pn+k+1}x) < \varepsilon + dist(D_{i+k}, D_{i+k+1}), m, n \geq n_1, k \in \{1, 2, \dots, p\}.$$

Therefore, the sequence $\{T^n x\}$ is a p-cyclic Cauchy sequence in (X, η). Since (X, η) is p-cyclic complete, there exists a $k \in \{1, 2, \dots, p\}$ such that $\{T^{pn+k}x\}$ converges to $x^* \in D_{i+k}$. By Theorem 4.3, x^* is best proximity point of T in D_j, where $j = i + k$. ■

Theorem 4.5. *Let $D_1, D_2, \dots, D_p, (p \geq 2)$ be nonempty subsets of a p-cyclic metric space (X, η). Suppose that $T : \cup_i^p D_i \to \cup_i^p D_i$ is a* **generalized p-cyclic weak (F, ψ, φ)-contraction** *(4.1) and there exists $y_0 \in D_i$. Define $y_{n+1} = Ty_n$ for any $n \in N$. Then $\eta(y_n, y_{n+1}) \to dist(D_i, D_{i+1})$, as $n \to \infty$.*

Proof. Let $d_n = \eta(y_n, y_{n+1})$. First we claim that the sequence $\{d_n\}$ is nonincreasing. By the assumption, we have

$$\begin{aligned} \psi(d_{n+1}) &= \psi(\eta(y_{n+1}, y_{n+2})) \\ &= \psi(\eta(Ty_n, Ty_{n+1})) \\ &\leq F\Big(\psi(\mathcal{M}(y_n, y_{n+1})) - \psi(dist(D_i, D_{i+1})), \\ &\quad \varphi(\mathcal{M}(y_n, y_{n+1})) - \varphi(dist(D_i, D_{i+1}))\Big) + \psi(dist(D_i, D_{i+1})), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \mathcal{M}(y_n, y_{n+1}) &= \max\{\eta(y_n, y_{n+1}), \rho(y_n, Ty_n), \eta(y_{n+1}, Ty_{n+1}), \\ &\quad \frac{1}{2}[\eta(y_n, Ty_{n+1}) + \eta(y_{n+1}, Ty_n)]\} \\ &= \max\{\eta(y_n, y_{n+1}), \eta(y_{n+1}, y_{n+2})\}. \end{aligned}$$

Assume that there exists $n_0 \in N$ such that $\mathcal{M}(y_{n_0}, y_{n_0+1}) = \eta(y_{n_0+1}, y_{n_0+2})$. From $\eta(y_{n_0+1}, y_{n_0+2}) > \eta(y_{n_0}, y_{n_0+1})$, we have

$$\begin{aligned} &\psi(\eta(y_{n_0+1}, y_{n_0+2})) \\ &\quad \leq F\Big(\psi(\eta(y_{n_0+1}, y_{n_0+2})) - \psi(dist(D_i, D_{i+1})), \\ &\quad\quad \varphi(\eta(y_{n_0+1}, y_{n_0+2})) - \varphi(dist(D_i, D_{i+1}))\Big) + \psi(dist(D_i, D_{i+1})) \\ &\quad \leq \psi(\eta(y_{n_0+1}, y_{n_0+2})). \end{aligned}$$

This implies that

$$\psi(\eta(y_{n_0+1}, y_{n_0+2})) - \psi(dist(D_i, D_{i+1})) = 0$$

or

$$\varphi(\eta(y_{n_0+1}, y_{n_0+2})) - \varphi(dist(D_i, D_{i+1})) = 0,$$

which is a contradiction. Hence, for $\forall n \in N$

$$\mathcal{M}(y_n, y_{n+1}) = \eta(y_n, y_{n+1}).$$

Then the expression (4.3) turns into

$$\begin{aligned}\psi(\eta(y_{n+1},y_{n+2})) \leq F\Big(&\psi(\eta(y_n,y_{n+1}))-\psi(dist(D_i,D_{i+1})),\\ &\varphi(\eta(y_n,y_{n+1}))-\varphi(dist(D_i,D_{i+1}))\Big)+\psi(dist(D_i,D_{i+1}))\\ &\leq \psi(\eta(y_n,y_{n+1})).\end{aligned} \tag{4.4}$$

Therefore,

$$\eta(y_{n+1},y_{n+2}) \leq \eta(y_n,y_{n+1}).$$

That is, the sequence $\{d_n\}$ is nonincreasing and bounded below, it is obvious that $\lim_{n\to\infty} d_n$ exists.

If $d_{n_0}=0$, for some $n_0 \in N$, obviously, $d_n \to 0$ and $dist(D_i,D_{i+1})=0$, that is, $d_n \to dist(D_i,D_{i+1})$.

If $d_n \neq 0$, for $\forall n \in N$. Put $d_n \to \gamma$, thus $\gamma \geq dist(D_i,D_{i+1})$. Since φ is a strictly increasing map, we have $\varphi(\gamma) \geq \varphi(dist(D_i,D_{i+1}))$. From the expression (4.4), we get that

$$\begin{aligned}\psi(\eta(y_n,y_{n+1})) \leq F\Big(&\psi(\eta(y_{n-1},y_n))-\psi(dist(D_i,D_{i+1})),\\ &\varphi(\eta(y_{n-1},y_n))-\varphi(dist(D_i,D_{i+1}))\Big)+\psi(dist(D_i,D_{i+1})),\end{aligned}$$

from which it follows that

$$\begin{aligned}\psi(\gamma) &\leq F\Big(\psi(\gamma)-\psi(dist(D_i,D_{i+1})),\varphi(\gamma)-\varphi(dist(D_i,D_{i+1}))\Big)\\ &\quad+\psi(dist(D_i,D_{i+1}))\\ &\leq \psi(\gamma).\end{aligned}$$

This implies that

$$\psi(\gamma)-\psi(dist(D_i,D_{i+1}))=0$$

or

$$\varphi(\gamma)-\varphi(dist(D_i,D_{i+1}))=0.$$

Therefore, $\gamma = dist(D_i,D_{i+1})$. That is, $d_n \to dist(D_i,D_{i+1})$. The proof is complete. ■

We give the following example in support of our main result.

Example 4.6. Consier the Euclidean plane $X =: \mathbb{R}^2$ with the usual Euclidean metric. Suppose

$$D_1=\{(0,1+x):0\leq x\leq 1\}, D_2=\{(1+x,0):0\leq x\leq 1\},$$
$$D_3=\{(0,-(1+x)):0\leq x\leq 1\} \text{ and } D_4=\{-(1+x,0):0\leq x\leq 1\}.$$

Let $\varphi(t)=\frac{1}{5}t, \psi(t)=t, \forall t\geq 0$ and $F(s,t)=s-t, s,t\geq 0$.

Define $T:\cup_{i=1}^4 D_i \to \cup_{i=1}^4 D_i$ by

$$T(0,1+x)=(1+\frac{x}{10},0)$$

$$T(1+x,0)=(0,-(1+\frac{x}{10}))$$

$$T(0,-(1+x))=(-(1+\frac{x}{10}),0)$$

$$T(-(1+x),0) = (0, 1+\frac{x}{10}).$$

Clearly $dist(D_1, D_2) = dist(D_2, D_3) = dist(D_3, D_4) = dist(D_4, D_1) = \sqrt{2}$. Since $T(D_1) \subseteq D_2, T(D_2) \subseteq D_3, T(D_3) \subseteq D_4$ and $T(D_4) \subseteq D_1$, obviously T is a 4-cyclic map. It is not hard to see that, if $x \in D_i, y \in D_i + 1, i = 1, 2, 3, 4$, we have

$$\begin{aligned} F\Big(&\psi(\mathcal{M}(x,y)) - \psi(dist(D_i, D_{i+1})), \varphi(\mathcal{M}(x,y)) - \varphi(dist(D_i, D_{i+1}))\Big) \\ &+ \psi(dist(D_i, D_{i+1})) - \psi(\eta(Tx, Ty)) \\ &= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) + \varphi(dist(D_i, D_{i+1}))\big) - \psi(\eta(Tx, Ty)) \\ &= \mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) + \varphi(dist(D_i, D_{i+1}))\big) - \eta(Tx, Ty) \\ &= \frac{4}{5}\mathcal{M}(x,y) + \frac{\sqrt{2}}{5} - \sqrt{(1+\frac{x}{10})^2 + ((1+\frac{y}{10})^2)} \geq 0, \end{aligned}$$

for all $x \in D_i, y \in D_{i+1}$, where

$$\mathcal{M}(x,y) = \max\{\eta(x,y), \eta(x,Tx), \eta(y,Ty), \frac{1}{2}[\eta(x,Ty) + \eta(y,Tx)]\}.$$

Hence,

$$\begin{aligned} \psi(\eta(Tx,Ty)) \leq F\Big(&\psi(\mathcal{M}(x,y)) - \psi(dist(D_i, D_{i+1})), \varphi(\mathcal{M}(x,y)) - \varphi(dist(D_i, D_{i+1}))\Big) \\ &+ \psi(dist(D_i, D_{i+1})), \end{aligned}$$

for all $x \in D_i, y \in D_{i+1}$, where

$$\mathcal{M}(x,y) = \max\{\eta(x,y), \eta(x,Tx), \eta(y,Ty), \frac{1}{2}[\eta(x,Ty) + \eta(y,Tx)]\}.$$

Therefore, T is a **generalized p-cyclic weak (F, ψ, φ)-contraction** (4.1), where $p = 4$. All the conditions of Theorem 4.5 hold true, and T has the best proximity point. Here $(0,1), (1,0), (0,-1)$ and $(-1,0)$ are unique best proximity point of T in D_1, D_2, D_3 and D_4 respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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