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Interpolative Rational Type Contractions in Bipolar Metric Spaces

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Abstract The main objective of this study is to introduce an extended interpolative rational contractions in bipolar metric spaces. We shall prove certain fixed point theorems in bipolar metric spaces. Our results considerably improve several well-known outcomes in the present literature. Some examples are also provided to illustrate the theorems.

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Keywords: interpolative contractions; rational type contractions; bipolar metric spaces

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1. Introduction

Fixed point theory is a powerful and fruitful instrument. It is an interdisciplinary branch of mathematics that is capable of being utilized in a variety of mathematical and non-mathematical fields, including game theory, mathematical economics, optimization problems, approximation theory, initial and boundary value problems in ordinary and partial differential equations, variational inequalities, biology, chemistry, physics, engineering, and others. In 1922, the Polish mathematician Stefan Banach [1] published the most significant finding in fixed point theory, which motivated many researchers. Indeed, he established a theorem that ensures a unique fixed point for any contraction mapping in the entire metric space. The Banach contraction principle or the Banach fixed point theorem is the result. Furthermore, a constructive demonstration of the Banach fixed point theorem is intriguing since it offers one of the iterative methods for determining a fixed point. Because discoveries of space and its features are always interesting to mathematicians, various academics have generalized the metric space structure by either weakening its fundamental qualities or adjusting the measure's domains and range. Mutlu et al. [2] also improved the metric space structure by modifying the domain definition of the function to incorporate the distance between points of two different sets rather than just one set. The concept is known as a bipolar metric space, and several fixed point theorems

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of metric spaces, which include the Banach contraction principle, are as well extended to the notion's settings (see [6–11] and references therein).

The interpolative contraction was proposed by [12]. In the setting of interpolation, there is a well-known Kannan type contraction. Following that, several renowned contractions (Ćirić[13], Reich [14], Rus [15], Hardy-Rogers[16], Kannan [17], Bianchini [18]) are returned in this new context, see [19–21].

In this paper, we involve all of these ideas and trends in the literature to create more general results on the topic in the literature. Interpolative contractions involving various rational forms appear, producing a fixed point in the context of bipolar metric spaces.

2. Preliminaries

Definition 2.1. [2] Let $\mathcal{H}, \mathcal{K} \neq \emptyset$ and $d : \mathcal{H} \times \mathcal{K} \to [0, \infty)$ be a function. d is called a bipolar metric on pair $(\mathcal{H}, \mathcal{K})$, if the following properties are satisfied

(b0) if $d(\mu, \vartheta) = 0$, then $\mu = \vartheta$;
(b1) if $\mu = \vartheta$, then $d(\mu, \vartheta) = 0$;
(b2) if $\mu, \vartheta \in \mathcal{H} \cap \mathcal{K}$, then $d(\mu, \vartheta) = d(\vartheta, \mu)$;
(b3) $d(\mu_1, \vartheta_2) \leq d(\mu_1, \vartheta_1) + d(\mu_2, \vartheta_1) + d(\mu_2, \vartheta_2)$
for all (μ, ϑ), (μ_1, ϑ_1), $(\mu_2, \vartheta_2) \in \mathcal{H} \times \mathcal{K}$.

Then the triple $(\mathcal{H}, \mathcal{K}, d)$ is called a bipolar metric space.

Definition 2.2. [2] Let $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ be pairs of sets and given a function $\mathcal{Q} : \mathcal{H}_1 \cup \mathcal{K}_1 \to \mathcal{H}_2 \cup \mathcal{K}_2$.

(i) If $\mathcal{Q}(\mathcal{H}_1) \subseteq \mathcal{H}_2$ and $\mathcal{Q}(\mathcal{K}_1) \subseteq \mathcal{K}_2$, then $\mathcal{Q}$ is called a covariant map from $(\mathcal{H}_1, \mathcal{K}_1)$ to $(\mathcal{H}_2, \mathcal{K}_2)$ and denoted this with $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1) \rightrightarrows (\mathcal{H}_2, \mathcal{K}_2)$.

(ii) If $\mathcal{Q}(\mathcal{H}_1) \subseteq \mathcal{K}_2$ and $\mathcal{Q}(\mathcal{K}_1) \subseteq \mathcal{H}_2$, then $\mathcal{Q}$ is called a contravariant map from $(\mathcal{H}_1, \mathcal{K}_1)$ to $(\mathcal{H}_2, \mathcal{K}_2)$ and denoted this $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1) \rightleftarrows (\mathcal{H}_2, \mathcal{K}_2)$.

If d_1 and d_2 are bipolar metrics on $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$, respectively, we also use the notations.

$$\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightrightarrows (\mathcal{H}_2, \mathcal{K}_2, d_2) \quad \text{and} \quad \mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightleftarrows (\mathcal{H}_2, \mathcal{K}_2, d_2).$$

Definition 2.3. [2] Let $(\mathcal{H}, \mathcal{K}, d)$ be a bipolar metric space.

(i) A point $\rho \in \mathcal{H} \cup \mathcal{K}$ is called a left point if $\rho \in \mathcal{H}$, a right point if $\rho \in \mathcal{K}$ and a central point if it is both left and right point.

(ii) A sequence $\{\mu_n\}$ on the set $\mathcal{H}$ is called a left sequence and a sequence $\{\vartheta_n\}$ on $\mathcal{K}$ is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

(iii) A sequence $\{\mu_n\}$ is called convergent to a point ρ, if $\{\mu_n\}$ is a left sequence, ρ is a right point and $\lim_{n\to\infty} d(\mu_n, \rho) = 0$ or $\{\mu_n\}$ is a right sequence, ρ is a left point and $\lim_{n\to\infty} d(\rho, \mu_n) = 0$. A bisequence $\{(\mu_n, \vartheta_n)\}$ on $(\mathcal{H}, \mathcal{K}, d)$ is a sequence on the set $\mathcal{H} \times \mathcal{K}$. If the sequences $\{\mu_n\}$ and $\{\vartheta_n\}$ are convergent, then the bisequence $\{(\mu_n, \vartheta_n)\}$ is called convergent, and if $\{\mu_n\}$ and $\{\vartheta_n\}$ converge to a common point, then $\{(\mu_n, \vartheta_n)\}$ is called biconvergent.

(iv) $\{(\mu_n, \vartheta_n)\}$ is a Cauchy bisequence, if $\lim_{n,m\to\infty} d(\mu_n, \vartheta_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

(v) A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.4. [2] Let $(\mathcal{H}_1, \mathcal{K}_1, d_1)$ and $(\mathcal{H}_2, \mathcal{K}_2, d_2)$ be bipolar metric spaces.

(i) A map $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightrightarrows (\mathcal{H}_2, \mathcal{K}_2, d_2)$ is called left-continuous at a point $\mu_0 \in \mathcal{H}_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_1(\mu_0, \vartheta) < \delta, d_2(\mathcal{Q}(\mu_0), \mathcal{Q}(\vartheta)) < \varepsilon \text{ as } \vartheta \in \mathcal{K}_1.$$

(ii) A map $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightrightarrows (\mathcal{H}_2, \mathcal{K}_2, d_2)$ is called right-continuous at a point $\vartheta_0 \in \mathcal{K}_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_1(\mu, \vartheta_0) < \delta, d_2(\mathcal{Q}(\mu), \mathcal{Q}(\vartheta_0)) < \varepsilon \text{ as } \mu \in \mathcal{H}_1.$$

(iii) A map $\mathcal{Q}$ is called continuous, if it is left-continuous at each point $\mu \in \mathcal{H}_1$ and right-continuous at each point $\vartheta \in \mathcal{K}_1$.

(iv) A contravariant map $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightleftarrows (\mathcal{H}_2, \mathcal{K}_2, d_2)$ is called continuous if it is continuous as a covariant map $\mathcal{Q} : (\mathcal{H}_1, \mathcal{K}_1, d_1) \rightrightarrows (\mathcal{K}_2, \mathcal{H}_2, d_2)$.

Definition 2.5. [3] For a nonempty set $\mathcal{H}$, let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ and $\omega : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be given mappings. We say that $\mathcal{Q}$ is ω-admissible, if for all $\mu, \vartheta \in \mathcal{H}$ we have $\omega(\mu, \vartheta) \geq 1$ implies $\omega(\mathcal{Q}\mu, \mathcal{Q}\vartheta) \geq 1$.

Definition 2.6. [4] Let $\mathcal{Q} : (\mathcal{H}, \mathcal{K}) \rightleftarrows (\mathcal{H}, \mathcal{K})$ and $\omega : \mathcal{H} \times \mathcal{K} \to [0, \infty)$. Then $\mathcal{Q}$ is called ω-admissible (contravariant) if for $\omega(\mu, \vartheta) \geq 1$,

$$\omega(\mathcal{Q}\vartheta, \mathcal{Q}\mu) \geq 1 \text{ for all } \mu \in \mathcal{H} \text{ and } \vartheta \in \mathcal{K}.$$

Definition 2.7. [5] Let $\omega : \mathcal{H} \times \mathcal{K} \to [0, \infty)$ be a mapping. A contravariant mapping $\mathcal{Q} : \mathcal{H} \cup \mathcal{K} \rightleftarrows \mathcal{H} \cup \mathcal{K}$ is said to be ω-orbital admissible if

$$\omega(\mu, \mathcal{Q}\mu) \geq 1 \Rightarrow \omega(\mathcal{Q}^2\mu, \mathcal{Q}\mu) \geq 1 \tag{2.1}$$

and

$$\omega(\mathcal{Q}\vartheta, \vartheta) \geq 1 \Rightarrow \omega(\mathcal{Q}\vartheta, \mathcal{Q}^2\vartheta) \geq 1, \tag{2.2}$$

for all $(\mu, \vartheta) \in \mathcal{H} \times \mathcal{K}$.

Definition 2.8. Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is nondecreasing.

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

3. Main Results

Definition 3.1. Let $(\mathcal{H}, \mathcal{K}, d)$ be a complete bipolar metric space and $\mathcal{Q} : \mathcal{H} \cup \mathcal{K} \rightleftarrows \mathcal{H} \cup \mathcal{K}$ be a contravariant mapping. Then $\mathcal{Q}$ is said to be a ω-interpolative rational type contravariant contraction if there exists $\psi \in \Psi, \omega : \mathcal{H} \times \mathcal{K} \to [0, \infty)$ such that

$$\begin{aligned}\omega(\mu, \vartheta) d(\mathcal{Q}\vartheta, \mathcal{Q}\mu) \leq \psi\Big(& [d(\mu, \vartheta)]^{\theta_1} [d(\mu, \mathcal{Q}\mu)]^{\theta_2} [d(\vartheta, \mathcal{Q}\vartheta)]^{\theta_3} \\ & \times \left[\frac{d(\mu, \mathcal{Q}\mu) d(\vartheta, \mathcal{Q}\vartheta)}{d(\mu, \vartheta)}\right]^{\theta_4} \\ & \times \left[\frac{d(\mu, \mathcal{Q}\mu) d(\mu, \mathcal{Q}\vartheta) + d(\vartheta, \mathcal{Q}\vartheta) d(\vartheta, \mathcal{Q}\mu)}{d(\mu, \mathcal{Q}\vartheta) + d(\vartheta, \mathcal{Q}\mu)}\right]^{\theta_5} \Big),\end{aligned} \tag{3.1}$$

where $\theta_i \geq 0,\ i = 1, 2, 3, 4, 5$ are such that $\sum_{i=1}^{5} \theta_i = 1$ for all $\mu, \vartheta \in \mathcal{H} \times \mathcal{K}$ which $\mu, \vartheta \notin \mathcal{F}ix(\mathcal{Q}) = \{\rho \in \mathcal{H} \cup \mathcal{K} : \mathcal{Q}\rho = \rho\}$.

Theorem 3.2. *Let $(\mathcal{H}, \mathcal{K}, d)$ be a complete bipolar metric space and $\mathcal{Q} : \mathcal{H} \cup \mathcal{K} \rightleftarrows \mathcal{H} \cup \mathcal{K}$ be a ω-interpolative rational type contravariant contraction satisfying the followings:*

(c1) *$\mathcal{Q}$ is ω-orbital admissible;*
(c2) *there exists $\mu_0 \in \mathcal{H}$ such that $\omega(\mu_0, \mathcal{Q}\mu_0) \geq 1$;*
(c3) *$\mathcal{Q}$ is continuous.*

Then $\mathcal{Q}$ has a fixed point.

Proof. Let $\mu_0 \in \mathcal{H}$ such that $\omega(\mu_0, \mathcal{Q}\mu_0) \geq 1$. We define a bisequence $\{(\mu_n, \vartheta_n)\}$ as $\mu_{n+1} = \mathcal{Q}\vartheta_n$ and $\vartheta_n = \mathcal{Q}\mu_n$ for all $n \in \mathbb{N}$. Because $\mathcal{Q}$ is ω-orbital admissible, from equation (2.1), we get.

$$\omega(\mu_0, \vartheta_0) = \omega(\mu_0, \mathcal{Q}\mu_0) \geq 1.$$

This implies that

$$\omega(\mathcal{Q}^2\mu_0, \mathcal{Q}\mu_0) = \omega(\mu_1, \vartheta_0) \geq 1. \tag{3.2}$$

Equation (3.2) implies that

$$\omega(\mu_1, \vartheta_0) = \omega(\mathcal{Q}\vartheta_0, \vartheta_0) \geq 1. \tag{3.3}$$

From equation (2.2) and (3.3) we get

$$\omega(\mathcal{Q}\vartheta_0, \vartheta_0) \Rightarrow \omega(\mathcal{Q}\vartheta_0, \mathcal{Q}^2\vartheta_0) \geq 1. \tag{3.4}$$

This implies that

$$\omega(\mu_1, \vartheta_1) \geq 1. \tag{3.5}$$

In the same way, we get

$$\omega(\mu_n, \vartheta_n) \geq 1 \text{ and } \omega(\mu_{n+1}, \vartheta_n) \geq 1. \tag{3.6}$$

Using equation (3.1), (3.6) and Definition 2.8, let $\mu = \mu_n$ and $\vartheta = \vartheta_{n-1}$. Then we have

$$\begin{aligned}
d(\mu_n, \vartheta_n) &= d(\mathcal{Q}\vartheta_{n-1}, \mathcal{Q}\mu_n) \\
&\leq \omega(\mu_n, \vartheta_{n-1}) d(\mathcal{Q}\vartheta_{n-1}, \mathcal{Q}\mu_n) \\
&\leq \psi\Big([d(\mu_n, \vartheta_{n-1})]^{\theta_1} [d(\mu_n, \vartheta_n)]^{\theta_2} [d(\vartheta_{n-1}, \mu_n)]^{\theta_3} \\
&\quad \times \left[\frac{d(\mu_n, \vartheta_n) d(\vartheta_{n-1}, \mu_n)}{d(\mu_n, \vartheta_{n-1})} \right]^{\theta_4} \\
&\quad \times \left[\frac{d(\mu_n, \vartheta_n) d(\mu_n, \mu_n) + d(\vartheta_{n-1}, \mu_n) d(\vartheta_{n-1}, \vartheta_n)}{d(\mu_n, \mu_n) + d(\vartheta_{n-1}, \vartheta_n)} \right]^{\theta_5} \Big), \\
&\leq \psi([d(\mu_n, \vartheta_{n-1})]^{\theta_1} [d(\mu_n, \vartheta_n)]^{\theta_2} [d(\vartheta_{n-1}, \mu_n)]^{\theta_3} \\
&\quad \times [d(\mu_n, \vartheta_n)]^{\theta_4} [d(\vartheta_{n-1,\mu_n})]^{\theta_5}) \\
&\leq \psi([d(\mu_n, \vartheta_{n-1})]^{\theta_1+\theta_3+\theta_5} [d(\mu_n, \vartheta_n)]^{\theta_2+\theta_4}) \\
&\leq [d(\mu_n, \vartheta_{n-1})]^{\theta_1+\theta_3+\theta_5} [d(\mu_n, \vartheta_n)]^{\theta_2+\theta_4}.
\end{aligned} \tag{3.7}$$

This implies that

$$[d(\mu_n, \vartheta_n)]^{1-\theta_2-\theta_4} \leq [d(\mu_n, \vartheta_{n-1})]^{\theta_1+\theta_3+\theta_5}. \tag{3.8}$$

Hence we have

$$[d(\mu_n, \vartheta_n)]^{1-\theta_2-\theta_4} \leq [d(\mu_n, \vartheta_{n-1})]^{1-\theta_2-\theta_4}. \tag{3.9}$$

From equation (3.9) implies that

$$d(\mu_n, \vartheta_n) \leq d(\mu_n, \vartheta_{n-1}). \tag{3.10}$$

Now, by using equation (3.10), we obtain

$$\begin{aligned}
[d(\mu_n, \vartheta_{n-1})]^{\theta_1+\theta_3+\theta_5} [d(\mu_n, \vartheta_n)]^{\theta_2+\theta_4} &\leq [d(\mu_n, \vartheta_{n-1})]^{\theta_1+\theta_3+\theta_5} [d(\mu_n, \vartheta_{n-1})]^{\theta_2+\theta_4} \\
&= d(\mu_n, \vartheta_{n-1}).
\end{aligned} \tag{3.11}$$

Using equation (3.7) and (3.11), we get

$$d(\mu_n, \vartheta_n) \leq \psi(d(\mu_n, \vartheta_{n-1})). \tag{3.12}$$

Using Definition 2.8 and equation (3.6) let $\mu = \mu_n$ and $\vartheta = \vartheta_n$. Then we have

$$\begin{aligned} d(\mu_{n+1}, \vartheta_n) &= d(\mathcal{Q}\vartheta_n, \mathcal{Q}\mu_n) \\ &\leq \omega(\mu_n, \vartheta_n) d(\mathcal{Q}\vartheta_n, \mathcal{Q}\mu_n) \\ &\leq \psi\Big([d(\mu_n, \vartheta_n)]^{\theta_1} [d(\mu_n, \vartheta_n)]^{\theta_2} [d(\vartheta_n, \mu_{n+1})]^{\theta_3} \\ &\quad \times \left[\frac{d(\mu_n, \vartheta_n) d(\vartheta_n, \mu_{n+1})}{d(\mu_n, \vartheta_n)}\right]^{\theta_4} \\ &\quad \times \left[\frac{d(\mu_n, \vartheta_n) d(\mu_n, \mu_{n+1}) + d(\vartheta_n, \mu_{n+1}) d(\vartheta_n, \vartheta_n)}{d(\mu_n, \mu_{n+1}) + d(\vartheta_n, \vartheta_n)}\right]^{\theta_5}\Big) \\ &\leq \psi([d(\mu_n, \vartheta_n)]^{\theta_1} [d(\mu_n, \vartheta_n)]^{\theta_2} [d(\vartheta_n, \mu_{n+1})]^{\theta_3} \\ &\quad \times [d(\mu_{n+1}, \vartheta_n)]^{\theta_4} [d(\vartheta_n, \mu_n)]^{\theta_5}) \\ &\leq \psi([d(\mu_n, \vartheta_n)]^{\theta_1+\theta_2+\theta_5} [d(\mu_{n+1}, \vartheta_n)]^{\theta_3+\theta_4}) \\ &\leq [d(\mu_n, \vartheta_n)]^{\theta_1+\theta_2+\theta_5} [d(\mu_{n+1}, \vartheta_n)]^{\theta_3+\theta_4}. \end{aligned} \tag{3.13}$$

This implies that

$$\begin{aligned} [d(\mu_{n+1}, \vartheta_n)]^{1-\theta_3-\theta_4} &\leq [d(\mu_n, \vartheta_n)]^{\theta_1+\theta_2+\theta_5} \\ &= [d(\mu_n, \vartheta_n)]^{1-\theta_3-\theta_4}. \end{aligned}$$

Hence we have

$$d(\mu_{n+1}, \vartheta_n) \leq d(\mu_n, \vartheta_n). \tag{3.14}$$

Again, using equation (3.13), we get

$$\begin{aligned} [d(\mu_n, \vartheta_n)]^{\theta_1+\theta_2+\theta_5} [d(\mu_{n+1}, \vartheta_n)]^{\theta_3+\theta_4} &\leq [d(\mu_n, \vartheta_n)]^{\theta_1+\theta_2+\theta_5} [d(\mu_n, \vartheta_n)]^{\theta_3+\theta_4} \\ &= d(\mu_n, \vartheta_n). \end{aligned} \tag{3.15}$$

From equation (3.13) and (3.15), we get

$$d(\mu_{n+1}, \vartheta_n) \leq \psi(d(\mu_n, \vartheta_n)). \tag{3.16}$$

From equation (3.10) and (3.16), we get

$$d(\mu_n, \vartheta_n) \leq \psi^n(d(\mu_1, \vartheta_0)) \text{ and } d(\mu_{n+1}, \vartheta_n) \leq \psi^{n+1}(d(\mu_0, \vartheta_0)), \tag{3.17}$$

for all $n \in \mathbb{N}$. Using Definition 2.8, it is clear that there exists $\epsilon > 0$ and $N(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq N(\epsilon)} \psi^n(d(\mu_1, \vartheta_0)) \leq \frac{\epsilon}{2} \text{ and } \sum_{n \geq N(\epsilon)} \psi^{n+1}(d(\mu_0, \vartheta_0)) \leq \frac{\epsilon}{2}. \tag{3.18}$$

For $n, m \in \mathbb{N}$ with $m > n > N(\epsilon)$, and using property (b3) of Definition 2.1, we get

$$\begin{aligned} d(\mu_n, \vartheta_m) &\leq d(\mu_n, \vartheta_n) + d(\mu_{n+1}, \vartheta_n) + d(\mu_{n+1}, \vartheta_{n+1}) \\ &\quad + \cdots + d(\mu_m, \vartheta_{m-1}) + d(\mu_m, \vartheta_m) \\ &= \sum_{k=n}^{m} d(\mu_k, \vartheta_k) + \sum_{k=n}^{m-1} d(\mu_{k+1}, \vartheta_k). \end{aligned}$$

Using equation (3.17), we get

$$\begin{aligned} d(\mu_n, \vartheta_m) &\leq \sum_{k=n}^{m} d(\mu_k, \vartheta_k) + \sum_{k=n}^{m-1} d(\mu_{k+1}, \vartheta_k) \\ &\leq \sum_{k=n}^{m} \psi^k(d(\mu_1, \vartheta_0)) + \sum_{k=n}^{m-1} \psi^{k+1}(d(\mu_0, \vartheta_0)) \\ &\leq \sum_{n\geq N(\epsilon)} \psi^n(d(\mu_1, \vartheta_0)) + \sum_{n\geq N(\epsilon)} \psi^{n+1}(d(\mu_0, \vartheta_0)). \end{aligned}$$

Hence we have

$$d(\mu_n, \vartheta_m) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.19}$$

For $n, m \in \mathbb{N}$ with $n > m > N(\epsilon)$, and using property (b3) of Definition 2.1, we get

$$\begin{aligned} d(\mu_n, \vartheta_m) &\leq d(\mu_n, \vartheta_{n-1}) + d(\mu_{n-1}, \vartheta_{n-1}) + d(\mu_{n-1}, \vartheta_{n-2}) \\ &\quad + \cdots + d(\mu_m, \vartheta_{m-1}) + d(\mu_m, \vartheta_m) \\ &= \sum_{k=m}^{n-1} d(\mu_k, \vartheta_k) + \sum_{k=m}^{n} d(\mu_k, \vartheta_{k-1}). \end{aligned}$$

Using equation (3.17), we get

$$\begin{aligned} d(\mu_n, \vartheta_m) &\leq \sum_{k=m}^{n-1} d(\mu_k, \vartheta_k) + \sum_{k=m}^{n} d(\mu_k, \vartheta_{k-1}) \\ &\leq \sum_{k=m}^{n-1} \psi^k(d(\mu_1, \vartheta_0)) + \sum_{k=m}^{n} \psi^k(d(\mu_0, \vartheta_0)) \\ &\leq \sum_{n\geq N(\epsilon)} \psi^n(d(\mu_1, \vartheta_0)) + \sum_{n\geq N(\epsilon)} \psi^{n+1}(d(\mu_0, \vartheta_0)). \end{aligned}$$

Hence we have

$$d(\mu_n, \vartheta_m) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.20}$$

Thus, $\{(\mu_n, \vartheta_n)\}$ is a Cauchy bisequence. Because $(\mathcal{H}, \mathcal{K}, d)$ is a complete bipolar metric space, $\{(\mu_n, \vartheta_n)\}$ is biconvergent. That is, there exists $\rho \in \mathcal{H} \cap \mathcal{K}$ such that $\mu_n \to \rho$ and $\vartheta_n \to \rho$ as $n \to \infty$. Because $\mathcal{Q}$ is continuous, $\mu_n \to \rho$ implies that $\vartheta_n = \mathcal{Q}\mu_n \to \rho$ and combining this with $\vartheta_n \to \rho$ gives $\mathcal{Q}\rho = \rho$. Hence ρ is the fixed point of $\mathcal{Q}$. ∎

Theorem 3.3. *Let $(\mathcal{H}, \mathcal{K}, d)$ be a complete bipolar metric space and $\mathcal{Q} : \mathcal{H} \cup \mathcal{K} \rightleftarrows \mathcal{H} \cup \mathcal{K}$ be a ω-interpolative rational type contravariant contraction satisfying the followings:*

- (c1) *$\mathcal{Q}$ is ω-orbital admissible;*
- (c2) *there exists $\mu_0 \in \mathcal{H}$ such that $\omega(\mu_0, \mathcal{Q}\mu_0) \geq 1$;*
- (c3) *if $\{(\mu_n, \vartheta_n)\}$ is a bisequence such that $\omega(\mu_n, \vartheta_n) \geq 1$ for all n and $\vartheta_n \to \rho \in \mathcal{H} \cap \mathcal{K}$ as $n \to \infty$, there exists $\{\mu_{n(k)}\}$ in $\{\mu_n\}$ such that*

  $$\omega(\mu_{n(k)}, \rho) \geq 1 \text{ for all } k \geq 1.$$

Then $\mathcal{Q}$ has a fixed point.

Proof. In the same way as proving the Theorem 3.2, we obtain that $\{(\mu_n, \vartheta_n)\}$ is a Cauchy bisequence. Because $(\mathcal{H}, \mathcal{K}, d)$ is a complete bipolar metric space, hence $\{(\mu_n, \vartheta_n)\}$ is biconvergent. That is, there exists $\rho \in \mathcal{H} \cap \mathcal{K}$ such that $\mu_n \to \rho$ and $\vartheta_n \to \rho$ as $n \to \infty$. Let $\mathcal{Q}\rho \neq \rho$. Then $d(\mathcal{Q}\rho, \rho) > 0$. Using equation (3.6) and condition (c3), we obtain $\omega(\mu_{n(k)}, \rho) \geq 1$ for all $k \geq 1$. Because $d(\mu_{n(k)}, \rho) \to 0$, $d(\mu_{n(k)}, \vartheta_{n(k)}) \to 0$ as $k \to \infty$ and $d(\mathcal{Q}\rho, \rho) > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$d(\mu_{n(k)}, \rho) < d(\mathcal{Q}\rho, \rho) \text{ and } d(\mu_{n(k)}, \vartheta_{n(k)}) < d(\mathcal{Q}\rho, \rho). \tag{3.21}$$

Using equation (3.1) and (3.21), we obtain

$$\begin{aligned}
d(\mathcal{Q}\rho, \vartheta_{n(k)}) &= d(\mathcal{Q}\rho, \mathcal{Q}\mu_{n(k)}) \\
&\leq \omega(\mu_{n(k)}, \rho) d(\mathcal{Q}\rho, \mathcal{Q}\mu_{n(k)}) \\
&\leq \psi\Big([d(\mu_{n(k)}, \rho)]^{\theta_1} [d(\mu_{n(k)}, \mathcal{Q}\mu_{n(k)})]^{\theta_2} [d(\rho, \mathcal{Q}\rho)]^{\theta_3} \\
&\quad \times \left[\frac{d(\mu_{n(k)}, \mathcal{Q}\mu_{n(k)}) d(\rho, \mathcal{Q}\rho)}{d(\mu_{n(k)}, \rho)} \right]^{\theta_4} \\
&\quad \times \left[\frac{d(\mu_{n(k)}, \mathcal{Q}\mu_{n(k)}) d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \mathcal{Q}\rho) d(\rho, \mathcal{Q}\mu_{n(k)})}{d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \mathcal{Q}\mu_{n(k)})} \right]^{\theta_5} \Big) \\
&\leq \psi\Big([d(\mu_{n(k)}, \rho)]^{\theta_1} [d(\mu_{n(k)}, \vartheta_{n(k)})]^{\theta_2} [d(\rho, \mathcal{Q}\rho)]^{\theta_3} \\
&\quad \times \left[\frac{d(\mu_{n(k)}, \vartheta_{n(k)}) d(\rho, \mathcal{Q}\rho)}{d(\mu_{n(k)}, \rho)} \right]^{\theta_4} \\
&\quad \times \left[\frac{d(\mu_{n(k)}, \vartheta_{n(k)}) d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \mathcal{Q}\rho) d(\rho, \vartheta_{n(k)})}{d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \vartheta_{n(k)})} \right]^{\theta_5} \Big) \\
&\leq \psi\Big([d(\mathcal{Q}\rho, \rho)]^{\theta_1} [d(\mathcal{Q}\rho, \rho)]^{\theta_2} [d(\rho, \mathcal{Q}\rho)]^{\theta_3} \left[\frac{d(\mathcal{Q}\rho, \rho) d(\rho, \mathcal{Q}\rho)}{d(\mathcal{Q}\rho, \rho)} \right]^{\theta_4} \\
&\quad \times \left[\frac{d(\mathcal{Q}\rho, \rho) d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \mathcal{Q}\rho) d(\rho, \vartheta_{n(k)})}{d(\mu_{n(k)}, \mathcal{Q}\rho) + d(\rho, \vartheta_{n(k)})} \right]^{\theta_5} \Big) \\
&= \psi\Big([d(\mathcal{Q}\rho, \rho)]^{\theta_1} [d(\mathcal{Q}\rho, \rho)]^{\theta_2} [d(\rho, \mathcal{Q}\rho)]^{\theta_3} \, [d(\mathcal{Q}\rho, \rho)]^{\theta_4} \, [d(\mathcal{Q}\rho, \rho)]^{\theta_5} \Big) \\
&= \psi(d(\mathcal{Q}\rho, \rho)) \\
&\leq d(\mathcal{Q}\rho, \rho).
\end{aligned}$$

Letting $k \to \infty$, we obtain $d(\mathcal{Q}\rho, \rho) < d(\mathcal{Q}\rho, \rho)$ which is a contradiction. Hence, $\mathcal{Q}\rho = \rho$. Therefore ρ is the fixed point of $\mathcal{Q}$. ∎

Corollary 3.4. *Let $(\mathcal{H}, \mathcal{K}, d)$ be a complete bipolar metric space and $\mathcal{Q} : \mathcal{H} \cup \mathcal{K} \rightleftarrows \mathcal{H} \cup \mathcal{K}$ be a contravariant mapping. If there exists $\psi \in \Psi$ such that*

$$\begin{aligned}
d(\mathcal{Q}\vartheta, \mathcal{Q}\mu) \leq \psi\Big(&[d(\mu, \vartheta)]^{\theta_1} [d(\mu, \mathcal{Q}\mu)]^{\theta_2} [d(\vartheta, \mathcal{Q}\vartheta)]^{\theta_3} \left[\frac{d(\mu, \mathcal{Q}\mu) d(\vartheta, \mathcal{Q}\vartheta)}{d(\mu, \vartheta)} \right]^{\theta_4} \\
&\times \left[\frac{d(\mu, \mathcal{Q}\mu) d(\mu, \mathcal{Q}\vartheta) + d(\vartheta, \mathcal{Q}\vartheta) d(\vartheta, \mathcal{Q}\mu)}{d(\mu, \mathcal{Q}\vartheta) + d(\vartheta, \mathcal{Q}\mu)} \right]^{\theta_5} \Big),
\end{aligned}$$

where $\theta_i \geq 0,\ i = 1,2,3,4,5,$ *are such that* $\sum_{i=1}^{5}\theta_i = 1$ *for all* $\mu,\vartheta \in \mathcal{H}\times\mathcal{K}$ *which* $\mu,\vartheta \notin \mathcal{F}ix(\mathcal{Q}) = \{\rho \in \mathcal{H}\cup\mathcal{K} : \mathcal{Q}\rho = \rho\}$, *then* $\mathcal{Q}$ *has a fixed point.*

Proof. Putting $\omega(\mu,\vartheta) = 1$ in Theorem 3.2, then we can prove the Corollary 3.4. ∎

Corollary 3.5. *Let* $(\mathcal{H},\mathcal{K},d)$ *be a complete bipolar metric space and* $\mathcal{Q} : \mathcal{H}\cup\mathcal{K} \rightleftarrows \mathcal{H}\cup\mathcal{K}$ *be a contravariant mapping. If there exists* $\gamma \in [0,1)$ *and* $\psi \in \Psi$ *such that*

$$d(\mathcal{Q}\vartheta,\mathcal{Q}\mu) \leq \gamma\Big([d(\mu,\vartheta)]^{\theta_1}[d(\mu,\mathcal{Q}\mu)]^{\theta_2}[d(\vartheta,\mathcal{Q}\vartheta)]^{\theta_3}\left[\frac{d(\mu,\mathcal{Q}\mu)d(\vartheta,\mathcal{Q}\vartheta)}{d(\mu,\vartheta)}\right]^{\theta_4} \times \left[\frac{d(\mu,\mathcal{Q}\mu)d(\mu,\mathcal{Q}\vartheta)+d(\vartheta,\mathcal{Q}\vartheta)d(\vartheta,\mathcal{Q}\mu)}{d(\mu,\mathcal{Q}\vartheta)+d(\vartheta,\mathcal{Q}\mu)}\right]^{\theta_5}\Big),$$

where $\theta_i \geq 0,\ i = 1,2,3,4,5,$ *are such that* $\sum_{i=1}^{5}\theta_i = 1$ *for all* $\mu,\vartheta \in \mathcal{H}\times\mathcal{K}$ *which* $\mu,\vartheta \notin \mathcal{F}ix(\mathcal{Q}) = \{\rho \in \mathcal{H}\cup\mathcal{K} : \mathcal{Q}\rho = \rho\}$, *then* $\mathcal{Q}$ *has a fixed point.*

Proof. Putting $\omega(\mu,\vartheta) = 1$ and $\psi(t) = \gamma t$, then we can prove the Corollary 3.5. ∎

Example 3.6. Let $\mathcal{H} = \{6,7,8\}$ and $\mathcal{K} = \{7,8,9\}$ with $d(\mu,\vartheta) = |\mu-\vartheta|$, for all $(\mu,\vartheta) \in \mathcal{H}\times\mathcal{K}$. It is clear that$(\mathcal{H},\mathcal{K},d)$ is a complete bipolar metric space. Define $\mathcal{Q} : \mathcal{H}\cup\mathcal{K} \rightleftarrows \mathcal{H}\cup\mathcal{K}$ by $\mathcal{Q}\rho = 8$ for all $\rho \in \mathcal{H}\cup\mathcal{K}$. Hence, $\mathcal{Q}$ is continuous. Putting $\omega(\mu,\vartheta) = 1$ for all $(\mu,\vartheta) \in \mathcal{H}\times\mathcal{K}$, then it is easy to verify that $\mathcal{Q}$ is ω-orbital admissible, and $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. From equation (3.1) holds for the above setup. Hence, all the conditions of Theorem 3.2 are satisfied. Thus, $\mathcal{Q}$ has a fixed point. Clearly, 8 is the fixed point of $\mathcal{Q}$. Therefore, Theorem 3.2 is verified.

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