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# A New Phase- and Amplification-Fitted Sixth-Order Explicit RKN Method to Solve Oscillating Systems

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**Abstract** An optimization of the sixth-order explicit Runge-Kutta-Nyström method with six stages derived by El-Mikkawy and Rahmo using the phase-fitted and amplification-fitted techniques with constant step-size is constructed in this paper. The new adapted method integrates exactly the common test: $y'' = -w^2 y$. The local truncation error of the new method is computed, showing that the order of convergence is maintained. The stability analysis is addressed, showing that the developed method is "almost" P-stable. The numerical experiments demonstrate the high performance of the proposed scheme compared to other existing explicit RKN codes with six stages and same order.

**MSC:** 65L05; 65L06
**Keywords:** phase-fitted and amplification-fitted schemes; RKN method; oscillatory systems; initial-value problems

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## 1. Introduction

It is the purpose of this paper to effectively solve the special second-order initial-value problem of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0, \tag{1.1}$$

assuming that their solutions are oscillatory, where $y \in \Re^d$ and $f : \Re \times \Re^d \to \Re^d$ are sufficiently differentiable functions. In recent and past years, the search of new numerical algorithms to effectively solve (1.1) has brought the attention of many researchers due to the great role this problem played in so many areas of applied sciences. To solve (1.1) directly, the class of Runge-Kutta-Nyström (RKN) methods has been largely used. Regarding the widespread use of these methods, some RKN methods of sixth-order with six stages have been developed in [1], [2], and [3]. A lot of adapted RKN methods have been developed, which are of less algebraic order than the method constructed in this paper. To mention a few, we cite those in [4–9]. Recently, Demba et al. [10, 11] derived two new explicit RKN methods trigonometrically adapted for solving the kind of problems in (1.1).

This work aims at the development of a new phase- and amplification-fitted sixth order explicit RKN method with six stages based on the sixth order method of the RKN6(4)6ER pair given in [3] for solving the problem in (1.1). The constructed method solves exactly the test equation $y'' = -w^2 y$. The numerical experiments reveal the effectiveness of the obtained method compared to standard RKN codes of sixth order with six stages.

The remaining part of this paper is organized as follows: the basic theory of explicit RKN methods, the definitions of phase-lag and amplification error, and the definitions regarding the stability analysis are addressed in Section 2. Section 3 is devoted to the construction of the new code, to determine the order and error analysis, and to bring some details about the periodicity property of the derived code. Some numerical examples are presented in Section 4, showing the good performance of the proposed scheme. Comments on the obtained results are given in Section 5, and finally, Section 6 gives a conclusion.

## 2. Fundamental Concepts

### 2.1. Explicit Runge-Kutta-Nyström Methods

An explicit RKN method with $r$ stages is generally expressed by the formulas:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{l=1}^{r} b_l f(x_n + c_l h, Y_l), \tag{2.1}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l f(x_n + c_l h, Y_l), \tag{2.2}$$

$$Y_l = y_n + c_l h y'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), \quad l = 1, \dots, r, \tag{2.3}$$

where as usual, $y_{n+1}$ and $y'_{n+1}$ denote approximations for $y(x_{n+1})$ and $y'(x_{n+1})$, respectively, and the grid points on the integration interval $[x_0, x_N]$ are given by $x_j = x_0 + jh$, $j = 0, 1, \dots, N$, with $h$ a fixed step-size.

The above method may be formulated compactly using the Butcher array in the form

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \end{array}$$

being $A = (a_{ij})_{r\times r}$ a matrix of coefficients, $c = (c_1, c_2, \ldots, c_r)^T$ the vector of stages, and $b = (b_1, b_2, \ldots, b_r)^T$, $d = (d_1, d_2, \ldots, d_r)^T$ two vectors containing the remaining coefficients of the method. For short, this can be denoted as $(c, A, b, d)$.

**Definition 2.1.** ([12]) An explicit Runge-Kutta-Nyström method as given in the equations (2.1)–(2.3) is said to have algebraic order $k$ if at any grid point $x_{n+1}$ it holds

$$\begin{cases} y_{n+1} - y(x_n + h) = O(h^{k+1}), \\ y'_{n+1} - y'(x_n + h) = O(h^{k+1}). \end{cases} \tag{2.4}$$

## 2.2. Analysis of Phase-Lag, Amplification Error and Stability

Applying the RKN method in (2.1)–(2.3) to the test equation $y^{''} = -w^2 y$, the phase-lag, amplification error and the linear stability analysis are derived. In particular, letting $\tilde{h} = (wh)^2$, the approximate solution provided by (2.1)–(2.3) verifies the recurrence equation:

$$L_{n+1} = E(\tilde{h}) L_n,$$

where

$$L_{n+1} = \begin{bmatrix} y_{n+1} \\ h y'_{n+1} \end{bmatrix}, \; L_n = \begin{bmatrix} y_n \\ h y'_n \end{bmatrix}, \; E(\tilde{h}) = \begin{bmatrix} 1 - \tilde{h} b^T N^{-1} e & 1 - \tilde{h} b^T N^{-1} c \\ -\tilde{h} d^T N^{-1} e & 1 - \tilde{h} d^T N^{-1} c \end{bmatrix},$$

$N = I + \tilde{h} A$, with $A = (a_{ij})_{r\times r}$, $b, c, d$ the corresponding matrix and vectors of coefficients, $I$ the identity matrix of dimension $r$, and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^r$.

For enough small values of $\mu = wh$, it can be assumed that the matrix $E(\tilde{h})$ possesses conjugate complex eigenvalues [13]. Under this assumption, an oscillatory numerical solution should be provided by the method. The oscillatory character depends on the eigenvalues of the stability matrix $E(\tilde{h})$. The characteristic equation of this matrix can be expressed as:

$$\lambda^2 - \lambda Tr(E(\tilde{h})) + Det(E(\tilde{h})) = 0. \tag{2.5}$$

**Theorem 2.2.** ([3]) *If we apply to the common test equation $y^{''} = -w^2 y$ the Runge-Kutta-Nyström scheme in (2.1)–(2.3), we get the formula for calculating directly the phase-lag (or dispersion error) $\Psi(\mu)$ given by:*

$$\Psi(\mu) = \mu - cos^{-1}\left(\frac{Tr(E(\tilde{h}))}{2\sqrt{Det(E(\tilde{h}))}}\right). \tag{2.6}$$

If $\Psi(\mu) = O(\mu^{l+1})$, then it is said that the method has a phase-lag of order $l$. For an explicit RKN method, $Tr(E(\tilde{h})$ and $Det(E(\tilde{h})$ are polynomials in $\mu$ (we note that in case of an implicit RKN method these would be rational functions).

**Definition 2.3.** An explicit Runge-Kutta-Nyström method as given in the equations (2.1)−(2.3) is said to be phase-fitted, if the phase-lag is zero.

**Definition 2.4.** ([3]) For the Runge-Kutta-Nyström method given in the equations (2.1)−(2.3), the value $\beta(\mu) = 1 - \sqrt{Det(E(\tilde{h}))}$ is known as the amplification error (or dissipative error). If $\beta(\mu) = O(\mu^{s+1})$, then it is said that the method has an amplification error of order $s$.

**Definition 2.5.** An explicit Runge-Kutta-Nyström method as given in the equations (2.1)−(2.3) is said to be amplification-fitted if the amplification error is zero.

We further study the stability property of the developed method when applied to the test equation, $y'' = -w^2 y$.

**Definition 2.6.** ([3]) The interval $I = (0, \tilde{h}_b)$, $\tilde{h}_b \in \mathbb{R}^+ \cup \{+\infty\}$, so that $\mu \in (0, \tilde{h}_b)$ is called

(1) the interval of stability of the RKN method, if $\tilde{h}_b$ is the highest value for which $|\lambda| < 1$. In this case, if $\tilde{h}_b = \infty$ then the RKN method is called A-stable.

(2) the interval of periodicity of the RKN method, if $\tilde{h}_b$ is the highest value for which $|\lambda| = 1$, and $[tr(E(\tilde{h}))]^2 - 4det(E(\tilde{h})) < 0$ (the eigenvalues are complex conjugate). In this case, if $\tilde{h}_b = \infty$ then the RKN method is called P-stable.

The adapted method developed is "almost" P-stable, but has no interval of stability. Using the Mathematica software we have found that the eigenvalues of the developed method are: $\lambda_1 = e^{-i\sqrt{H}}$, $\lambda_2 = e^{i\sqrt{H}}$, and also the trace and the determinant of the matrix $E(\tilde{h})$ for the developed method are, $tr(E(\tilde{h})) = 2\cos(\sqrt{H})$ and $det(E(\tilde{h})) = 1$. From the very construction we have that $\Psi(\mu) = \mu - \arccos(\frac{2\cos(\mu)}{2\sqrt{1}}) = 0$, $\beta(\mu) = 1 - \sqrt{det(E(\tilde{h}))} = 0$, which implies that the method is dispersive of order infinity and dissipative of order infinity.

**Theorem 2.7.** *The new phase- and amplification-fitted method developed in this paper is "almost" P-stable, that is, the conditions for P-stability are verified except for the set* $\delta := \{v^2 \in \mathbb{R} : v^2 \neq (n\pi)^2, n \in \mathbb{N}\}$.

The theorem can be proved by considering the fact that the adapted method developed here has respectively the following, $tr(E(\tilde{h})) = 2\cos(\mu)$ and $det(E(\tilde{h})) = 1$. According to [14], an adapted method is "almost" P-stable, if and only if for every $\mu > 0$, $det(E(\tilde{h})) = 1$ and $|tr(E(\tilde{h}))| < 2$. Since for the proposed method it is $|tr(E(\tilde{h}))| = |2\cos(v)| < 2$, for $v \neq n\pi, n \in \mathbb{N}$, this gives the desired result.

## 3. Development of the New Scheme

In this section, we will obtain a sixth order explicit phase- and amplification-fitted RKN scheme based on the higher-order method in the RKN6(4)6ER embedded pair derived by El-Mikkawy and Rahmo in [2], which we named as RKN6-6ER. The coefficients of the sixth order RKN method in [2] are shown in Table 1 with the correct value of $a_{54}$ as given in [3].

TABLE 1. Coefficients of the RKN6-6ER method in [2]

| | | | | | | |
|---|---|---|---|---|---|---|
| $0$ | | | | | | |
| $\frac{1}{77}$ | $\frac{1}{11858}$ | | | | | |
| $\frac{1}{3}$ | $-\frac{7189}{17118}$ | $\frac{4070}{8559}$ | | | | |
| $\frac{2}{3}$ | $\frac{4007}{2403}$ | $-\frac{589655}{355644}$ | $\frac{25217}{118548}$ | | | |
| $\frac{13}{15}$ | $-\frac{4477057}{843750}$ | $\frac{13331783894}{2357015625}$ | $-\frac{281996}{5203125}$ | $\frac{563992}{7078125}$ | | |
| $1$ | $\frac{17265}{2002}$ | $-\frac{1886451746}{212088107}$ | $\frac{22401}{31339}$ | $\frac{2964}{127897}$ | $\frac{178125}{5428423}$ | |
| | $-\frac{341}{780}$ | $\frac{386683451}{661053840}$ | $\frac{2853}{11840}$ | $\frac{267}{3020}$ | $\frac{9375}{410176}$ | $0$ |
| | $-\frac{341}{780}$ | $\frac{29774625727}{50240091840}$ | $\frac{8559}{23680}$ | $\frac{801}{3020}$ | $\frac{140625}{820352}$ | $\frac{847}{18240}$ |

In order to get the new adapted scheme, we equate to zero the phase-lag $\Psi(\mu)$ and the amplification error $\beta(\mu)$, and we get the system:

$$\begin{cases} \Psi(\mu) = 0 \\ \beta(\mu) = 0. \end{cases} \tag{3.1}$$

We solve this system considering the coefficients in Table 1 except two of them which are taking as unknowns. Specifically, we take $b_5$ and $d_5$ as unknowns. After solving the system in (3.1) we obtain the following values:

$$\begin{aligned} b_5 \;=\; \frac{2503125}{410176M}\Big( & -12586322337073683034636800000000 + 52499468404370638714802508000 0\,\mu^2 \end{aligned}$$

$$+380278327832939254939061688 00\,\mu^4 - 42305110040020986855472545000\,\mu^6$$

$$+6389496350903753079525017100\,\mu^8 - 396360945814751886526623990\,\mu^{10}$$

$$+12393674919826270714885995\,\mu^{12} - 163757382111950819488686\,\mu^{14} + 443880244626070278520\,\mu^{16}$$

$$-3556135517458913619310080000\,\mu^2\cos(\mu) + 7969295957655526325216985600\,\mu^6\cos(\mu)$$

$$-12571802032109736088632960 0\,\mu^8\cos(\mu) - 742693155585909485806937088 00\,\mu^4\cos(\mu)$$

$$+12586322337073683034636800000000\,\cos(\mu)\Big),$$

$$
\begin{aligned}
d_5 = & \frac{625}{820352M}\Big(488268269088677306372\,0\,\mu^{20} - 1766435438191731348692196\,\mu^{18} \\
& +14267128649887801201534956\,0\,\mu^{16} - 10126226143892166109616015370\,\mu^{14} \\
& +475493904396311527376632326825\,\mu^{12} + 546881973050840782778527104\,00\,\mu^{10}\cos(\mu) \\
& -10391680199125544879555652445650\,\mu^{10} - 103819002965894624674923296640\,00\,\mu^{8}\cos(\mu) \\
& +113086760758089573241298829586500\,\mu^{8} + 2536902040490601057324033984000\,00\,\mu^{6}\cos(\mu) \\
& -38172183245988102106377647719500\,0\,\mu^{6} - 1775893905546681693988359573660000\,\mu^{4} \\
& -6305505579734821871351779238400\,00\,\mu^{4}\cos(\mu) + 3199753041551405164628715874500000\,0\,\mu^{2} \\
& -6721096127997346740496051200000\,00\,\mu^{2}\cos(\mu) + 7561233143997015083058057600000000\,0\cos(\mu) \\
& -7561233143997015083058057600000000\,0\Big),
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
M = \mu^2\Big( & -28803310743425593080234375000000 + 48005517905709321800390625000\,00\,\mu^2 \\
& +24098647278210084739510312500\,0\,\mu^4 - 21157585474732123459365303750\,0\,\mu^6 \\
& +276933794694142245743227927\,50\,\mu^8 - 1543565245575968927989765335\,\mu^{10} \\
& +5515885104849964144936935\,0\,\mu^{12} - 861578557170344748268248\,\mu^{14} \\
& +24413413454433865318\,60\,\mu^{16}\Big).
\end{aligned} \tag{3.3}
$$

The corresponding Taylor series expansions in powers of $\mu$ result in

$$
\begin{aligned}
b_5 = & \frac{9375}{410176} - \frac{261461}{93847723200}\mu^6 + \frac{20361401}{369525410100000}\mu^8 - \frac{177044709462626977}{866977960060782108000000\,0}\mu^{10} \\
& + \frac{1134755857534331292255\,7}{88756868661222568306500000000\,0}\mu^{12} - \frac{1014777911601836484322385\,39}{136685577738282755192010000000000000}\mu^{14} + \cdots, \\
d_5 = & \frac{140625}{820352} - \frac{1}{213290280}\mu^6 - \frac{618923}{739050820200}\mu^8 - \frac{1251344791}{93120403345200000}\mu^{10} \\
& - \frac{1902976380761163252\,19}{7396405721768547358875000000}\mu^{12} + \frac{352769454320927392403167\,9}{99407692900569276503280000000000\,0}\mu^{14} + \cdots.
\end{aligned} \tag{3.4}
$$

As $\mu \to 0$, the newly obtained coefficients $b_5, d_5$ become the coefficients of the counterpart scheme in the original method. The new adapted RKN scheme will be named as PFAFRKN6-6ER.

### 3.1. Order of Convergence

This section is devoted to present the local truncation error of the proposed method and to get the algebraic order of convergence. This is accomplished by using the usual tool of Taylor expansions. The local truncation errors (LTE) at the point $x_1$ of the solution and the first derivative are given respectively by:

$$\begin{aligned} LTE &= y(x_0+h) - y_1, \\ LTE_{der} &= y^{'}(x_0+h) - y_1^{'}. \end{aligned} \tag{3.5}$$

**Proposition 3.1.** *The corresponding LTEs of the formulas to provide the solution and the derivative with the new RKN method are, respectively:*

$$\begin{aligned} LTE &= \tfrac{h^7}{213290280}(f_y)^2(f_x + f_y y') + O(h^8), \\ LTE_{der} &= \tfrac{h^7}{5040}(f_{xxxxxx} + 15(y')^4 f_{yyyy}y'' + 60(y')^3 f_{xyyyy}y'' + 60y' f_{xxxyy}y'' \\ &\quad +90(y')^2 f_{xxyyy}y'' + 21 f_y f_{yxx}y'' + 60y'' f_{xyy}f_x + 15y'' f_{yy}f_{xx} + 18(y'')^2 f_{yy}f_y \\ &\quad +90y' f_{xyyy}(y'')^2 + 45(y')^2 f_{yyyy}(y'')^2 + 33(y')^2 (f_{yy})^2 y'' + 48y' f_{xy}f_{yxx} \\ &\quad +10 f_y f_{xy} f_x + 12(f_y)^2 y' f_{xy} + 60y' f_{xxyy}f_x + 60(y')^2 f_{xyyy}f_x + 20(y')^3 f_{yyyy}f_x \\ &\quad +24 f_y y' f_{xxxy} + 30y' f_{xyy}f_{xx} + 15(y')^2 f_{yyy}f_{xx} + 6y' f_{yy}f_{xxx} + 78(y')^2 f_{xyy}f_{xy} \\ &\quad +66(y')^2 f_y f_{xxyy} + 33(y')^2 f_{yy}f_{yxx} + 64(y')^3 f_y f_{xyyy} + 36(y')^3 f_{yyy}f_{xy} \\ &\quad +48(y')^3 f_{yy}f_{xyy} + 21(f_y)^2(y')^2 f_{yy} + 21(y')^4 f_y f_{yyyy} + 21(y')^4 f_{yyy}f_{yy} \\ &\quad +15(y'')^3 f_{yyy} + 45(y'')^2 f_{xxyy} + 15y'' f_{xxxxy} + 18y''(f_{xy})^2 + (f_y)^3 y'' \\ &\quad +(y')^6 f_{yyyyyy} + 6(y')^5 f_{xyyyyy} + (f_y)^2 f_{xx} + 6 f_{xxx}f_{xy} + f_y f_{xxxx} \\ &\quad +20 f_x f_{xxxy} + 6y' f_{xxxxxy} + 15 f_{yxx}f_{xx} + 15(y')^4 f_{xxyyyy} + 15(y')^2 f_{xxxxyy} \\ &\quad +20(y')^3 f_{xxxyyy} + 10 f_{yy}(f_x)^2 + 81(y')^2 f_{yyy}f_y y'' + 60y' f_{yyy}f_x y'' \\ &\quad +102y' f_y f_{xyy}y'' + 66y' f_{yy}f_{xy}y'' + 30y' f_{yy}f_y f_x) + O(h^8), \end{aligned} \tag{3.6}$$

*from which we can infer that the PFAFRKN6-6ER method has algebraic order six.*

## 4. Some Numerical Examples

To assess the performance of the new scheme, we have considered the following RKN codes of the same order and stages to get fair comparisons:

- PFAFRKN6-6ER: The adapted explicit RKN code developed here,
- RKN6-6ER: An explicit sixth-order six stage RKN method presented in [2],
- RKN6-6ER-PFAF: An optimized explicit sixth-order six stage RKN method derived by Anastassi and Kosti in [3],
- RKN6-6FM: An explicit sixth-order six stage RKN method developed by Dormand et al. in [1].

We will consider different oscillatory problems appeared in the literature to test the performance of the above methods:

**Problem 1**. Homogeneous Problem in [15]

$$y'' = -w^2 y, \; y(0)=1, \; y'(0)=-2, \; x\in[0,4000],$$

with a known solution given by

$$y(x) = -\frac{1}{4}\sin(8x)+\cos(8x).$$

To use the adapted methods we have taken the parameter value $w=8$.

**Problem 2**. Non-homogeneous Problem in [16]

$$y'' = -v^2 y+(v^2-1)\sin x, \; y(0)=1, \; y'(0)=v+1, \; v=10, \; x\in[0,4000],$$

with a known solution given by

$$y(x) = \sin(10x)+\cos(10x)+\sin(x).$$

Now, in the adapted methods we have taken the value $v=w=10$.

**Problem 3**. Non-linear System in [17]

$$\begin{aligned} y_1''+w^2y_1 &= \frac{2y_1y_2-\sin(2wx)}{({y_1}^2+{y_2}^2)^{\frac{3}{2}}}, \; y_1(0)=1, y_1'(0)=0,\\ y_2''+w^2y_2 &= \frac{{y_1}^2-{y_2}^2-\sin(2wx)}{({y_1}^2+{y_2}^2)^{\frac{3}{2}}}, \; y_2(0)=0, y_2'(0)=w, \quad x\in[0,4000], \end{aligned}$$

with a known solution given by

$$\begin{aligned} y_1(x) &= \cos(wx),\\ y_2(x) &= \sin(wx). \end{aligned}$$

To use the adapted methods we have taken the parameter value $w=5$.

**Problem 4**. Non-homogeneous System in [18]

$$\begin{aligned} &y_1'' = -m^2y_1(x)+m^2g(x)+g''(x), \; y_1(0)=b+g(0), \; y_1'(0)=g'(0),\\ &y_2'' = -m^2y_2(x)+m^2g(x)+g''(x), \; y_2(0)=g(0), \; y_2'(0)=mb+g'(0), \; x\in[0,4000], \end{aligned}$$

with a known solution given by

$$\begin{aligned} y_1(x) &= b\cos(mx)+g(x)\\ y_2(x) &= b\sin(mx)+g(x). \end{aligned}$$

For the numerical computations we have taken $w=m=20, g(x)=e^{-0.05x}$, and $b=0.1$.

**Problem 5**. Non-homogeneous Problem in [19]

$$y'' = -25y+100\cos 5x, \; y(0)=1, \; y'(0)=5, \; x\in[0,4000],$$

with a known solution given by

$$y(x) = \sin(5x)+\cos(5x)+10x\sin(5x).$$

Now, in the adapted methods we have taken the value $w = 5$.

We have considered the integration interval $[x_0, x_N]$, with step-length, $h = \frac{x_N - x_0}{N}$, taking different final points, as $x_N = 100, 1000, 4000$. The obtained maximum absolute errors, $Max\, global\, error$, are given in Tables 2 to 6, considering different step-sizes $h$.

Table 2. Maximum absolute errors corresponding to Problem 1

| h | Methods | $x_N = 100$ | $x_N = 1000$ | $x_N = 4000$ |
|---|---|---|---|---|
| 0.05 | PFAFRKN6-6ER | 8.376888(-10) | 5.297163(-9) | 4.047332(-8) |
| | RKN6-6ER | 1.876489(-6) | 1.889563(-5) | 7.556011(-5) |
| | RKN6-6ER-PFAF | 5.667095(-5) | 5.659304(-4) | 2.266264(-3) |
| | RKN6-6FM | 5.240274(-6) | 5.275814(-5) | 2.111010(-4) |
| 0.075 | PFAFRKN6-6ER | 4.061289(-8) | 2.393823(-7) | 9.391945(-7) |
| | RKN6-6ER | 3.238321(-5) | 3.236094(-4) | 1.296214(-3) |
| | RKN6-6ER-PFAF | 3.689401(-2) | 3.396261(-1) | 1.006993(+0) |
| | RKN6-6FM | 6.058584(-5) | 6.084909(-4) | 2.434357(-3) |
| 0.1 | PFAFRKN6-6ER | 9.005675(-7) | 7.208692(-6) | 2.830818(-5) |
| | RKN6-6ER | 2.394757(-4) | 2.436649(-3) | 9.745593(-3) |
| | RKN6-6ER-PFAF | 1.153997(+0) | 1.153997(+0) | 1.153997(+0) |
| | RKN6-6FM | 3.445139(-4) | 3.481070(-3) | 1.392376(-2) |
| 0.125 | PFAFRKN6-6ER | 1.149804(-5) | 1.031363(-4) | 4.085795(-4) |
| | RKN6-6ER | 1.171023(-3) | 1.169972(-2) | 4.616601(-2) |
| | RKN6-6ER-PFAF | 1.070101(+0) | 1.070101(+0) | 1.070101(+0) |
| | RKN6-6FM | 1.342387(-3) | 1.355532(-2) | 5.432584(-2) |

Table 3. Maximum absolute errors corresponding to Problem 2

| h | Methods | $x_N = 100$ | $x_N = 1000$ | $x_N = 4000$ |
|---|---|---|---|---|
| 0.05 | PFAFRKN6-6ER | 6.087944(-9) | 4.514620(-8) | 1.029183(-7) |
| | RKN6-6ER | 1.549647(-5) | 1.547221(-4) | 6.194987(-4) |
| | RKN6-6ER-PFAF | 3.434323(-3) | 3.429801(-2) | 1.353524(-1) |
| | RKN6-6FM | 3.452593(-5) | 4.472916(-4) | 1.390353(-3) |
| 0.075 | PFAFRKN6-6ER | 5.291679(-7) | 5.508185(-6) | 2.220847(-5) |
| | RKN6-6ER | 2.643734(-4) | 2.664580(-3) | 1.063403(-2) |
| | RKN6-6ER-PFAF | 1.370402(+0) | 1.611646(+0) | 1.611646(+0) |
| | RKN6-6FM | 4.034230(-4) | 4.032180(-3) | 1.614168(-2) |
| 0.1 | PFAFRKN6-6ER | 1.730785(-5) | 1.744420(-4) | 6.981273(-4) |
| | RKN6-6ER | 2.008096(-3) | 2.001580(-2) | 7.876678(-2) |
| | RKN6-6ER-PFAF | 1.448751(+0) | 1.448751(+0) | 1.448751(+0) |
| | RKN6-6FM | 2.314834(-3) | 2.329405(-2) | 9.338362(-2) |
| 0.125 | PFAFRKN6-6ER | 2.430470(-4) | 2.523091(-3) | 1.017188(-2) |
| | RKN6-6ER | 9.339244(-3) | 9.430920(-2) | 3.480040(-1) |
| | RKN6-6ER-PFAF | 4.595800(+306) | 4.595800(+306) | 4.595800(+306) |
| | RKN6-6FM | 9.020961(-3) | 9.181146(-2) | 3.698289(-1) |

Table 4. Maximum absolute errors corresponding to Problem 3

| **h** | **Methods** | $x_N$ **= 100** | $x_N$ **= 1000** | $x_N$ **= 4000** |
|---|---|---|---|---|
| 0.05 | PFAFRKN6-6ER | 3.802533(-10) | 2.155096(-9) | 9.277232(-9) |
| | RKN6-6ER | 4.282131(-8) | 1.632977(-7) | 1.632977(-7) |
| | RKN6-6ER-PFAF | 1.899990(-8) | 7.334019(-8) | 7.667858(-8) |
| | RKN6-6FM | 1.953175(-7) | 7.542634(-7) | 7.542634(-7) |
| 0.075 | PFAFRKN6-6ER | 9.475666(-9) | 3.697725(-8) | 3.697725(-8) |
| | RKN6-6ER | 7.249395(-7) | 2.799656(-6) | 2.804762(-6) |
| | RKN6-6ER-PFAF | 1.214422(-5) | 4.709903(-5) | 4.713322(-5) |
| | RKN6-6FM | 2.236286(-6) | 8.631087(-6) | 8.634346(-6) |
| 0.1 | PFAFRKN6-6ER | 9.349917(-8) | 3.600327(-7) | 3.600327(-7) |
| | RKN6-6ER | 5.491464(-6) | 2.106615(-5) | 2.106615(-5) |
| | RKN6-6ER-PFAF | 1.223785(-3) | 4.727541(-3) | 4.727541(-3) |
| | RKN6-6FM | 1.261951(-5) | 4.880468(-5) | 4.881228(-5) |
| 0.125 | PFAFRKN6-6ER | 5.305980(-7) | 2.048570(-6) | 2.048570(-6) |
| | RKN6-6ER | 2.598789(-5) | 1.008581(-4) | 1.008581(-4) |
| | RKN6-6ER-PFAF | 4.319413(-2) | 2.315899(-1) | 2.324977(-1) |
| | RKN6-6FM | 4.804257(-5) | 1.875787(-4) | 1.876650(-4) |

Table 5. Maximum absolute errors corresponding to Problem 4

| **h** | **Methods** | $x_N$ **= 100** | $x_N$ **= 1000** | $x_N$ **= 4000** |
|---|---|---|---|---|
| 0.0125 | PFAFRKN6-6ER | 2.826968(-11) | 2.890083(-9) | 4.628854(-8) |
| | RKN6-6ER | 1.703302(-8) | 1.700487(-7) | 6.883544(-7) |
| | RKN6-6ER-PFAF | 7.477886(-9) | 7.738988(-8) | 2.531203(-7) |
| | RKN6-6FM | 7.573732(-8) | 7.559689(-7) | 3.081923(-6) |
| 0.025 | PFAFRKN6-6ER | 1.149865(-9) | 7.538132(-9) | 9.932046(-9) |
| | RKN6-6ER | 2.182914(-6) | 2.190211(-5) | 8.758873(-5) |
| | RKN6-6ER-PFAF | 4.874585(-4) | 4.831134(-3) | 1.875728(-2) |
| | RKN6-6FM | 4.906952(-6) | 4.912734(-5) | 1.965290(-4) |
| 0.05 | PFAFRKN6-6ER | 2.578029(-6) | 2.484148(-5) | 9.901855(-5) |
| | RKN6-6ER | 2.849681(-4) | 2.818441(-3) | 1.084435(-2) |
| | RKN6-6ER-PFAF | 1.052562(-1) | 1.052562(-1) | 1.052562(-1) |
| | RKN6-6FM | 3.285820(-4) | 3.298875(-3) | 1.320687(-2) |
| 0.075 | PFAFRKN6-6ER | 3.205100(-4) | 3.238187(-3) | 1.347178(-2) |
| | RKN6-6ER | 4.903214(-3) | 4.137844(-2) | 9.520212(-2) |
| | RKN6-6ER-PFAF | 4.561856(+306) | 4.561856(+306) | 4.561856(+306) |
| | RKN6-6FM | 4.036525(-3) | 4.092547(-2) | 1.578012(-1) |

Table 6. Maximum absolute errors corresponding to Problem 5

| h | Methods | $x_N = 100$ | $x_N = 1000$ | $x_N = 4000$ |
|---|---|---|---|---|
| 0.05 | PFAFRKN6-6ER | 2.213611(-7) | 1.740843(-5) | 9.767030(-4) |
| | RKN6-6ER | 2.111063(-5) | 2.131245(-3) | 3.396483(-2) |
| | RKN6-6ER-PFAF | 9.114216(-6) | 9.145261(-4) | 1.585036(-2) |
| | RKN6-6FM | 9.504147(-5) | 9.493622(-3) | 1.507729(-1) |
| 0.075 | PFAFRKN6-6ER | 3.713266(-6) | 2.999952(-5) | 8.333719(-4) |
| | RKN6-6ER | 3.594272(-4) | 3.642403(-2) | 5.833073(-1) |
| | RKN6-6ER-PFAF | 6.123057(-3) | 6.095651(-1) | 9.779085(+0) |
| | RKN6-6FM | 1.088588(-3) | 1.082964(-1) | 1.737337(+0) |
| 0.1 | PFAFRKN6-6ER | 2.970377(-5) | 3.585056(-4) | 2.668772(-3) |
| | RKN6-6ER | 2.687021(-3) | 2.733629(-1) | 4.380288(+0) |
| | RKN6-6ER-PFAF | 6.105609(-1) | 6.076355(+1) | 9.693221(+2) |
| | RKN6-6FM | 6.162659(-3) | 6.127625(-1) | 9.821965(+0) |
| 0.125 | PFAFRKN6-6ER | 1.554099(-4) | 2.540054(-3) | 2.320075(-2) |
| | RKN6-6ER | 1.295125(-2) | 1.302793(+0) | 2.094574(+1) |
| | RKN6-6ER-PFAF | 2.153911(+1) | 1.993890(+3) | 2.502423(+4) |
| | RKN6-6FM | 2.341523(-2) | 2.338821(+0) | 3.782324(+1) |

To show the efficiency of the developed PFAFRKN6-6ER code, we present in Figures 1 to 5 the efficiency curves for the considered problems. For each problem, the logarithm of the maximum absolute global error has been plotted versus the logarithm of the total number of function evaluations. It can be observed the good behavior of the new code.

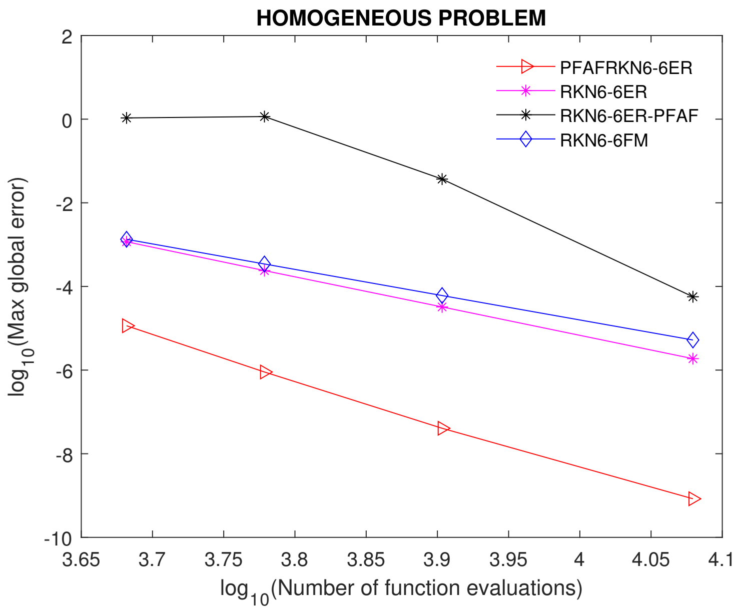


Figure 1. Efficiency curves corresponding to Problem 1

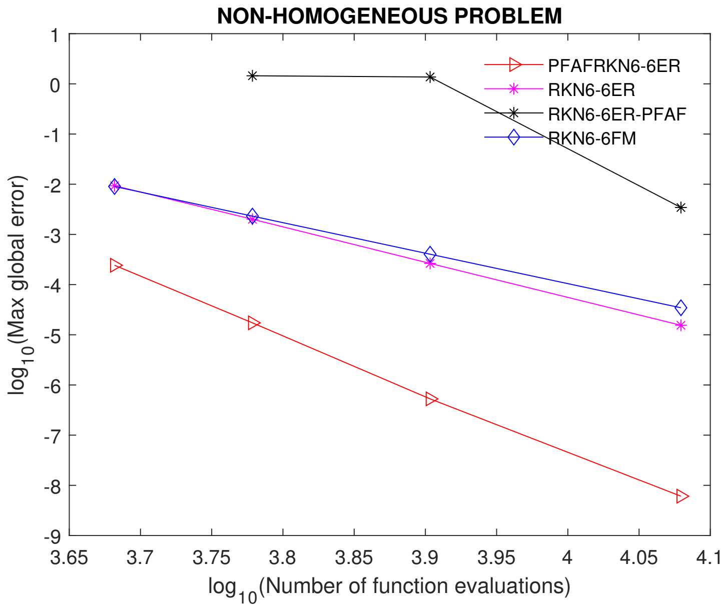


FIGURE 2. Efficiency curves corresponding to Problem 2

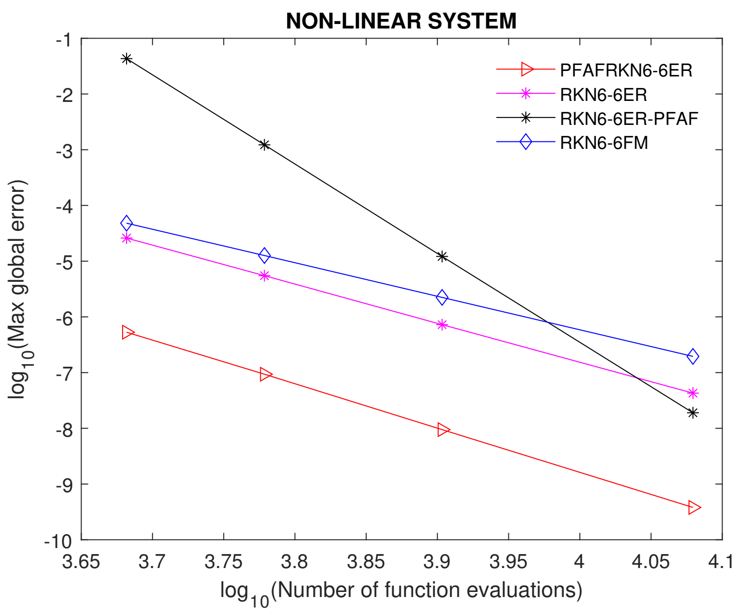


FIGURE 3. Efficiency curves corresponding to Problem 3

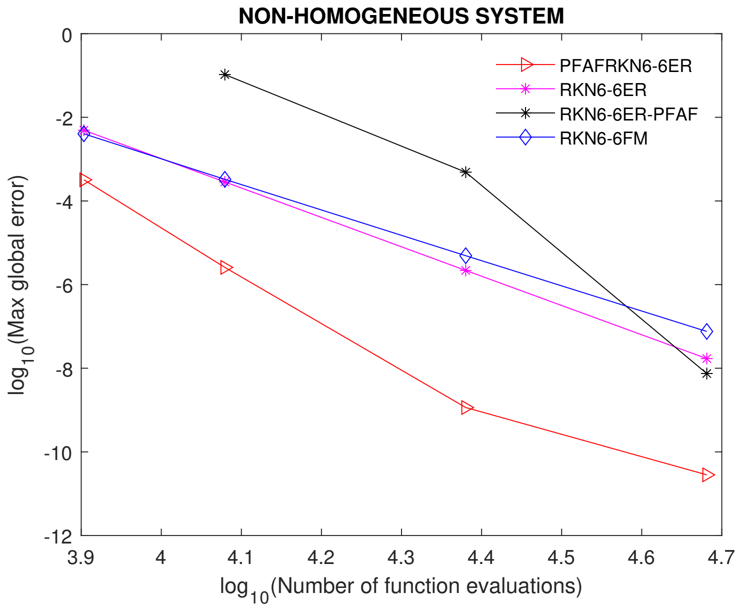


FIGURE 4. Efficiency curves corresponding to Problem 4

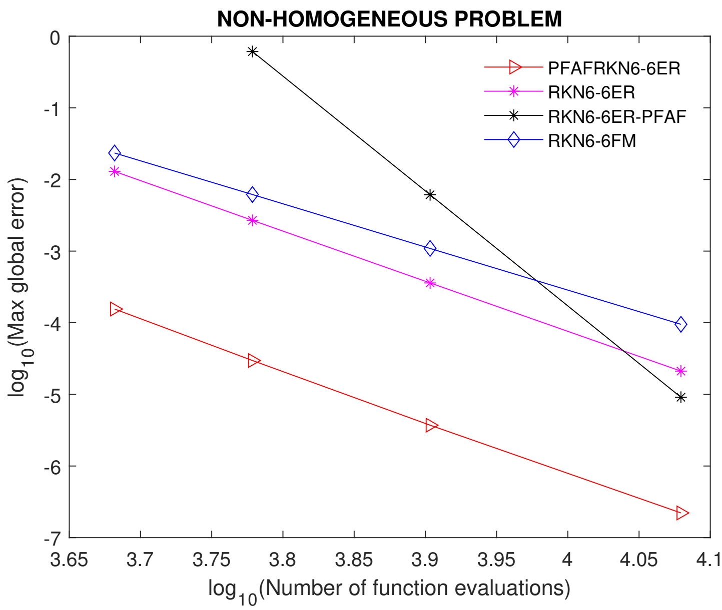


FIGURE 5. Efficiency curves corresponding to Problem 5

To further demonstrate the efficiency of the constructed PFAFRKN6-6ER code, we present in Figures 6 to 10 the efficiency curves for the considered problems. Now, the logarithm of the maximum absolute global error has been plotted versus the CPU time used. It can be observed the good behavior of the new code.

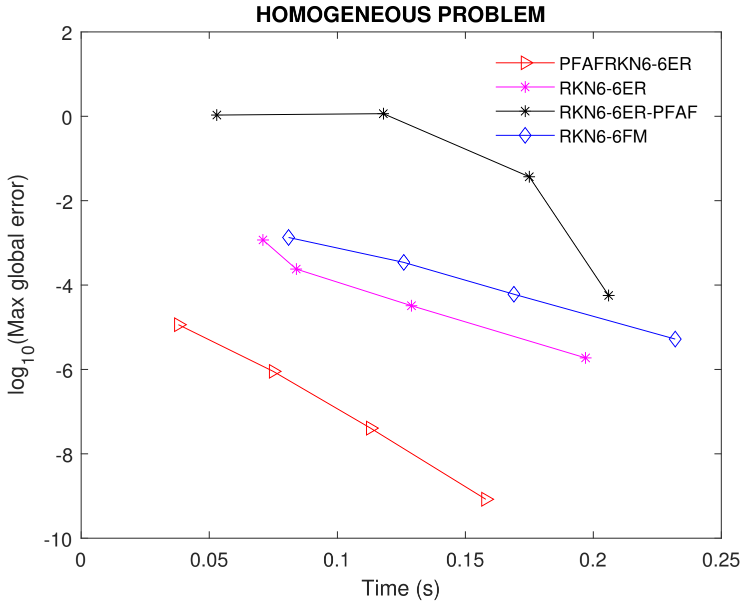


FIGURE 6. Efficiency curves corresponding to Problem 1

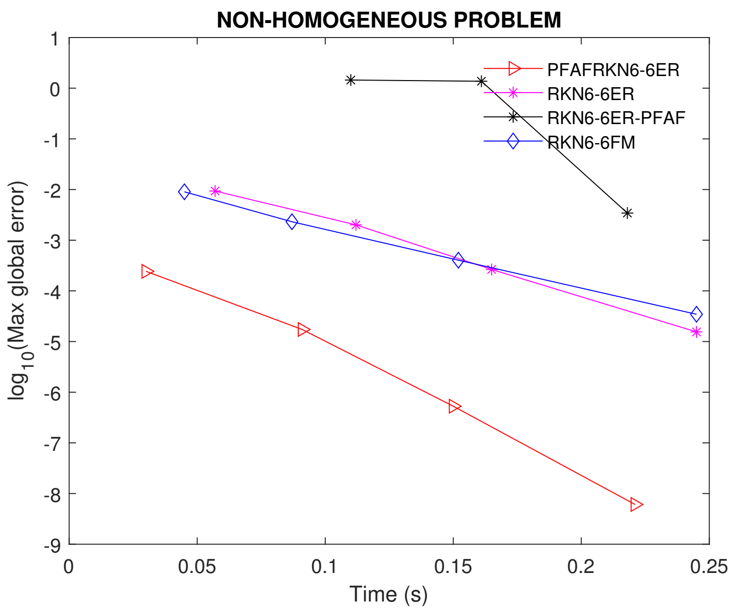


FIGURE 7. Efficiency curves corresponding to Problem 2

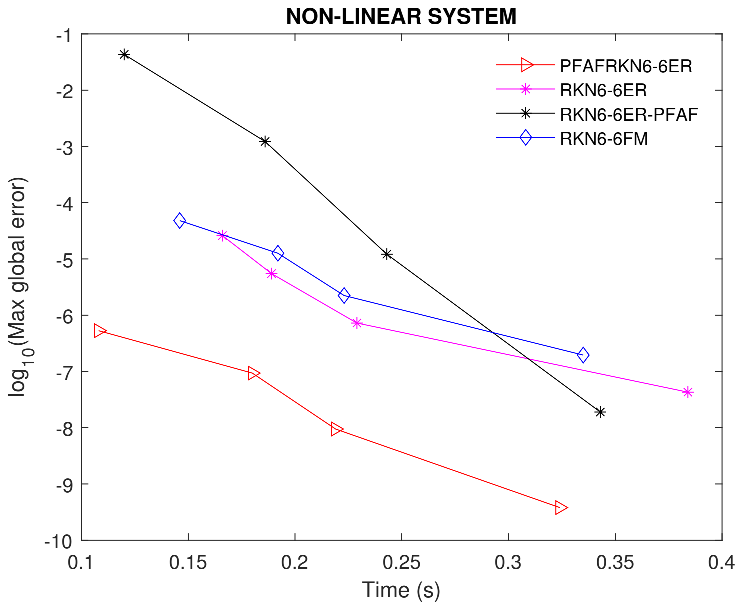


FIGURE 8. Efficiency curves corresponding to Problem 3

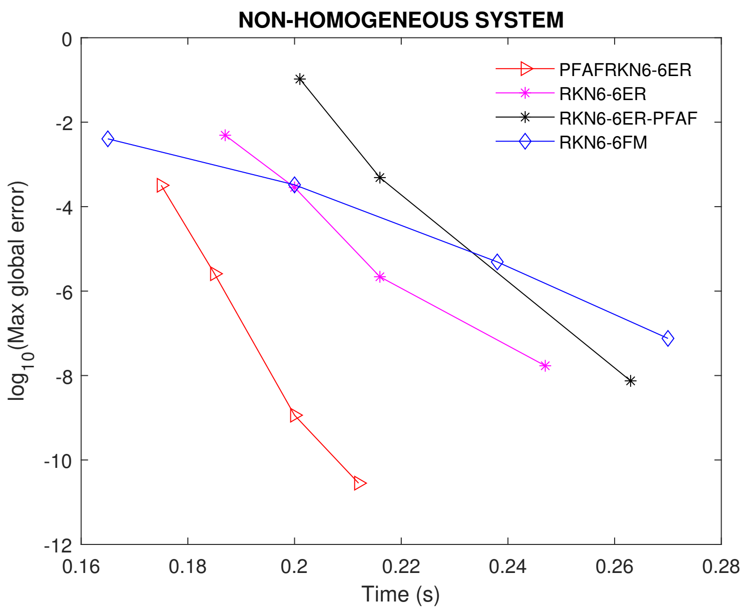


FIGURE 9. Efficiency curves corresponding to Problem 4

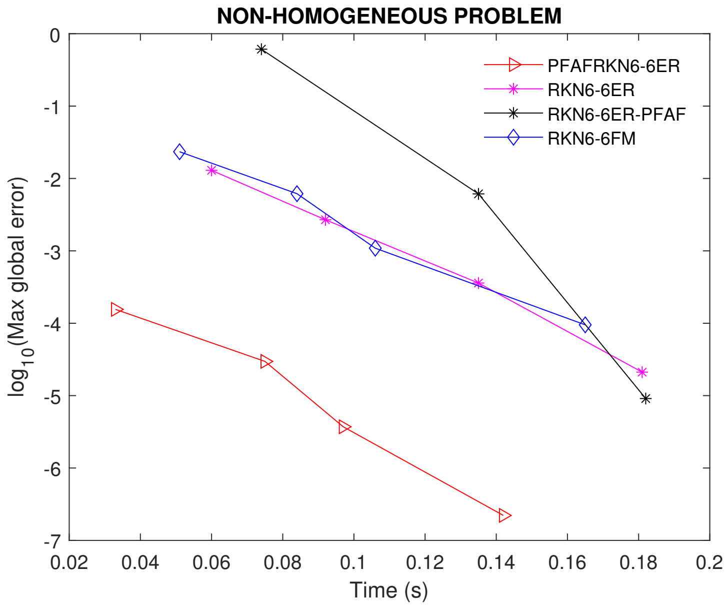


Figure 10. Efficiency curves corresponding to Problem 5

## 5. Discussion

The new PFAFRKN6-6ER code gives minimum error norm, minimum number of function evaluations, and minimum computational cost (Time(s)). Tables 2 − 6 and Figures 1 − 10 put an evidence that PFAFRKN6-6ER is a very efficient scheme. Therefore, we can say that PFAFRKN6-6ER is more appropriate for solving the type of problem in (1.1) than the other existing RKN methods of order 6 with six stages in the literature.

## 6. Conclusion

In this study, we have used the methodology for constructing the phase-fitted and amplification-fitted methods to develop an efficient explicit phase- and amplification-fitted RKN code based on the RKN6-6ER method due to El-Mikkawy and Rahmo [2]. The new developed method has two variable coefficients depending on the parameter $\mu = wh$, which is usually known as the parameter frequency [20, 21]. We computed the local truncation error of the new method, confirming that the algebraic order of convergence of the underlying code is maintained. In addition, the stability analysis of the new code revealed that it is "almost" P-stable. The numerical results obtained clearly show that PFAFRKN6-6ER is more accurate and efficient than other sixth-order six-stage RKN codes in the literature.

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