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# Applicability of Fixed Point Results in Boundary Value Problems Satisfying Generalized Geraghty-Type Contraction

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**Abstract** We establish a unique common fixed point theorem for $G$-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. With the help of the obtain results, we derive a coupled fixed point result and achieve the solution for periodic boundary value problems. We also give an example to show the applicability of the obtained results. Our results modify and generalize various well-known results.

**MSC:** 47H10; 54H25

**Keywords:** fixed point; coincidence point; generalized Geraghty-type contraction; partially ordered metric space; $G$-non-decreasing mapping; ordinary differential equations

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## 1. Introduction

As the Banach contraction principle is a powerful tool for solving many problems in applied mathematics and sciences, it has been improved and extended in many ways. In particular, Geraghty proved in [1] an interesting generalization of Banach contraction principle which had a lot of applications. Kadelburg et al. [2] proved some common coupled fixed point theorems for Geraghty-type contraction mappings by using monotone and $G-$monotone property instead of mixed monotone and $G-$mixed monotone property. For more details one can consult ([3–17]).

In this paper, we establish a fixed point theorem for $G$-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. With the help of the obtain results, we construct a coupled fixed point result and achieve the solution for periodic boundary value problems. We also give an example to validate our results. Our results modify and sharpen the results of Kadelburg et al. [2] and various well-known results in the recent literature.

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## 2. Main Results

In the sequel, $X$ is a non-empty set. Given $n \in \mathbb{N}$ where $n \geq 2$, let $X^n$ be the nth Cartesian product $X \times X \times ... \times X$ (n times). Let $G : X \to X$ be a mapping. For simplicity, we denote $G(x)$ by $Gx$ where $x \in X$.

**Definition 2.1.** Let $X$ be a non-empty set. A fixed point of a mapping $F : X \to X$ is a point $x \in X$ such that $x = Fx$.

**Definition 2.2.** ([18, 19]). A coincidence point of two mappings $F,\, G : X \to X$ is a point $x \in X$ such that $Fx = Gx$.

**Definition 2.3.** ([15]). A partially ordered metric space $(X,\, d,\, \preceq)$ is a metric space $(X,\, d)$ provided with a partial order $\preceq$ .

**Definition 2.4.** ([15]). An ordered metric space $(X,\, d,\, \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X,\, d,\, \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

**Definition 2.5.** ([15]). Let $(X,\, \preceq)$ be a partially ordered set and let $F,\, G : X \to X$ be two mappings. We say that $F$ is $(G,\, \preceq)$-non-decreasing if $Fx \preceq Fy$ for all $x,\, y \in X$ such that $Gx \preceq Gy$. If $G$ is the identity mapping on $X$, we say that $F$ is $\preceq$-non-decreasing. If $F$ is $(G,\, \preceq)$-non-decreasing and $Gx = Gy$, then $Fx = Fy$. It follows that

$$Gx = Gy \Rightarrow \left\{ \begin{array}{c} Gx \preceq Gy, \\ Gy \preceq Gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} Fx \preceq Fy, \\ Fy \preceq Fx \end{array} \right\} \Rightarrow Fx = Fy.$$

**Definition 2.6.** ([20]). Two self-mappings $F$ and $G$ of a non-empty set $X$ are said to be commutative if $FGx = GFx$ for all $x \in X$.

**Definition 2.7.** ([21]). Let $(X,\, d,\, \preceq)$ be a partially ordered metric space. Two mappings $F,\, G : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(GFx_n,\ FGx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n \in X.$$

**Definition 2.8.** ([22]). Two self-mappings $G$ and $F$ of a non-empty set $X$ are said to be weakly compatible if they commute at their coincidence points, that is, if $Gx = Fx$ for some $x \in X$, then $GFx = FGx$.

In [2], Kadelburg et al. introduced the class $\Theta$ of all functions $\theta : [0,\, +\infty) \to [0,\, 1)$ satisfying that for any sequence $\{s_n\}$ of non-negative real numbers $\theta(s_n) \to 1$ implies that $s_n \to 0$.

The following are examples of some functions belonging to $\Theta$.
(1) $\theta(s) = k$ for all $s \geq 0$, where $k \in [0,\, 1)$.
(2) $\theta(s) = \left\{ \begin{array}{ll} \dfrac{\ln(1+s)}{s} & s > 0, \\ r \in [0,\ 1), & s = 0. \end{array} \right.$

(3) $\theta(s) = \begin{cases} \dfrac{\ln(1+ks)}{ks} & s > 0, \\ r \in [0,\ 1), & s = 0, \end{cases}$ where $k \in [0,\ 1)$.

**Theorem 2.9.** *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F,\ G : X \to X$ be two mappings such that $F$ is $(G,\ \preceq)$-non-decreasing, $F(X) \subseteq G(X)$ and there exists $\theta \in \Theta$ such that*

$$d(Fx,\ Fy) \leq \theta(d(Gx,\ Gy))d(Gx,\ Gy), \tag{2.1}$$

*for all $x,\ y \in X$ where $Gx \preceq Gy$. There exists $x_0 \in X$ such that $Gx_0 \preceq Fx_0$. Also assume that one of the following conditions holds.*

*(a) $(X,\ d)$ is complete, $F$ and $G$ are continuous and the pair $(F,\ G)$ is compatible,*

*(b) $(G(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,*

*(c) $(X,\ d)$ is complete, $G$ is continuous and monotone non-decreasing, the pair $(F,\ G)$ is compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.*

*Then $F$ and $G$ have a coincidence point. Furthermore, if for every $x,\ y \in X$ there exists $u \in X$ such that $Fu$ is comparable to $Fx$ and $Fy$ and the pair $(F,\ G)$ is weakly compatible. Then $F$ and $G$ have a unique common fixed point.*

*Proof.* Let $x_0 \in X$ be arbitrary. Since $F(X) \subseteq G(X)$, therefore there exists $x_1 \in X$ such that $Fx_0 = Gx_1$, then $Gx_0 \preceq Fx_0 = Gx_1$. As $F$ is $(G,\ \preceq)$-non-decreasing and so $Fx_0 \preceq Fx_1$. Repeating this procedure, we get a sequence $\{x_n\}_{n\geq 0}$ such that $\{Gx_n\}$ is $\preceq$-non-decreasing, $Gx_{n+1} = Fx_n \preceq Fx_{n+1} = Gx_{n+2}$ and

$$Gx_{n+1} = Fx_n, \text{ for all } n \geq 0. \tag{2.2}$$

Let $\zeta_n = d(Gx_n,\ Gx_{n+1})$, for all $n \geq 0$. By using contractive condition (2.1), we have

$$d(Gx_{n+1},\ Gx_{n+2}) = d(Fx_n,\ Fx_{n+1}) \leq \theta(d(Gx_n,\ Gx_{n+1}))d(Gx_n,\ Gx_{n+1}), \tag{2.3}$$

which, by the fact that $\theta < 1$, implies

$$d(Gx_{n+1},\ Gx_{n+2}) < d(Gx_n,\ Gx_{n+1}), \text{ that is, } \zeta_{n+1} < \zeta_n \text{ for all } n \geq 0.$$

Thus the sequence $\{\zeta_n\}_{n\geq 0}$ is decreasing. Hence there exists an $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(Gx_n,\ Gx_{n+1}) = \zeta. \tag{2.4}$$

We claim that $\zeta = 0$. If possible, suppose $\zeta > 0$. Then from (2.3), we obtain

$$\frac{\zeta_{n+1}}{\zeta_n} \leq \theta(\zeta_n) < 1.$$

On taking limit as $n \to \infty$, we get

$$\theta(\zeta_n) \to 1 \text{ as } n \to \infty.$$

Using the properties of function $\theta$, we have

$$\zeta_n = d(Gx_n,\ Gx_{n+1}) \to 0 \text{ as } n \to \infty,$$

which contradicts the assumption that $\zeta > 0$. Hence, by (2.4), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(Gx_n,\ Gx_{n+1}) = 0. \tag{2.5}$$

We now claim that $\{Gx_n\}_{n\geq 0}$ is a Cauchy sequence in $(X,\ d)$. Suppose, to the contrary, that the sequence $\{Gx_n\}_{n\geq 0}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}_{n\geq 0}$ with $n(k) > m(k) \geq k$ such that

$$d(Gx_{n(k)},\ Gx_{m(k)}) \geq \varepsilon. \tag{2.6}$$

Let $n(k)$ be the smallest positive integer satisfying (2.6). Then

$$d(Gx_{n(k)-1},\ Gx_{m(k)}) < \varepsilon. \tag{2.7}$$

By using (2.6), (2.7) and triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq r_k = d(Gx_{n(k)},\ Gx_{m(k)}) \\ &\leq d(Gx_{n(k)},\ Gx_{n(k)-1}) + d(Gx_{n(k)-1},\ Gx_{m(k)}) \\ &< d(Gx_{n(k)},\ Gx_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.5), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(Gx_{n(k)},\ Gx_{m(k)}) = \varepsilon. \tag{2.8}$$

By the triangle inequality, we have

$$\begin{aligned} r_k &= d(Gx_{n(k)},\ Gx_{m(k)}) \\ &\leq d(Gx_{n(k)},\ Gx_{n(k)+1}) + d(Gx_{n(k)+1},\ Gx_{m(k)+1}) + d(Gx_{m(k)+1},\ Gx_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + d(Fx_{n(k)},\ Fx_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + \theta(d(Gx_{n(k)},\ Gx_{m(k)}))d(Gx_{n(k)},\ Gx_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + r_k. \end{aligned}$$

This shows that

$$r_k \leq \delta_{n(k)} + \delta_{m(k)} + \theta(r_k)r_k \leq \delta_{n(k)} + \delta_{m(k)} + r_k.$$

On taking limit as $n \to \infty$ in the above inequality, by using (2.5) and (2.8), we get

$$\theta(r_k) \to 1.$$

Using the properties of function $\theta$, we obtain

$$r_k = d(Gx_{n(k)},\ Gx_{m(k)}) \to 0 \text{ as } k \to \infty,$$

which implies

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(Gx_{n(k)},\ Gx_{m(k)}) = 0,$$

which contradicts the fact that $\varepsilon > 0$. Consequently $\{Gx_n\}_{n\geq 0}$ is a Cauchy sequence in $X$. We claim that $F$ and $G$ have a coincidence point between cases $(a) - (c)$.

Suppose $(a)$ holds, that is, $(X,\ d)$ is complete, $F$ and $G$ are continuous and the pair $(F,\ G)$ is compatible. Since $(X,\ d)$ is complete, therefore there exists $z \in X$ such that $\{Gx_n\} \to z$. It follows, from (2.2), that $\{Fx_n\} \to z$. Since $F$ and $G$ are continuous, therefore $\{FGx_n\} \to Fz$ and $\{GGx_n\} \to Gz$. As the pair $(F,\ G)$ is compatible and so we conclude that

$$d(Gz,\ Fz) = \lim_{n\to\infty} d(GGx_{n+1},\ FGx_n) = \lim_{n\to\infty} d(GFx_n,\ FGx_n) = 0,$$

that is, $z$ is a coincidence point of $F$ and $G$.

Suppose now $(b)$ holds, that is, $(G(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{Gx_n\}_{n\geq 0}$ is a Cauchy sequence in the complete space $(G(X),\ d)$ and so there exists $y \in G(X)$ such that $\{Gx_n\} \to y$. Let $z \in X$ be any point such that $y = Gz$, then $\{Gx_n\} \to Gz$. Also, since $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{Gx_n\}$ is $\preceq$-non-decreasing which converging to $Gz$, therefore we get $Gx_n \preceq Gz$ for all $n \geq 0$. Applying the contractive condition (2.1), we have

$$d(Gx_{n+1},\ Fz) = d(Fx_n,\ Fz) \leq \theta(d(Gx_n,\ Gz))d(Gx_n,\ Gz),$$

which, by the fact $\theta < 1$, implies

$$d(Gx_{n+1},\ Fz) \leq d(Gx_n,\ Gz).$$

Letting $n \to \infty$ in the above inequality and using $\lim_{n\to\infty} Gx_n = Gz$, we get $d(Gz,\, Fz) = 0$, that is, $z$ is a coincidence point of $F$ and $G$.

Suppose now that $(c)$ holds, that is, $(X,\, d)$ is complete, $G$ is continuous and monotone non-decreasing, the pair $(F,\, G)$ is compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular. As $(X,\, d)$ is complete and so there exists $z \in X$ such that $\{Gx_n\} \to z$. It follows, from (2.2), that $\{Fx_n\} \to z$. Since $G$ is continuous, therefore $\{GGx_n\} \to Gz$. Also, since the pair $(F,\, G)$ is compatible, it means that $\{FGx_n\} \to Gz$.

As $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{Gx_n\}$ is $\preceq$-non-decreasing which converging to $z$, we obtain that $Gx_n \preceq z$, which, by the monotonicity of $G$, implies $GGx_n \preceq Gz$. Applying the contractive condition (2.1), we get

$$d(FGx_n,\ Fz) \leq \theta(d(GGx_n,\ Gz))d(GGx_n,\ Gz),$$

which, by the fact $\theta < 1$, implies

$$d(FGx_n,\ Fz) \leq d(GGx_n,\ Gz).$$

On taking $n \to \infty$, by using $\{GGx_n\} \to Gz$ and $\{FGx_n\} \to Gz$ as $n \to \infty$, we get $d(Gz,\, Fz) = 0$, that is, $z$ is a coincidence point of $F$ and $G$.

It is obvious that the set of coincidence points of $F$ and $G$ is non-empty. Suppose $x$ and $y$ are coincidence points of $F$ and $G$, that is, $Gx = Fx$ and $Gy = Fy$. Now, we show $Gx = Gy$. By the assumption, there exists $u \in X$ such that $Fu$ is comparable with $Fx$ and $Fy$. Put $u_0 = u$ and choose $u_1 \in X$ so that $Gu_0 = Fu_1$. Then, we can inductively define sequence $\{Gu_n\}$ where $Gu_{n+1} = Fu_n$ for all $n \geq 0$. Hence $Fx = Gx$ and $Fu = Fu_0 = Gu_1$ are comparable. Suppose that $Gu_1 \preceq Gx$ (the proof is similar to that in the other case). We claim that $Gu_n \preceq Gx$ for each $n \in \mathbb{N}$. In fact, we will use mathematical induction. Since $Gu_1 \preceq Gx$, our claim is true for $n = 1$.

Assume that $Gu_n \preceq Gx$ holds for some $n > 1$. Since $F$ is $G$-nondecreasing with respect to $\preceq$, we get $Gu_{n+1} = Fu_n \preceq Fx = Gx$ and this proves our claim.

Let $\xi_n = d(Gu_n,\, Gx)$ for all $n \geq 0$. Since $Gu_n \preceq Gx$, therefore by using contractive condition (2.1), we have

$$d(Gu_{n+1},\ Gx) = d(Fu_n,\ Fx) \leq \theta(d(Gu_n,\ Gx))d(Gu_n,\ Gx), \tag{2.9}$$

which, by the fact $\theta < 1$, implies

$$d(Gu_{n+1},\ Gx) < d(Gu_n,\ Gx), \text{ that is, } \xi_{n+1} < \xi_n \text{ for all } n \geq 0.$$

Thus the sequence $\{\xi_n\}_{n\geq 0}$ is decreasing. Hence there exists an $\xi \geq 0$ such that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} d(Gu_n,\ Gx) = \xi. \tag{2.10}$$

Now, we show that $\xi = 0$. Suppose that $\xi > 0$. Then, from (2.9), we obtain that

$$\frac{\xi_{n+1}}{\xi_n} \leq \theta(\xi_n) < 1.$$

On taking limit as $n \to \infty$, we get

$$\theta(\xi_n) \to 1 \text{ as } n \to \infty.$$

Using the properties of function $\theta$, we have

$$\xi_n = d(Gu_n,\ Gx) \to 0 \text{ as } n \to \infty.$$

which contradicts the assumption that $\xi > 0$. Hence, by (2.10), we get

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} d(Gu_n,\ Gx) = 0. \tag{2.11}$$

Similarly, we show that

$$\lim_{n\to\infty} d(Gu_n,\ Gy) = 0. \tag{2.12}$$

Hence, by (2.11) and (2.12), we get

$$Gx = Gy. \tag{2.13}$$

As $Gx = Fx$ and so by weak compatibility of $G$ and $F$, we have $GGx = GFx = FGx$. Let $z = Gx$, then $Gz = Fz$, that is, $z$ is a coincidence point of $G$ and $F$. Then by using (2.13) with $y = z$, we get $Gx = Gz$, that is, $z = Gz = Fz$. Therefore, $z$ is a common fixed point of $G$ and $F$. To prove the uniqueness, suppose $w$ is another common fixed point of $G$ and $F$. Then by (2.13) we have $w = Gw = Gz = z$. Hence the common fixed point of $G$ and $F$ is unique. ∎

Taking $\theta(s) = k$ with $k \in [0,\ 1)$ for all $s \geq 0$ in Theorem 2.9, we obtain the following corollary.

**Corollary 2.10.** *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $F, G : X \to X$ be two mappings such that $F$ is $(G,\ \preceq)$-non-decreasing, $F(X) \subseteq G(X)$ and there exists $k \in [0,\ 1)$ such that*

$$d(Fx,\ Fy) \leq kd(Gx,\ Gy),$$

*for all $x,\ y \in X$ where $Gx \preceq Gy$. There exists $x_0 \in X$ such that $Gx_0 \preceq Fx_0$. Also assume that one of the conditions $(a) - (c)$ of Theorem 2.9 holds. Then $F$ and $G$ have a coincidence point. Furthermore, if for every $x,\ y \in X$ there exists $u \in X$ such that $Fu$ is comparable to $Fx$ and $Fy$ and the pair $(F,\ G)$ is weakly compatible. Then $F$ and $G$ have a unique common fixed point.*

Put $G = I$ (the identity mapping) in Corollary 2.10, we obtain the following corollary:

**Corollary 2.11.** *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space and let $F : X \to X$ be a $\preceq$-non-decreasing mapping such that*

$$d(Fx,\ Fy) \leq kd(x,\ y),$$

*for all $x,\ y \in X$ where $x \preceq y$ and $k \in [0,\ 1)$. There exists $x_0 \in X$ such that $x_0 \preceq Fx_0$. Suppose that*
*(a) $F$ is continuous or,*
*(b) $(X,\ d,\ \preceq)$ is regular.*
*Then $F$ has a fixed point.*

**Example 2.12.** Suppose that $X = \mathbb{R}$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$ . Let $F, G : X \to X$ be defined as

$$Fx = \ln\left(1 + x^2\right) \text{ and } Gx = x^2, \text{ for all } x \in X.$$

Define $\theta : [0, +\infty) \to [0, 1)$ as follows

$$\theta(s) = \begin{cases} \dfrac{\ln(1+s)}{s}, & s > 0, \\ 0, & s = 0. \end{cases}$$

Firstly, we shall show that the contractive condition of Theorem 2.9 should satisfy by the mappings $F$ and $G$. Let $x, y \in X$ such that $Gx \preceq Gy$, we have

$$\begin{aligned} d(Fx, Fy) &= |Fx - Fy| \\ &= \left|\ln\left(1+x^2\right) - \ln\left(1+y^2\right)\right| \\ &= \left|\ln \frac{1+x^2}{1+y^2}\right| \\ &= \left|\ln\left(1 + \frac{(x^2-y^2)}{1+y^2}\right)\right| \\ &\leq \ln\left(1 + \left|x^2 - y^2\right|\right) \\ &\leq \ln(1 + |Gx - Gy|) \\ &\leq \ln(1 + d(Gx, Gy)) \\ &\leq \frac{\ln(1 + d(Gx, Gy))}{d(Gx, Gy)} \times d(Gx, Gy) \\ &\leq \theta(d(Gx, Gy))d(Gx, Gy). \end{aligned}$$

Thus the contractive condition of Theorem 2.9 is satisfied for all $x, y \in X$. Furthermore, all the other conditions of Theorem 2.9 are satisfied and $z = 0$ is a unique common fixed point of $F$ and $G$.

## 3. Two Dimensional Results

Consider the partially ordered metric space $(X^2, \delta, \sqsubseteq)$, if $(X, d, \preceq)$ is a partially ordered metric space, then $\delta : X^2 \times X^2 \to [0, +\infty)$ defined as follows

$$\delta((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}, \text{ for all } (x, y), (u, v) \in X^2.$$

Then $\delta$ is metric on $X^2$ and $(X, d)$ is complete if and only if $(X^2, \delta)$ is complete. Also $\sqsubseteq$ partial order on $X^2$ defined by

$$(u, v) \sqsubseteq (x, y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u, v), (x, y) \in X^2.$$

Define the mappings $T_F, T_G : X^2 \to X^2$, for all $(x, y) \in X^2$, by

$$T_F(x, y) = (F(x, y), F(y, x)) \text{ and } T_G(x, y) = (Gx, Gy).$$

**Definition 3.1.** ([23]). Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of $F$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

**Definition 3.2.** ([24]). Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that $F$ has the mixed monotone property if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

**Definition 3.3.** ([25]). Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings $F$ and $g$ if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy.$$

**Definition 3.4.** ([25]). Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a common coupled fixed point of the mappings $F$ and $g$ if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy.$$

**Definition 3.5.** ([25]). Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that $F$ has the mixed $g-$monotone property if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

If $g$ is the identity mapping on $X$, then $F$ satisfies the mixed monotone property.

**Definition 3.6.** ([25]). Let $X$ be a nonempty set. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if

$$gF(x,\ y) = F(gx,\ gy), \text{ for all } (x,\ y) \in X^2.$$

**Definition 3.7.** ([26]). Let $(X, d)$ be a metric space. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y, \text{ for some } x,\ y \in X. \end{aligned}$$

**Definition 3.8.** ([20]). Let $X$ be a nonempty set. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be weakly compatible if they commute at their coupled coincidence points, that is, $F(x, y) = gx$ and $y = F(y, x) = gy$ for some $(x, y) \in X^2$, then $gF(x, y) = F(gx, gy)$.

**Lemma 3.9.** ([4]). *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^2 \to X$ and $G : X \to X$ be two mappings. Then*

*(1) $(X, d)$ is complete if and only if $(X^2, \delta)$ is complete.*

*(2) If $(X, d, \preceq)$ is regular, then $(X^2, \delta, \sqsubseteq)$ is also regular.*

*(3) If $F$ is $d$-continuous, then $T_F$ is $\delta$-continuous.*

*(4) $F$ has the mixed monotone property with respect to $\preceq$ if and only if $T_F$ is $\sqsubseteq$ −non-decreasing.*

*(5) $F$ has the mixed $G$−monotone property with respect to $\preceq$ if and only if $T_F$ is $(T_G, \sqsubseteq)$-non-decreasing.*

*(6) If there exist two elements $x_0, y_0 \in X$ with $Gx_0 \preceq F(x_0, y_0)$ and $Gy_0 \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.*

*(7) If $F(X^2) \subseteq G(X)$, then $T_F(X^2) \subseteq T_G(X^2)$.*

*(8) If $F$ and $G$ are commuting in $(X, d, \preceq)$, then $T_F$ and $T_G$ are also commuting in $(X^2, \delta, \sqsubseteq)$.*

*(9) If $F$ and $G$ are compatible in $(X, d, \preceq)$, then $T_F$ and $T_G$ are also compatible in $(X^2, \delta, \sqsubseteq)$.*

*(10) If $F$ and $G$ are weak compatible in $(X, d, \preceq)$, then $T_F$ and $T_G$ are also weak compatible in $(X^2, \delta, \sqsubseteq)$.*

*(11) A point $(x, y) \in X^2$ is a coupled coincidence point of $F$ and $G$ if and only if it is a coincidence point of $T_F$ and $T_G$.*

*(12) $(x, y) \in X^2$ is a coupled fixed point of $F$ if and only if it is a fixed point of $T_F$.*

**Theorem 3.10.** *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric $d$ on $X$. Assume $F : X^2 \to X$ and $G : X \to X$ are two mappings such that $F$ has the mixed $G$−monotone property with respect to $\preceq$ on $X$ for which there exists $\theta \in \Theta$ such that*

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \theta(\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\})\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\}, \end{aligned} \tag{3.1}$$

*for all $x, y, u, v \in X$, with $Gx \preceq Gu$ and $Gy \succeq Gv$. Suppose that $F(X^2) \subseteq G(X)$, $G$ is continuous and monotone non-decreasing and the pair $\{F, G\}$ is compatible. Also suppose that either*

*(a) $F$ is continuous or*

*(b) $(X, d, \preceq)$ is regular.*

*If there exist two elements $x_0, y_0 \in X$ with*

$$Gx_0 \preceq F(x_0,\ y_0) \text{ and } Gy_0 \succeq F(y_0,\ x_0).$$

*Then $F$ and $G$ have a coupled coincidence point. Furthermore, suppose that for every $(x, y), (x^*, y^*) \in X^2$ there exists $(u, v) \in X^2$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, and also the pair $(F, G)$ is weakly compatible. Then $F$ and $G$ have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X^2$ such that $x = Gx = F(x, y)$ and $y = Gy = F(y, x)$.*

*Proof.* Let $x, y, u, v \in X$ be such that $Gx \preceq Gu$ and $Gy \succeq Gv$. Then by using (3.1), we have

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \theta(\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\})\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\}. \end{aligned}$$

Furthermore $Gy \succeq Gv$ and $Gx \preceq Gu$, the contractive condition (3.1) also guarantees that

$$\begin{aligned} & d(F(y,\ x),\ F(v,\ u)) \\ \leq\ & \theta(\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\})\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\}. \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\} \\ \leq\ & \theta(\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\})\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\}. \end{aligned} \tag{3.2}$$

Thus, by using (3.2), we get

$$\begin{aligned} & \delta(T_F(x,\ y),\ T_F(u,\ v))) \\ =\ & \delta((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u))) \\ =\ & \max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\} \\ \leq\ & \theta(\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\})\max\{d(Gx,\ Gu),\ d(Gy,\ Gv)\} \\ \leq\ & \theta(\delta(F_G(x,\ y),\ F_G(u,\ v)))\delta(F_G(x,\ y),\ F_G(u,\ v)). \end{aligned}$$

It is only require to use Theorem 2.9 to the mappings $F = T_F$ and $G = T_G$ in the partially ordered metric space $(X^2,\ \delta,\ \sqsubseteq)$ with the help of Lemma 3.9. ■

**Corollary 3.11.** *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric $d$ on $X$. Assume $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exists $\theta \in \Theta$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq \theta(\max\{d(x,\ u),\ d(y,\ v)\})\max\{d(x,\ u),\ d(y,\ v)\},$$

*for all $x,\ y,\ u,\ v \in X$, with $x \preceq u$ and $y \succeq v$. Also suppose that either*
*(a) $F$ is continuous or*
*(b) $(X,\ d,\ \preceq)$ is regular.*
*If there exist two elements $x_0,\ y_0 \in X$ with*

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

*Then $F$ has a coupled fixed point.*

Put $\theta(s) = k$ with $k \in [0,\ 1)$ for all $s \geq 0$ in Corollary 3.11, we obtain the following corollary:

**Corollary 3.12.** *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric $d$ on $X$. Assume $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exists $k \in [0,\ 1)$ such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq k\max\{d(x,\ u),\ d(y,\ v)\},$$

*for all $x,\ y,\ u,\ v \in X$ with $x \preceq u$ and $y \succeq v$. Also suppose that either*
*(a) $F$ is continuous or*
*(b) $(X,\ d,\ \preceq)$ is regular.*
*If there exist two elements $x_0,\ y_0 \in X$ with*

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

*Then $F$ has a coupled fixed point.*

## 4. Application to Ordinary Differential Equations

In this fragment, we first study the existence of a solution for the following first-order periodic problem:

$$\begin{cases} u'(t) = f(t,\ u(t)),\ t \in [0,\ T], \\ u(0) = u(T), \end{cases} \tag{4.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Considered the space $X = C(I,\ \mathbb{R})$ $(I = [0,\ T])$ of all continuous functions from $I$ to $\mathbb{R}$. It is visible that $X$ is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{t\in I} |x(t) - y(t)|\,, \text{ for all } x,\ y \in X.$$

Also $X$ can be furnished with a partial order given by

$$x \preceq y \Longleftrightarrow x(t) \le y(t), \text{ for all } x,\ y \in X \text{ and } t \in I. \tag{4.2}$$

**Definition 4.1.** A lower solution for (4.1) is a function $\alpha \in C^1(I,\ \mathbb{R})$ such that

$$\alpha'(t) \le f(t,\ \alpha(t)) \text{ for } t \in I,\ \alpha(0) = \alpha(T) = 0.$$

**Theorem 4.2.** *Consider problem (4.1) with $f : I \times \mathbb{R} \to \mathbb{R}$ continuous and for $x$, $y$, $u$, $v \in X$ with $x \succeq y$,*

$$0 \le f(t,\ x) + \lambda x - f(t,\ y) - \lambda y \le \frac{\lambda}{2} x - y).$$

*Then the existence of a coupled upper-lower solution of (4.1) provides the existence of a solution of (4.1).*

*Proof.* Problem (4.1) is equivalent to the integral equation

$$u(t) = \int_0^T G(t,\ s)[f(s,\ u(s)) + \lambda u(s)]ds,$$

where $G(t,\ s)$ is the Green function given by

$$G(t,\ s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1},\ 0 \le s < t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1},\ 0 \le t < s \le T, \end{cases}$$

Define the mapping $F : X \to X$ by

$$F(x)(t) = \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s)]ds.$$

If $x_1 \succeq x_2$, then by using our assumption, we have

$$f(t,\ x_1) + \lambda x_1 \ge f(t,\ x_2) + \lambda x_2.$$

It follows from $G(t,\ s) > 0$ for $t \in I$, that

$$F(x_1)(t) \quad = \quad \int_0^T G(t,\ s)[f(s,\ x_1(s)) + \lambda x_1(s)]ds$$

$$\geq \int_0^T G(t,\ s)[f(s,\ x_2(s)) + \lambda x_2(s)]ds = F(x_2)(t).$$

Thus the mapping $F$ is non-decreasing. Now, for all $x \succeq y$, we have

$$\begin{aligned}
& d(Fx,\ Fy) \\
= \ & \sup_{t\in I} |F(x)(t) - F(y)(t)| \\
= \ & \sup_{t\in I} \left| \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s) - f(s,\ y(s)) - \lambda y(s)]ds \right| \\
\leq \ & \sup_{t\in I} \left| \int_0^T G(t,\ s) \cdot \frac{\lambda}{2}((x(s) - y(s))ds \right| \\
\leq \ & \frac{\lambda}{2} d(x,\ y) \sup_{t\in I} \left| \int_0^T G(t,\ s)ds \right| \\
\leq \ & \frac{\lambda}{2} d(x,\ y) \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right| \\
\leq \ & \frac{1}{2} d(x,\ y).
\end{aligned}$$

Thus

$$d(Fx,\ Fy) \leq \frac{1}{2} d(x,\ y).$$

Thus the contractive condition of Corollary 2.11 satisfied with $k = 1/2 < 1$. Finally, let $\alpha \in X$ be a lower solution of (4.1), then

$$\alpha'(s) + \lambda\alpha(s) \leq f(s,\ \alpha(s)) + \lambda\alpha(s),\ \text{for } t \in I.$$

Multiplying by $G(t,\ s)$, we get

$$\int_0^T \alpha'(s)G(t,\ s)ds + \lambda \int_0^T \alpha(s)G(t,\ s)ds \leq F(\alpha)(t),\ \text{for } t \in I.$$

Then, for all $t \in I$, we have

$$\int_0^t \alpha'(s)\frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \alpha'(s)\frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds + \lambda \int_0^T \alpha(s)G(t,\ s)ds \leq F(\alpha)(t).$$

Using integration by parts and since $\alpha(0) = \alpha(T) = 0$ for all $t \in I$, we get

$$\alpha(t) \leq F(\alpha)(t).$$

It follows that $\alpha \preceq F\alpha$. Hence all the hypothesis of Corollary 2.11 are satisfied. Consequently, $F$ has a fixed point $x \in X$ which is the solution to (4.1) in $X = C(I,\ \mathbb{R})$. ■

Now, we apply our main results to study the existence and uniqueness of solution to the two-point boundary value problem.

$$\begin{cases} -x''(t) = f(t,\ x(t),\ x(t)),\ x \in (0,\ +\infty),\ t \in [0,\ 1], \\ x(0) = x(1) = 0. \end{cases} \tag{4.3}$$

where $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $X = C(I,\ \mathbb{R})$ $(I = [0,\ 1])$ denote the space of all continuous functions from $I$ to $\mathbb{R}$. It is crystal clear that $X$ is a regular complete partial ordered metric space with respect to the sup metric

$$d(x,\ y) = \sup_{t\in I} |x(t) - y(t)|\,,\ \text{for all}\ x,\ y \in X,$$

where partial order is given by (4.2).

**Theorem 4.3.** *Under the assumptions*

(*a*) $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(*b*) *Suppose that there exists* $0 \le \gamma \le 8$ *such that for all* $t \in I$, $x \succeq u$ *and* $y \preceq v$,

$$0 \le f(t,\ x,\ y) - f(t,\ u,\ v) \le \frac{\gamma}{2}(g(x-u) + g(y-v)),$$

*where* $g(t) : [0,\ +\infty) \to [0,\ +\infty)$ *is a right upper semi-continuous and non-decreasing function with* $g(0) = 0$, $g(t) \le \ln(1+t)$, *for all* $t > 0$.

(*c*) *There exists* $(\alpha,\ \beta) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ *solution to*

$$\begin{cases} -\alpha''(t) \le f(t,\ \alpha(t),\ \beta(t)),\ t \in [0,\ 1], \\ -\beta''(t) \ge f(t,\ \beta(t),\ \alpha(t)),\ t \in [0,\ 1], \\ \alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0. \end{cases} \tag{4.4}$$

*Problem* (4.3) *has one and only one solution in* $C^2(I,\ \mathbb{R})$.

*Proof.* It is clear that the solution (in $C^2(I,\ \mathbb{R})$) of problem (4.3) is equivalent to the solution (in $C(I,\ \mathbb{R})$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ x(s)) ds \text{ for } t \in [0,\ 1],$$

where $G(t,\ s)$ is the Green function of differential operator $-\frac{d^2}{dt^2}$ with Dirichlet boundary condition $x(0) = x(1) = 0$, that is,

$$G(t,\ s) = \begin{cases} t(1-s),\ 0 \le t \le s \le 1, \\ s(1-t),\ 0 \le s \le t \le 1. \end{cases} \tag{4.5}$$

Define $\theta : [0,\ +\infty) \to [0,\ 1)$ as follows

$$\theta(s) = \begin{cases} \frac{\ln(1+s)}{s},\ s > 0, \\ 0,\ s = 0. \end{cases}$$

and $F : X^2 \to X$ is define by

$$F(x,\ y)(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ y(s)) ds,\ t \in [0,\ 1] \text{ and } x,\ y \in X.$$

From $(b)$, it is clear that $F$ has the mixed monotone property with respect to the partial order $\preceq$ in $X$. Let $x,\ y,\ u,\ v\in X$ such that $x\succeq u$ and $y\preceq v$. From $(b)$, we have

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
= \ & \sup_{t\in I}|F(x,\ y)(t)-F(u,\ v)(t)| \\
= \ & \sup_{t\in I}\int_0^1 G(t,\ s)[f(s,\ x(s),\ y(s))-f(s,\ u(s),\ v(s))]ds \\
\leq \ & \frac{\gamma}{2}\sup_{t\in I}\int_0^1 G(t,\ s)\cdot(g(x(s)-u(s))+g(y(s)-v(s)))ds \\
\leq \ & \gamma\left(\frac{g(d(x,\ u))+g(d(y,\ v))}{2}\right)\sup_{t\in I}\int_0^1 G(t,\ s)ds.
\end{aligned}$$

Now, since $g$ is non-decreasing, we have

$$\begin{aligned}
g(d(x,\ u)) &\leq g(\max\{d(x,\ u),\ d(y,\ v)\}), \\
g(d(y,\ v)) &\leq g(\max\{d(x,\ u),\ d(y,\ v)\}),
\end{aligned}$$

which implies

$$\frac{g(d(x,\ u))+g(d(y,\ v))}{2}\leq g(\max\{d(x,\ u),\ d(y,\ v)\}).$$

Thus

$$d(F(x,\ y),\ F(u,\ v))\leq\gamma(g(\max\{d(x,\ u),\ d(y,\ v)\}))\sup_{t\in I}\int_0^1 G(t,\ s)ds. \tag{4.6}$$

Clearly

$$\int_0^1 G(t,\ s)ds=-\frac{t^2}{2}+\frac{t}{2}\ \text{and}\ \sup_{t\in[0,\ 1]}\int_0^1 G(t,\ s)ds=\frac{1}{8}.$$

Thus, the inequality (4.6) and the hypothesis $0<\gamma\leq 8$ implies

$$\begin{aligned}
d(F(x,\ y),\ F(u,\ v)) &\leq \frac{\gamma}{8}(g(\max\{d(x,\ u),\ d(y,\ v)\})) \\
&\leq g(\max\{d(x,\ u),\ d(y,\ v)\}) \\
&\leq \ln(1+\max\{d(x,\ u),\ d(y,\ v)\}).
\end{aligned}$$

Thus

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq \ & \ln(1+\max\{d(x,\ u),\ d(y,\ v)\}) \\
\leq \ & \frac{\ln(1+\max\{d(x,\ u),\ d(y,\ v)\})}{\max\{d(x,\ u),\ d(y,\ v)\}}\times\max\{d(x,\ u),\ d(y,\ v)\} \\
\leq \ & \theta(\max\{d(x,\ u),\ d(y,\ v)\})\max\{d(x,\ u),\ d(y,\ v)\}),
\end{aligned}$$

which is the contractive condition of Corollary 3.11. Again, let $(\alpha,\ \beta) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ be a solution to (4.3). Then

$$-\alpha''(s) \leq f(s,\ \alpha(s),\ \beta(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t,\ s)$, we get

$$\int_0^1 -\alpha''(s)G(t,\ s)ds \leq F(\alpha,\ \beta)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0,\ 1]$, we have

$$-(1-t)\int_0^t s\alpha''(s)ds - t\int_t^1 (1-s)\alpha''(s)ds \leq F(\alpha,\ \beta)(t).$$

Using integration by parts and since $\alpha(0) = \alpha(1) = 0$ for all $t \in [0,\ 1]$, we get

$$-(1-t)(t\alpha'(t) - \alpha(t)) - t(-(1-t)\alpha'(t) - \alpha(t)) \leq F(\alpha,\ \beta)(t).$$

Thus, we have

$$\alpha(t) \preceq F(\alpha, \beta)(t),\ \text{for } t \in [0,\ 1].$$

It follows that $\alpha \preceq F(\alpha,\ \beta)$. Similarly, one can easily prove that $\beta \succeq F(\beta,\ \alpha)$. Hence all the hypothesis of Corollary 3.11 are satisfied. Consequently, $F$ has a coupled fixed point $(x,\ y) \in X^2$ which is the solution to (4.3) in $X = C(I,\ \mathbb{R})$. ∎

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