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Strong Convergence Theorems of Halpern's Type for Families of Nonexpansive Mappings in Hilbert Spaces

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Abstract : Let C be a nonempty closed convex subset of a real Hilbert space and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that the set of all common fixed points of $\{T_n\}$ is nonempty. We consider a sequence $\{x_n\}$ generated by $x_1 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$. Then, we give the conditions of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Keywords : Strong convergence; Nonexpansive; Proximal point algorithm; W-mapping; Nonexpansive semigroup; Splitting method; Variational inequality.
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1 Introduction

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and let $\mathbf{N}$ and $\mathbf{R}$ be the set of all positive integers and the set of all real numbers, respectively. Let C be a nonempty closed convex subset of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself with $F := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n. Halpern [7] considered the following iteration:

$$x_1 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)T_n x_n\ (\forall n \in \mathbf{N}),$$

where $\{\alpha_n\} \subset [0,1)$. Wittmann [37] proved a strong convergence theorem when $T_n = T$ $(\forall n \in \mathbf{N})$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, where T is a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, Bauschke

[2], Shimizu and Takahashi [24], Shioji and Takahashi [27], Kamimura and Takahashi [12] and Iiduka and Takahashi [9, 10, 11] studied the strong convergence by Halpern's type iteration in Hilbert spaces and Shioji and Takahashi [26, 28, 29, 30], Kamimura and Takahashi [13], Shimoji and Takahashi [25] and Takahashi, Tamura and Toyoda [36] studied the strong convergence by Halpern's type iteration in Banach spaces. Recently, Bauschke and Combettes [3] considered the following coherent condition: For every bounded sequence $\{z_n\} \subset C$, $\sum_{n=1}^{\infty} \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|z_n - T_n z_n\|^2 < \infty$ imply $\omega_w(z_n) \subset F$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$ and proved a weak convergence theorem and a strong convergence theorem by the hybrid Haugazeau's method.
Motivated by Halpern's type iteration and [3], in this paper, we consider the following iteration:

$$x_1 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n) T_n(\beta_n x + (1-\beta_n) x_n)\ (\forall n \in \mathbf{N}) \tag{1.1}$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Further, we consider the following conditions:

(I) There exists $\{a_n\} \subset [0,\infty)$ with $\sum_{n=1}^{\infty} a_n < \infty$ such that for every bounded subset B of C, there exists $M_B > 0$ such that $\|T_n x - T_{n+1} x\| \leq a_n M_B$ holds for all $n \in \mathbf{N}$ and $x \in B$;

(II) for each bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ implies $\omega_w(z_n) \subset F$;

(III) for every bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_{n+1} - T_n z_n\| = 0$ implies $\omega_w(z_n) \subset F$.

Then, we prove that if (i) $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and (I) and (II) hold or (ii) (III) holds, $\{x_n\}$ converges strongly to $P_F x$, where P_F is the metric projection onto F. These results generalize the results of [9, 11, 12, 24, 37]. Further, we get a new result for splitting methods (see [22, 15, 18] and references therein) by using these results.

2 Preliminaries

We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize the strong convergence. We know that H satisfies Opial's condition [21], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. Let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. T is said to be firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(I-T)x-(I-T)y\|^2$ for every $x, y \in C$. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. If T is firmly nonexpansive, T is nonexpansive. We know that the metric projection P_C of H onto C is firmly nonexpansive and for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $(x-z, z-u) \geq 0$ for all $u \in C$. It is known that $F(T)$ is closed and convex if T is nonexpansive of C into itself. We have the following lemma by Opial's condition; see [5].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then, $T_n = T$ ($\forall n \in \boldsymbol{N}$) satisfy the conditions (I) and (II) with $a_n = 0$ ($\forall n \in \boldsymbol{N}$).*

Proof. By $T_n = T_{n+1}$ for every $n \in \mathbf{N}$, (I) holds. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. Without loss of generality, let $z_n \rightharpoonup w$. Suppose that $w \neq Tw$. By Opial's condition,

$$\begin{aligned} \liminf_{n\to\infty} \|z_n - w\| &< \liminf_{n\to\infty} \|z_n - Tw\| \leq \liminf_{n\to\infty} (\|z_n - Tz_n\| + \|Tz_n - Tw\|) \\ &\leq \liminf_{n\to\infty} (\|z_n - Tz_n\| + \|z_n - w\|) = \liminf_{n\to\infty} \|z_n - w\|. \end{aligned}$$

This is a contradiction. So, $\omega_w(z_n) \subset F(T)$. □

An operator $A : H \longrightarrow 2^H$ is said to be monotone if $(x_1 - x_2, y_1 - y_2) \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal if and only if $R(I + \lambda A) = H$ for every $\lambda > 0$, where $R(I + \lambda A)$ is the range of $I + \lambda A$. We also know that a monotone operator A is maximal if and only if for $(u, v) \in H \times H$, $(x - u, y - v) \geq 0$ for every $(x, y) \in A$ implies $v \in Au$. And we have that for a maximal monotone operator, $A^{-1}0 = \{x \in H : 0 \in Ax\}$ is closed and convex. If A is monotone, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda A) \longrightarrow D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$, where $D(A)$ is the domain of A. J_λ is called the resolvent of A. We also define the Yosida approximation A_λ by $A_\lambda = (I - J_\lambda)/\lambda$. We know that $A^{-1}0 = F(J_\lambda)$ holds and J_λ is firmly nonexpansive for every $\lambda > 0$. It is also known that for $\lambda > 0$, $\|A_\lambda x - A_\lambda y\| \leq \frac{2}{\lambda}\|x-y\|$ for each $x, y \in R(I+\lambda A)$; see [33, 34] for more details. We have the following results.

Lemma 2.2. *Let $A : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Then, the following hold:*

(i) $T_n = J_{\lambda_n}$ $(\forall n \in \boldsymbol{N})$ *with* $\{\lambda_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ *satisfy the conditions (I) and (II) with* $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \boldsymbol{N})$*;*

(ii) $T_n = J_{\lambda_n}$ $(\forall n \in \boldsymbol{N})$ *with* $\{\lambda_n\} \subset (0, \infty)$ *and* $\lim_{n\to\infty} \lambda_n = \infty$ *satisfy the condition (III).*

Proof. (i). By [6, Lemma 2.1], we have

$$\|J_{\lambda_n}x - J_{\lambda_{n+1}}x\| \leq \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_n}\|x - J_{\lambda_n}x\| \leq \frac{|\lambda_n - \lambda_{n+1}|}{c}\{2\|x-u\|\}$$

for every $n \in \mathbf{N}$ and $x \in H$, where $u \in A^{-1}0$ and $c = \inf_{n\in\mathbf{N}} \lambda_n\,(> 0)$. So, for each bounded subset B of H, there exists $M_B > \frac{2}{c}\sup_{x\in B}\|x-u\|$ such that $\|T_nx - T_{n+1}x\| \leq a_nM_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \mathbf{N})$. Next, let $\{z_n\}$ be a bounded sequence in H such that $\lim_{n\to\infty}\|z_n - J_{\lambda_n}z_n\| = 0$. Without loss of generality, let $z_n \rightharpoonup w$. Since A is monotone, we get

$$(J_{\lambda_n}z_n - u, -v) \geq \frac{1}{\lambda_n}(J_{\lambda_n}z_n - u, J_{\lambda_n}z_n - z_n) \geq -\frac{1}{c}\|J_{\lambda_n}z_n - u\| \cdot \|J_{\lambda_n}z_n - z_n\|$$

for every $(u, v) \in A$ and $n \in \mathbf{N}$. $J_{\lambda_n}z_n \rightharpoonup w$ and $\{J_{\lambda_n}z_n - u\}$ is bounded. So, we get $(w - u, -v) \geq 0$ for all $(u, v) \in A$ which implies $w \in A^{-1}0$ by maximality of A. Therefore, $\omega_w(z_n) \subset A^{-1}0 = \cap_{n=1}^{\infty}F(T_n)$.
(ii). Let $\{z_n\} \subset H$ be a bounded sequence such that $\lim_{n\to\infty}\|z_{n+1} - J_{\lambda_n}z_n\| = 0$. Let $m \in \mathbf{N}$. By [20, Corollary 3.4], we have

$$\begin{aligned}
\|z_{n+1} - J_{\lambda_m}z_{n+1}\| &\leq \|z_{n+1} - J_{\lambda_n}z_n\| + \|J_{\lambda_n}z_n - J_{\lambda_m}J_{\lambda_n}z_n\| \\
&\quad + \|J_{\lambda_m}J_{\lambda_n}z_n - J_{\lambda_m}z_{n+1}\| \\
&\leq 2\|z_{n+1} - J_{\lambda_n}z_n\| + \frac{\lambda_m}{\lambda_n}\|z_n - J_{\lambda_n}z_n\| \\
&\leq 2\|z_{n+1} - J_{\lambda_n}z_n\| + \frac{\lambda_m}{\lambda_n}\{2\|z_n - u\|\}
\end{aligned}$$

for every $n \in \mathbf{N}$, where $u \in A^{-1}0$. Since a sequence $\{z_n - u\}$ is bounded and $\lim_{n\to\infty}\lambda_n = \infty$, we have $\lim_{n\to\infty}\|z_n - J_{\lambda_m}z_n\| = 0$ and hence $\omega_w(z_n) \subset A^{-1}0 = \cap_{n=1}^{\infty}F(T_n)$ by Opial's condition. □

Let $\alpha > 0$ and C be a nonempty closed convex subset of H. An operator $A : C \longrightarrow H$ is said to be α-inverse-strongly-monotone [4, 16, 18] if $(x - y, Ax - Ay) \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$. Let $A : H \longrightarrow H$ be an α-inverse-strongly-monotone operator and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A + B)^{-1}0 \neq \emptyset$. Then, we know that $A + B$ is maximal monotone and for every $\lambda > 0$, $(A + B)^{-1}0 = F(J_\lambda^B(I - \lambda A))$, where J_λ^B is the resolvent of B. It is also known that $J_\lambda^B(I - \lambda A)$ is nonexpansive of H into itself when $0 < \lambda \leq 2\alpha$; see [18, 19]. We have the following result.

Lemma 2.3. *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be an α-inverse-strongly-monotone operator and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Then, $T_n = J^B_{\lambda_n}(I - \lambda_n A)$ $(\forall n \in \boldsymbol{N})$ with $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ satisfy the conditions (I) and (II) with $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \boldsymbol{N})$.*

Proof. Let $u \in (A+B)^{-1}0$ and $\lambda_n > 0$. Then we have $J^B_{\lambda_n}(I - \lambda_n A)u = u$. This implies $A^B_{\lambda_n}(I - \lambda_n A)u = -Au$ for all $n \in \mathbf{N}$, where $A^B_{\lambda_n}$ is the Yosida approximation of B. So, by [6, Lemma 2.1], we get

$$\begin{aligned}
&\|J^B_{\lambda_n}(I - \lambda_n A)x - J^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x\| \\
&\quad\leq \|J^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x - J^B_{\lambda_n}(I - \lambda_{n+1} A)x\| \\
&\qquad + \|J^B_{\lambda_n}(I - \lambda_{n+1} A)x - J^B_{\lambda_n}(I - \lambda_n A)x\| \\
&\quad\leq \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_{n+1}} \|(I - \lambda_{n+1} A)x - J^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x\| + |\lambda_n - \lambda_{n+1}| \|Ax\| \\
&\quad\leq |\lambda_n - \lambda_{n+1}| \|A^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x\| + |\lambda_n - \lambda_{n+1}| \Big(\frac{1}{\alpha}\|x - u\| + \|Au\|\Big) \\
&\quad\leq |\lambda_n - \lambda_{n+1}| \{\|A^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x + Au\| + \|Au\|\} \\
&\qquad + |\lambda_n - \lambda_{n+1}| \Big(\frac{1}{\alpha}\|x - u\| + \|Au\|\Big) \\
&\quad= |\lambda_n - \lambda_{n+1}| \Big\{\|A^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)x - A^B_{\lambda_{n+1}}(I - \lambda_{n+1} A)u\| + \|Au\| \\
&\qquad + \Big(\frac{1}{\alpha}\|x - u \mid + \|Au\|\Big)\Big\} \\
&\quad\leq |\lambda_n - \lambda_{n+1}| \Big\{\frac{2}{\lambda_{n+1}} \|(I - \lambda_{n+1} A)x - (I - \lambda_{n+1} A)u\| + \|Au\| \\
&\qquad + \Big(\frac{1}{\alpha}\|x - u\| + \|Au\|\Big)\Big\} \\
&\quad\leq |\lambda_n - \lambda_{n+1}| \Big\{\frac{2}{\lambda_{n+1}} (\|x - u\| + \lambda_{n+1}\|Ax - Au\|) + \|Au\| \\
&\qquad + \Big(\frac{1}{\alpha}\|x - u\| + \|Au\|\Big)\Big\} \\
&\quad\leq |\lambda_n - \lambda_{n+1}| \Big\{\frac{2}{a}\|x - u\| + \frac{2}{\alpha}\|x - u\| + \|Au\| + \Big(\frac{1}{\alpha}\|x - u\| + \|Au\|\Big)\Big\} \\
&\quad= |\lambda_n - \lambda_{n+1}| \Big\{\Big(\frac{2}{a} + \frac{3}{\alpha}\Big)\|x - u\| + 2\|Au\|\Big\}
\end{aligned}$$

for each $n \in \mathbf{N}$ and $x \in H$. So, for every bounded subset B of H, there exists $M_B > \sup_{x \in B}\Big\{\Big(\frac{2}{a} + \frac{3}{\alpha}\Big)\|x - u\| + 2\|Au\|\Big\}$ such that $\|T_n x - T_{n+1} x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \mathbf{N})$. Next, let $\{z_n\}$ be a bounded sequence in H such that $\lim_{n\to\infty} \|z_n - J^B_{\lambda_n}(I - \lambda_n A)z_n\| = 0$. Without loss of generality, let $z_n \rightharpoonup w$. Let $v_n = J^B_{\lambda_n}(I - \lambda_n A)z_n$. Then, we obtain

$(v_n - u, \frac{1}{\lambda_n}\{(z_n - \lambda_n A z_n) - v_n\} + Au - v) \geq 0$ and hence

$$\begin{aligned}(v_n - u, -v) &\geq \Big(v_n - u, \frac{1}{\lambda_n}(v_n - z_n) + Az_n - Au\Big)\\ &= \frac{1}{\lambda_n}(v_n - u, (I - \lambda_n A)v_n - (I - \lambda_n A)z_n) + (v_n - u, Av_n - Au)\\ &\geq -\frac{1}{a}\|v_n - u\| \cdot \|(I - \lambda_n A)v_n - (I - \lambda_n A)z_n\|\end{aligned}$$

for all $(u, v) \in A + B$ and $n \in \mathbf{N}$ since A and B are monotone.

$$\begin{aligned}\|(I - \lambda_n A)v_n - (I - \lambda_n A)z_n\|^2 &= \|v_n - z_n\|^2 - 2\lambda_n(v_n - z_n, Av_n - Az_n)\\ &\quad +\lambda_n^2\|Av_n - Az_n\|^2\\ &\leq \|v_n - z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Av_n - Az_n\|^2\\ &\leq \|v_n - z_n\|^2\end{aligned}$$

for each $n \in \mathbf{N}$, $v_n \rightharpoonup w$ and $\{v_n - u\}$ is bounded. So, we have $(w - u, -v) \geq 0$ for every $(u, v) \in A + B$ which implies $w \in (A + B)^{-1}0$ by maximality of $A + B$. Therefore, $\omega_w(z_n) \subset (A + B)^{-1}0 = \cap_{n=1}^{\infty} F(T_n)$. □

Let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Then, an element x in C is a solution of the variational inequality of A if $(y - x, Ax) \geq 0$ for all $y \in C$. It is known that for $\lambda > 0$, $x \in C$ is a solution of the variational inequality of A if and only if $x = P_C(I - \lambda A)x$. We denote by $VI(C, A)$ the set of all solutions of the variational inequality of A. We know that $VI(C, A)$ is a closed convex subset of C if A is monotone and continuous. We have the following two lemmas.

Lemma 2.4. *Let $\alpha > 0$ and C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator with $VI(C, A) \neq \emptyset$. Then, for every $\lambda > 0$, $x \in C$ and $z \in VI(C, A)$, $\|P_C(I - \lambda A)x - z\|^2 \leq \|x - z\|^2 - \frac{2\alpha - \lambda}{2\alpha}\|x - P_C(I - \lambda A)x\|^2$.*

Proof. Let $\lambda > 0$, $x \in C$ and $z \in VI(C, A)$. We have

$$\begin{aligned}&\|P_C(I - \lambda A)x - z\|^2\\ &\leq \|(I - \lambda A)x - (I - \lambda A)z\|^2 - \|(I - P_C)(I - \lambda A)x - (I - P_C)(I - \lambda A)z\|^2\\ &= \|(x - z) - \lambda(Ax - Az)\|^2 - \|(x - P_C(I - \lambda A)x) - \lambda(Ax - Az)\|^2\\ &\leq \|x - z\|^2 - 2\alpha\lambda\|Ax - Az\|^2 + 2\lambda\|Ax - Az\| \cdot \|x - P_C(I - \lambda A)x\|\\ &\quad -\|x - P_C(I - \lambda A)x\|^2\\ &= \|x - z\|^2 - 2\alpha\lambda\Big\{\|Ax - Az\| - \frac{1}{2\alpha}\|x - P_C(I - \lambda A)x\|\Big\}^2\\ &\quad -\frac{2\alpha - \lambda}{2\alpha}\|x - P_C(I - \lambda A)x\|^2\\ &\leq \|x - z\|^2 - \frac{2\alpha - \lambda}{2\alpha}\|x - P_C(I - \lambda A)x\|^2.\end{aligned}$$

□

Lemma 2.5. *Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator and let T be a nonexpansive mapping of C into itself with $F(T) \cap VI(C, A) \neq \emptyset$. Then the following hold:*

(i) $TP_C(I - \lambda A)$ and $P_C(I - \lambda A)T$ are nonexpansive of C into itself when $0 < \lambda \leq 2\alpha$;

(ii) $T_n = TP_C(I - \lambda_n A)$, $0 < a \leq \lambda_n \leq b < 2\alpha$ ($\forall n \in \boldsymbol{N}$) and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ satisfy the conditions (I) and (II) with $a_n = |\lambda_n - \lambda_{n+1}|$ ($\forall n \in \boldsymbol{N}$) and $\cap_{n=1}^{\infty} F(T_n) = F(T) \cap VI(C, A)$;

(iii) $T_n = P_C(I - \lambda_n A)T$, $0 < a \leq \lambda_n \leq b < 2\alpha$ ($\forall n \in \boldsymbol{N}$) and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ satisfy the conditions (I) and (II) with $a_n = |\lambda_n - \lambda_{n+1}|$ ($\forall n \in \boldsymbol{N}$) and $\cap_{n=1}^{\infty} F(T_n) = F(T) \cap VI(C, A)$.

Proof. (i). We have

$$\begin{aligned}
&\|TP_C(I - \lambda A)x - TP_C(I - \lambda A)y\|^2 \leq \|P_C(I - \lambda A)x - P_C(I - \lambda A)y\|^2 \\
&\leq \|(x - y) - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda(x - y, Ax - Ay) + \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2
\end{aligned}$$

for every $x, y \in C$. So, $TP_C(I - \lambda A)$ is nonexpansive. Similarly, $P_C(I - \lambda A)T$ is nonexpansive.

(ii). Let $y \in C$. We have

$$\begin{aligned}
&\|TP_C(I - \lambda_n A)x - TP_C(I - \lambda_{n+1} A)x\| \\
&\leq \|P_C(I - \lambda_n A)x - P_C(I - \lambda_{n+1} A)x\| \leq |\lambda_n - \lambda_{n+1}| \cdot \|Ax\| \\
&\leq |\lambda_n - \lambda_{n+1}| \cdot \left(\frac{1}{\alpha}\|x - y\| + \|Ay\|\right)
\end{aligned}$$

for every $n \in \mathbf{N}$ and $x \in C$. So, for each bounded subset B of C, there exists $M_B > \sup_{x \in B}\{\frac{1}{\alpha}\|x - y\| + \|Ay\|\}$ such that $\|T_n x - T_{n+1} x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\lambda_n - \lambda_{n+1}|$ ($\forall n \in \mathbf{N}$). Next, let $z \in F(T) \cap VI(C, A)$. We have $T_n z = TP_C(I - \lambda_n A)z = Tz = z$. So, $F(T) \cap VI(C, A) \subset F(T_n)$ for every $n \in \mathbf{N}$. Conversely, let $z \in F(T_n)$ and $u \in F(T) \cap VI(C, A)$. By Lemma 2.4, we get

$$\begin{aligned}
\|z - u\|^2 &= \|TP_C(I - \lambda_n A)z - Tu\|^2 \leq \|P_C(I - \lambda_n A)z - u\|^2 \\
&\leq \|z - u\|^2 - \frac{2\alpha - \lambda_n}{2\alpha}\|z - P_C(I - \lambda_n A)z\|^2
\end{aligned}$$

which implies $z = P_C(I - \lambda_n A)z$, that is, $z \in VI(C, A)$. Further, we obtain $Tz = TP_C(I - \lambda_n A)z = z$ and hence, $z \in F(T) \cap VI(C, A)$. So, $F(T_n) \subset F(T) \cap VI(C, A)$

for each $n \in \mathbf{N}$. Therefore, $F(T_n) = F(T) \cap VI(C, A)$ for all $n \in \mathbf{N}$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - TP_C(I - \lambda_n A)z_n\| = 0$. Without loss of generality, let $z_n \rightharpoonup w$. Let $z \in F(T) \cap VI(C, A)$. By Lemma 2.4, we have

$$\begin{aligned} \|z_n - z\|^2 &\le \|z_n - TP_C(I - \lambda_n A)z_n\|(\|z_n - TP_C(I - \lambda_n A)z_n\| \\ &\quad +2\|TP_C(I - \lambda_n A)z_n - z\|) + \|TP_C(I - \lambda_n A)z_n - z\|^2 \\ &\le \|z_n - TP_C(I - \lambda_n A)z_n\|(\|z_n - TP_C(I - \lambda_n A)z_n\| + 2\|z_n - z\|) \\ &\quad + \|z_n - z\|^2 - \frac{2\alpha - \lambda_n}{2\alpha}\|z_n - P_C(I - \lambda_n A)z_n\|^2 \end{aligned}$$

for every $n \in \mathbf{N}$. So, we get $\lim_{n\to\infty} \|z_n - P_C(I - \lambda_n A)z_n\| = 0$. Let $v_n = P_C(I - \lambda_n A)z_n$. For all $u \in C$, we obtain $(z_n - \lambda_n Az_n - v_n, v_n - u) \ge 0$ and hence,

$$\begin{aligned} (Au, u - v_n) &\ge (Av_n - Au, v_n - u) + \frac{1}{\lambda_n}((I - \lambda_n A)v_n - (I - \lambda_n A)z_n, v_n - u) \\ &\ge -\frac{1}{a}\|(I - \lambda_n A)v_n - (I - \lambda_n A)z_n\| \cdot \|v_n - u\| \end{aligned}$$

for every $n \in \mathbf{N}$ since A is monotone. $\|(I - \lambda_n A)v_n - (I - \lambda_n A)z_n\|^2 \le \|v_n - z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Av_n - Az_n\|^2 \le \|v_n - z_n\|^2$ for each $n \in \mathbf{N}$, $v_n \rightharpoonup w$ and $\{v_n - u\}$ is bounded. So, we have $(Au, u - w) \ge 0$ for all $u \in C$. By continuity of A, $(Aw, u - w) \ge 0$ for every $u \in C$, that is, $w \in VI(C, A)$. Further, from $\|z_n - Tz_n\| \le \|z_n - TP_C(I - \lambda_n A)z_n\| + \|P_C(I - \lambda_n A)z_n - z_n\|$ for each $n \in \mathbf{N}$, $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. By Opial's condition, $w \in F(T)$. Therefore, $\omega_w(z_n) \subset F(T) \cap VI(C, A) = \cap_{n=1}^{\infty} F(T_n)$.

(iii). Let $y \in C$. As in (ii), for every bounded subset B of C, there exists $M_B > \sup_{x\in B}\{\frac{1}{\alpha}\|x - y\| + \|ATy\|\}$ such that $\|T_n x - T_{n+1}x\| \le a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \mathbf{N})$. As in the proof of (ii), we have $F(T_n) = F(T) \cap VI(C, A)$ for each $n \in \mathbf{N}$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - P_C(I - \lambda_n A)Tz_n\| = 0$. Without loss of generality, let $z_n \rightharpoonup w$. As in (ii), we get $\lim_{n\to\infty} \|Tz_n - P_C(I - \lambda_n A)Tz_n\| = 0$. And hence, we obtain $\lim_{n\to\infty} \|z_n - Tz_n\| = \lim_{n\to\infty} \|z_n - P_C(I - \lambda_n A)z_n\| = 0$. So, $w \in F(T) \cap VI(C, A)$. Therefore, $\omega_w(z_n) \subset F(T) \cap VI(C, A) = \cap_{n=1}^{\infty} F(T_n)$. $\square$

Let C be a nonempty closed convex subset of H. Let $S_1, S_2, \cdots$ be infinite nonexpansive mappings of C into itself and let $\beta_1, \beta_2, \cdots$ be real numbers such that $0 \le \beta_i \le 1$ for every $i \in \mathbf{N}$. Then, for any $n \in \mathbf{N}$, Takahashi [25, 32, 34] introduced a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n.n+1} &= I, \\
U_{n,n} &= \beta_n S_n U_{n,n+1} + (1-\beta_n)I, \\
U_{n,n-1} &= \beta_{n-1} S_{n-1} U_{n,n} + (1-\beta_{n-1})I, \\
&\vdots \\
U_{n,k} &= \beta_k S_k U_{n,k+1} + (1-\beta_k)I, \\
&\vdots \\
U_{n,2} &= \beta_2 S_2 U_{n,3} + (1-\beta_2)I, \\
W_n = U_{n,1} &= \beta_1 S_1 U_{n,2} + (1-\beta_1)I.
\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\beta_n, \beta_{n-1}, \cdots, \beta_1$. We know that if $\cap_{i=1}^n F(S_i) \neq \emptyset$ and $0 < \beta_i < 1$ for every $i = 2, 3, \cdots, n$ and $0 < \beta_1 \leq 1$, $F(W_n) = \cap_{i=1}^n F(S_i)$; see [34, 35]. We also have that if $\cap_{n=1}^\infty F(S_n) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbf{N}$ for some $b \in (0,1)$, $\lim_{n\to\infty} U_{n,k}x$ exists for every $x \in C$ and $k \in \mathbf{N}$; see [25]. By this, we define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-mapping generated by $S_1, S_2, \cdots$ and $\beta_1, \beta_2, \cdots$. And we have that if $\cap_{i=1}^\infty F(S_i) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbf{N}$ for some $b \in (0,1)$, $F(W) = \cap_{i=1}^\infty F(S_i)$; see [25]. We know the following result.

Lemma 2.6. *Let C be a nonempty closed convex subset of H. Let $S_1, S_2, \cdots$ be infinite nonexpansive mappings of C into itself with $\cap_{n=1}^\infty F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \cdots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \mathbf{N}$ for some $b \in (0,1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\beta_n, \beta_{n-1}, \cdots, \beta_1$ for every $n \in \mathbf{N}$. Then, $T_n = W_n$ ($\forall n \in \mathbf{N}$) satisfy the conditions (I) and (II) with $a_n = b^{n+1}$ ($\forall n \in \mathbf{N}$).*

Proof. Let $u \in \cap_{n=1}^\infty F(S_n)$. we have

$$\begin{aligned}
&\|W_n x - W_{n+1}x\| = \|\beta_1 S_1 U_{n,2}x - \beta_1 S_1 U_{n+1,2}x\| \\
&\leq \ \beta_1 \|U_{n,2}x - U_{n+1,2}x\| \\
&= \ \beta_1 \|\beta_2 S_2 U_{n,3}x - \beta_2 S_2 U_{n+1,3}x\| \leq \beta_1\beta_2 \|U_{n,3}x - U_{n+1,3}x\| \\
&\leq \ \cdots \leq \beta_1\beta_2\cdots\beta_n\beta_{n+1}\|x - S_{n+1}x\| \leq b^{n+1}\{2\|x-u\|\}
\end{aligned}$$

for every $n \in \mathbf{N}$ and $x \in C$. So, for each bounded subset B of C, there exists $M_B > 2 \cdot \sup_{x\in B}\|x-u\|$ such that $\|T_n x - T_{n+1}x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = b^{n+1}$ ($\forall n \in \mathbf{N}$). From [14, Theorem 3.1], $\{T_n\}$ satisfies the condition (II). □

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Then, for every $s \in S$ and $f \in B(S)$, we can define $r_s f \in B(S)$ and $l_s f \in B(S)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for each $t \in S$, respectively. We also denote by r_s^* and l_s^* the conjugate operators of r_s and l_s, respectively. Let D be a subspace of $B(S)$ containing constants and let μ be an element of D^*. A linear functional μ is called a mean on D if $\|\mu\| = \mu(1) = 1$. Let C be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(st) = T(s)T(t)$ for all $s, t \in S$;

(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for every $s \in S$ and $x, y \in C$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \cap_{s\in S} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex. We have that if $F(\mathcal{S}) \neq \emptyset$ and $(T(\cdot)x, y) \in D$ for every $x \in C$ and $y \in H$, there exists a unique element $T_\mu x$ in C such that $(T_\mu x, z) = \mu_s(T(s)x, z)$ for all $z \in H$ for any mean μ on D and $x \in C$; see [8, 31]. We also know that T_μ is a nonexpansive mapping of C into itself. Further, we have the following [1]: Let C be a nonempty bounded closed convex subset of H and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C and let D be a subspace of $B(S)$ containing constants and invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $z \in H$, the function $t \mapsto (T(t)x, z)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Then, $\lim_{n\to\infty} \sup_{x\in C} \|T_{\mu_n} x - T(t)T_{\mu_n} x\| = 0$ for all $t \in S$.

Lemma 2.7. *Let C be a nonempty closed convex subset of H and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $z \in H$, the function $t \mapsto (T(t)x, z)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Then, $T_n = T_{\mu_n}$ ($\forall n \in \boldsymbol{N}$) satisfy the condition (III) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{S})$.*

Proof. $F(\mathcal{S}) \subset \cap_{n=1}^\infty F(T_n)$ is trivial. Let $u \in \cap_{n=1}^\infty F(T_n)$. We have $\lim_{n\to\infty} \|T_{\mu_n} u - T(t)T_{\mu_n} u\| = 0$ for every $t \in S$ and hence, $u \in F(\mathcal{S})$. So, we get $F(\mathcal{S}) = \cap_{n=1}^\infty F(T_n)$. Next, let $\{z_n\} \subset C$ be a bounded sequence such that $\lim_{n\to\infty} \|z_{n+1} - T_{\mu_n} z_n\| = 0$. We obtain

$$\begin{aligned} \|z_{n+1} - T(t)z_{n+1}\| &\leq \|z_{n+1} - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T(t)T_{\mu_n} z_n\| \\ &\quad + \|T(t)T_{\mu_n} z_n - T(t)z_{n+1}\| \\ &\leq 2\|z_{n+1} - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T(t)T_{\mu_n} z_n\| \end{aligned}$$

for every $t \in S$ and $n \in \mathbf{N}$. Since we have $\lim_{n\to\infty} \|T_{\mu_n} z_n - T(t)T_{\mu_n} z_n\| = 0$ for all $t \in S$, $\lim_{n\to\infty} \|z_n - T(t)z_n\| = 0$ for each $t \in S$. So, $\{T_n\}$ satisfies the condition (III) by Opial's condition. □

Let C be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for every $s, t \ge 0$;

(iii) $\|T(s)x - T(s)y\| \le \|x - y\|$ for each $s \ge 0$ and $x, y \in C$;

(iv) for all $x \in C$, $s \longmapsto T(s)x$ is continuous.

3 Main Results

Using an idea of [37] (see also [34, Theorem 5.1.2]), we have the following two theorems.

Theorem 3.1. *Let C be a nonempty closed convex subset of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the conditions (I) and (II). Let $\{x_n\}$ be a sequence generated by (1), where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is the metric projection of H onto F.*

Proof. Let $u \in F$. We have $\|x_n - u\| \le \|x - u\|$ for every $n \in \mathbf{N}$. In fact, suppose that $\|x_n - u\| \le \|x - u\|$ for some $n \in \mathbf{N}$. We get

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n) - u\| \\ &\le \alpha_n\|x-u\| + (1-\alpha_n)\{\beta_n\|x-u\| + (1-\beta_n)\|x_n - u\|\} \\ &\le \|x-u\|. \end{aligned}$$

So, $\{x_n\}$ is bounded. Next, we obtain

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n) - \alpha_{n-1}x \\ &\quad -(1-\alpha_{n-1})T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &= \|(\alpha_n - \alpha_{n-1})x + (1-\alpha_n)\{T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\} \\ &\quad + (1-\alpha_n)\{T_{n-1}(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\} \\ &\quad + (\alpha_{n-1} - \alpha_n)T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &\le |\alpha_n - \alpha_{n-1}| \cdot \|x - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &\quad + (1-\alpha_n)\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \\ &\quad + (1-\alpha_n)\|\{\beta_n x + (1-\beta_n)x_n\} - \{\beta_{n-1}x + (1-\beta_{n-1})x_{n-1}\}\| \\ &\le |\alpha_n - \alpha_{n-1}| \cdot M_1 + (1-\alpha_n)\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \\ &\quad + (1-\alpha_n)\{|\beta_n - \beta_{n-1}| \cdot (\|x\| + \|x_{n-1}\|) + (1-\beta_n)\|x_n - x_{n-1}\|\} \end{aligned}$$

for each $n = 2, 3, \cdots$, where $M_1 = \sup_{n\in\mathbf{N}\setminus\{1\}} \|x - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\|$. Since a sequence $\{\beta_n x + (1-\beta_n)x_n\}$ is bounded, there exists $M_2 > 0$ such that

$$\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \le a_{n-1}M_2$$

for all $n = 2, 3, \cdots$ by the condition (I). Therefore, we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + a_{n-1})M \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - x_{n-1}\| \end{aligned} \tag{3.1}$$

for every $n = 2, 3, \cdots$, where $M = \max\{M_1, M_2, \sup_{n\in\mathbf{N}\setminus\{1\}}\{\|x\| + \|x_{n-1}\|\}\}$. Let $m, n \in \mathbf{N}$. By (2), we obtain

$$\begin{aligned} &\|x_{n+m+1} - x_{n+m}\| \\ &\le (|\alpha_{n+m} - \alpha_{n+m-1}| + |\beta_{n+m} - \beta_{n+m-1}| + a_{n+m-1})M \\ &\quad + (1-\alpha_{n+m})(1-\beta_{n+m})\|x_{n+m} - x_{n+m-1}\| \\ &\le (|\alpha_{n+m} - \alpha_{n+m-1}| + |\beta_{n+m} - \beta_{n+m-1}| + a_{n+m-1})M \\ &\quad + (1-\alpha_{n+m})(1-\beta_{n+m})\{(|\alpha_{n+m-1} - \alpha_{n+m-2}| \\ &\quad + |\beta_{n+m-1} - \beta_{n+m-2}| + a_{n+m-2})M \\ &\quad + (1-\alpha_{n+m-1})(1-\beta_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\|\} \\ &\le \{(|\alpha_{n+m} - \alpha_{n+m-1}| + |\alpha_{n+m-1} - \alpha_{n+m-2}|) \\ &\quad + (|\beta_{n+m} - \beta_{n+m-1}| + |\beta_{n+m-1} - \beta_{n+m-2}|) + (a_{n+m-1} + a_{n+m-2})\}M \\ &\quad + (1-\alpha_{n+m})(1-\beta_{n+m})(1-\alpha_{n+m-1})(1-\beta_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\| \\ &\le \cdots \\ &\le M \cdot \sum_{k=m}^{n+m-1} (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k| + a_k) \\ &\quad + \|x_{m+1} - x_m\| \cdot \prod_{k=m+1}^{n+m} (1-\alpha_k)(1-\beta_k). \end{aligned}$$

So, we have

$$\begin{aligned} \limsup_{n\to\infty} \|x_{n+1} - x_n\| &= \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \\ &\le M \cdot \sum_{k=m}^{\infty} (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k| + a_k) \end{aligned}$$

for each $m \in \mathbf{N}$. Therefore, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. It follows from

$$\begin{aligned} \|x_n - T_n x_n\| &\le \|x_n - T_n(\beta_n x + (1-\beta_n)x_n)\| + \|T_n(\beta_n x + (1-\beta_n)x_n) - T_n x_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|x - T_n(\beta_n x + (1-\beta_n)x_n)\| + \beta_n\|x - x_n\| \end{aligned}$$

for all $n \in \mathbf{N}$ that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. By the condition (II), we get $\omega_w(x_n) \subset F$. From $\lim_{n\to\infty} \|x_n - T_n(\beta_n x + (1-\beta_n)x_n)\| = 0$,

$$\limsup_{n\to\infty}(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) = \limsup_{n\to\infty}(x - P_F x, x_n - P_F x).$$

There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}(x - P_F x, x_n - P_F x) = \lim_{k\to\infty}(x - P_F x, x_{n_k} - P_F x)$. Since $\{x_{n_k}\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup z \in F$. So, we obtain

$$\limsup_{n\to\infty}(x - P_F x, x_n - P_F x) = (x - P_F x, z - P_F x) \leq 0.$$

Let $\varepsilon > 0$. There exists $m_0 \in \mathbf{N}$ such that $\alpha_n\|x - P_F x\|^2 < \frac{\varepsilon}{2}$, $\beta_n\|x - P_F x\|^2 < \frac{\varepsilon}{2}$, $(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) < \frac{\varepsilon}{4}$ and $(x - P_F x, x_n - P_F x) < \frac{\varepsilon}{4}$ for every $n \geq m_0$. So, we have

$$\begin{aligned}
&\|x_{n+1} - P_F x\|^2 = \|\alpha_n(x - P_F x) + (1-\alpha_n)\{T_n(\beta_n x + (1-\beta_n)x_n) - P_F x\}\|^2 \\
&\quad= \alpha_n^2\|x - P_F x\|^2 + (1-\alpha_n)^2\|T_n(\beta_n x + (1-\beta_n)x_n) - P_F x\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) \\
&\quad\leq \alpha_n^2\|x - P_F x\|^2 + (1-\alpha_n)^2\{\beta_n^2\|x - P_F x\|^2 + (1-\beta_n)^2\|x_n - P_F x\|^2 \\
&\qquad + 2\beta_n(1-\beta_n)(x - P_F x, x_n - P_F x)\} \\
&\qquad +2\alpha_n(1-\alpha_n)(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) \\
&\quad= \{\alpha_n^2 + \beta_n^2(1-\alpha_n)^2\}\|x - P_F x\|^2 + 2(1-\alpha_n)^2\beta_n(1-\beta_n)(x - P_F x, x_n - P_F x) \\
&\qquad + 2\alpha_n(1-\alpha_n)(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) \\
&\qquad +(1-\alpha_n)^2(1-\beta_n)^2\|x_n - P_F x\|^2 \\
&\quad\leq \{\alpha_n + \beta_n(1-\alpha_n)^2 + \beta_n(1-\beta_n)(1-\alpha_n)^2 + \alpha_n(1-\alpha_n)\}\frac{\varepsilon}{2} \\
&\qquad +(1-\alpha_n)(1-\beta_n)\|x_n - P_F x\|^2 \\
&\quad\leq \{\alpha_n + \beta_n(1-\alpha_n) + \beta_n(1-\alpha_n) + \alpha_n\}\frac{\varepsilon}{2} + (1-\alpha_n)(1-\beta_n)\|x_n - P_F x\|^2 \\
&\quad= \{1 - (1-\alpha_n)(1-\beta_n)\}\varepsilon + (1-\alpha_n)(1-\beta_n)\|x_n - P_F x\|^2
\end{aligned}$$

for each $n \geq m_0$. Therefore, we get

$$\begin{aligned}
\|x_{n+1} - P_F x\|^2 \quad\leq\quad &\{1 - \prod_{k=m_0}^{n}(1-\alpha_k)(1-\beta_k)\}\varepsilon \\
&+\|x_{m_0} - P_F x\|^2 \prod_{k=m_0}^{n}(1-\alpha_k)(1-\beta_k)
\end{aligned}$$

for all $n \geq m_0$. So, we obtain $\limsup_{n\to\infty}\|x_{n+1} - P_F x\|^2 \leq \varepsilon$. Since ε is arbitrary, we have $x_n \to P_F x$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty}F(T_n) \neq \emptyset$ which satisfies the condition (III). Let $\{x_n\}$ be a sequence generated by (1), where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$.*

Proof. As in the proof of Theorem 3.1, we have $\{x_n\}$ is bounded. And it follows from

$$\begin{aligned}\|x_{n+1} - T_n x_n\| &\le \|x_{n+1} - T_n(\beta_n x + (1-\beta_n)x_n)\| \\ &\quad + \|T_n(\beta_n x + (1-\beta_n)x_n) - T_n x_n\| \\ &\le \alpha_n \|x - T_n(\beta_n x + (1-\beta_n)x_n)\| + \beta_n \|x - x_n\|\end{aligned}$$

for all $n \in \mathbf{N}$ that $\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = 0$. By the condition (III), we get $\omega_w(x_n) \subset F$. Since $\lim_{n\to\infty} \|x_{n+1} - T_n(\beta_n x + (1-\beta_n)x_n)\| = 0$, we have $\limsup_{n\to\infty}(x - P_F x, T_n(\beta_n x + (1-\beta_n)x_n) - P_F x) = \limsup_{n\to\infty}(x - P_F x, x_n - P_F x)$. As in the proof of Theorem 3.1, we obtain $x_n \to P_F x$. □

4 Applications

In this section, using Theorems 3.1 and 3.2, we improve well-known strong convergence theorems. We first have the following theorem which generalizes the result of [37] by Lemma 2.1 and Theorem 3.1.

Theorem 4.1. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)T(\beta_n x + (1-\beta_n)x_n)$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$.*

We get the following theorem for proximal point algorithms (see [23, 12]) by Lemma 2.2 (i) and Theorem 3.1 (see also [17, 38]).

Theorem 4.2. *Let $A : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$, $x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}(\beta_n x + (1-\beta_n)x_n)$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and $\{\lambda_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$.*

We get the following theorem for proximal point algorithms which generalizes the result of [12] by Lemma 2.2 (ii) and Theorem 3.2.

Theorem 4.3. *Let $A : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$, $x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}(\beta_n x + (1-\beta_n)x_n)$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\{\lambda_n\} \subset (0,\infty)$ satisfies $\lim_{n\to\infty} \lambda_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$.*

We have the new theorem for splitting methods by Lemma 2.3 and Theorem 3.1.

Theorem 4.4. *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be an α-inverse-strongly-monotone operator and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$, $x_{n+1} = \alpha_n x + (1-\alpha_n)J^B_{\lambda_n}(I - \lambda_n A)(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$ satisfies $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{(A+B)^{-1}0}x$.*

We get the following theorem which generalizes the result of [9] by Lemma 2.5 (i), (ii) and Theorem 3.1.

Theorem 4.5. *Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator and let T be a nonexpansive mapping of C into itself with $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)TP_C(I - \lambda_n A)(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ with $a \leq b$ satisfies $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(C,A)}x$.*

We also have the following theorem which generalizes the result of [11] by Lemma 2.5 (i), (iii) and Theorem 3.1.

Theorem 4.6. *Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator and let T be a nonexpansive mapping of C into itself with $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)P_C(I - \lambda_n A)T(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ with $a \leq b$ satisfies $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(C,A)}x$.*

We have the following theorem for the W-mapping by Lemma 2.6 and Theorem 3.1 (see also [25]).

Theorem 4.7. *Let C be a nonempty closed convex subset of H. Let $S_1, S_2, \cdots$ be infinite nonexpansive mappings of C into itself with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \cdots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \boldsymbol{N}$ for some $b \in (0,1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\beta_n, \beta_{n-1}, \cdots \beta_1$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)W_n(\gamma_n x + (1-\gamma_n)x_n)$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\gamma_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\gamma_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_{\cap_{n=1}^{\infty} F(S_n)}x$.*

We have the following theorem for nonexpansive semigroups by Lemma 2.7 and Theorem 3.2 (see also [27]).

Theorem 4.8. *Let C be a nonempty closed convex subset of H and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $z \in H$, the function $t \mapsto (T(t)x, z)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)T_{\mu_n}(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{S})}x$.*

From Theorem 4.8, we get the following theorems.

Theorem 4.9. *([24]) Let C be a nonempty closed convex subset of H and let T_1, T_2 be nonexpansive mappings of C into itself such that $T_1T_2 = T_2T_1$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} T_1^iT_2^j x_n$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T_1)\cap F(T_2)}x$.*

Proof. Let $S = \{0,1,2,\cdots\} \times \{0,1,2,\cdots\}$, $\mathcal{S} = \{T_1^iT_2^j : (i,j) \in S\}$, $D = B(S)$ and
$$\mu_n(f) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} f(i,j)$$
for every $n \in \mathbf{N}$ and $f \in D$. Then, as in [1, Corollary 3.7], $\{\mu_n\}$ is a sequence of means on D and $\lim_{n\to\infty} \|\mu_n - l_{(l,m)}^*\mu_n\| = 0$ for each $(l,m) \in S$. By Theorem 4.8, we get Theorem 4.9. □

Theorem 4.10. *([24]) Let C be a nonempty closed convex subset of H and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset [0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\{t_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{S})}x$.*

Proof. Let $S = \{s \in \mathbf{R} : 0 \leq s\}$, $\mathcal{S} = \{T(s) : s \in S\}$ and let D be the Banach space $C(S)$ of all bounded continuous functions on S with supremum norm. Let $\lambda_s(f) = \frac{1}{s}\int_0^s f(t)\,dt$ for every $s > 0$ and $f \in D$. Then, $\lim_{s\to\infty} \|\lambda_s - l_k^*\lambda_s\| = 0$ for each $k \in (0,\infty)$ from [1, Corollary 3.8]. We also have $T_{\lambda_s}x = \frac{1}{s}\int_0^s T(t)x\,dt$ for every $x \in C$. By Theorem 4.8, we get Theorem 4.10. □

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