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Amenability and Weak Amenability of Some Banach Algebras

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Abstract In this paper, we suppose that H is a compact subgroup of locally compact topological group G and G/H is a homogeneous space which is equipped with a strongly quasi-invariant Radon measure μ.Then in the group algebra $L^1(G)$, we replace the homogeneouse space G/H instead of G and consider the new Banach algebra $L^1(G/H)$. We study this Banach algebra and it's dual. At the end, by characterization of $L^\infty(G/H)$ and the left and right dual $L^1(G/H)$-module actions of $L^\infty(G/H)$, we give a necessary and sufficient conditions for amenability and weak amenability of this Banach algebra.

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1. Introduction

Let G be a locally compact group and H be a closed subgroup of G. Then the space G/H consisting of all left cosets of H in G is a locally compact Hausdorff topological space that G acts on it transitively from the left. The term homogeneous space means a transitive G-space which is topologically isomorphic to G/H, for some closed subgroup H of G. It has been shown that if G is σ-compact, then every transitive G-space is homeomorphic to the quotient space G/H for some closed subgroup H (cf.[1], Subsection 2.6). We know that the homogeneous space G/H is not a group when H is not normal. However, over the last decades, the principal part of the classical harmonic analysis on locally compact topological groups carries over the homogeneous spaces G/H and it is quite well studied by several authors and have been achieved many interesting applications in geometric analysis, mathematical physics, differential geometry, geometric analysis (cf. [2–7]).

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In the following paper we aim to further develop the abstract results over the some Banach function algebras related to homogeneous spaces (coset spaces) of a locally compact group. For a locally compact group G with the left Haar measur m_G, it is well known that $L^1(G)$ is an involutive Banach algebra with a bounded approximate identity. The standard convolution for $f, g \in L^1(G)$ is given by

$$f *_{L^1(G)} g(x) = \int_G f(y)g(y^{-1}x)dm_G \qquad \text{(a.e } x \in G), \tag{1.1}$$

(cf.[1]). In [8], assuming that H is a compact subgroup of G with the normalized Haar measure m_H and μ is a stongly quasi-invariant Radon measure on G/H arising from the rho-function ρ, it is shown that there is a well defined convolution on $L^1(G/H, \mu)$. This convolution for $\varphi, \psi \in L^1(G/H, \mu)$ is given by

$$\varphi * \psi(xH) = \int_H \varphi_\rho *_{L^1(G)} g(xh)(\rho(xh))^{-1}dm_H \qquad \text{(a.e } xH \in G/H),$$

where $\varphi_\rho = \rho(\varphi \circ q)$ and g is any function in $L^1(G)$ which

$$\psi(xH) = \int_H g(xh)(\rho(xh))^{-1}dm_H \qquad \text{(a.e } xH \in G/H).$$

Also, $L^1(G/H, \mu)$ with this convolution becomes a Banach algebra which has a bounded right approximate identity and it is involutive Banach algebra if and only if H is normal in G. For any $\varphi \in L^1(G/H, \mu)$ and $a \in G$, the left and right translations are defined respectively as

$$L_a\varphi(xH) = \int_H \mathcal{L}_a\varphi_\rho(xh)(\rho(xh))^{-1}dm_H \qquad \text{(a.e } xH \in G/H),$$

and

$$R_a\varphi(xH) = \int_H \mathcal{R}_a\varphi_\rho(xh)(\rho(xh))^{-1}dm_H \qquad \text{(a.e } xH \in G/H),$$

where $\mathcal{L}_a$ $\big($resp. $\mathcal{R}_a\big)$ is the left translation on $L^1(G)$ which is given by $\mathcal{L}_a f(x) = f(a^{-1}x)$ $\big($resp. $\mathcal{R}_a f(x) = f(xa)\big)$ for $f \in L^1(G)$ and $x \in G$.

It is well known that $L^1(G)$ as a Banach algebra is amenable if and only if G is amenable (The well known Johnson's theorem). Also, $L^1(G)$ is always weakly amenable (see [9]). In this paper, motivated by the amenability and weak amenability of $L^1(G)$, we consider $L^1(G/H)$ as a Banach algebra where H is a compact subgroup of G and G/H is a homogeneous space which it is not necessarily a locally compact group. Then we characterize $L^\infty(G/H)$ as dual of the Banach algebra $L^1(G/H)$ and we obtain the left and the right dual $L^1(G/H)$-module actions of $L^\infty(G/H)$ and study the amenability and weak amenability $L^1(G/H)$. Finally, we find necessary and sufficient conditions for amenability and weak amenablity of the Banach algebra $L^1(G/H)$.

2. Preliminaries

In this section, for the readers convenience, we provide a summary of the mathematical notations and definitions which will be used in the sequel. (For details, we refer the reader to the general reference [9, 10], or any other standard book of harmonic analysis.)

For a locally compact Hausdorff space X equipped with a positive Radon measure m_X, we mean the space of containing all of continuous complex-valued functions on X which have compact support by $C_c(X)$. For each $1 \leq p < \infty$, we denote the Banach space of equivalence classes of m_X-measurable complex valued functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p = \big(\int_X |f(x)|^p dm_X(x)\big)^{1/p} < \infty,$$

by $L^p(X, m_X)$ and in brief by $L^p(X)$ which contains $C_c(X)$ as a $\|\cdot\|_p$-dense subspace. We denote the Banach space of all equivalence classes of locally measurable functions on X which are locally essentially bounded, by $L^\infty(X)$. The functions f, g in $L^p(X)$ are equal if they are equal almost everywhere and we just write $f = g$ for $1 \leq p < \infty$. Also, for $f, g \in L^\infty(X)$ the equality $f = g$ means that they are equal locally almost everywhere.

Let A be a Banach algebra and E be a Banach A-bimodule. Then the dual Banach space E^* of E is a Banach A-bimodule, with the dual actions given by

$$(a \cdot f)(x) = f(xa) \text{ and } (f \cdot a)(x) = f(ax) \ (f \in E^*, a \in A, x \in E).$$

In particular, A^* is a Banach A-bimodule. For example, for a locally compact topological group G, it is well known that $L^\infty(G)$ as dual of $L^1(G)$ is a Banach $L^1(G)$-bimodule and for each $f \in L^1(G)$ and $\psi \in L^\infty(G)$ the left and right $L^1(G)$-module actions of $L^\infty(G)$ are given by

$$\psi \cdot f = \tilde{f} * \psi \text{ and } f \cdot \psi = \psi * \check{f}$$

in which $\tilde{f}(x) = f(x^{-1})/\Delta(x)$ and $\check{f}(x) = f(x^{-1})$ and Δ is the modular function of G.

A linear map $D : A \to E^*$ is a derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)(a, b \in A)$. For example, if $\varphi \in E^*$, then the map $d_\varphi : a \mapsto a \cdot \varphi - \varphi \cdot a$ is a derivation. The derivations such as d_φ are called inner. The set of all derivations and inner drivations from A into E are denoted by $Z^1(A, E)$ and $B^1(A, E)$, respectively. Also, the quotient space $H^1(A, E) = Z^1(A, E)/B^1(A, E)$ is called first cohomology group of A.

Let A be a Banach algebra. Then A is called amenable if $H^1(A, E^*) = 0$ for every Banach A-bimodule E. Also, A is called weakly amenable if $H^1(A, A^*) = 0$, i.e, a Banach algebra A is weakly amenable if every continuous derivation from A into A^* is inner. For example in [9], we can see that the group algebra $L^1(G)$ is amenable if and only if G is amenable and also, $L^1(G)$ is always weakly amenable.

When G is a locally compact topological group and H is a closed subgroup of G, then the quotient space G/H consisting of all left cosets of H in G, is a homogeneous space that G acts on it from the left. Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$ for all Borel subset $E \subseteq G/H$. The measure μ is called strongly quasi-invariant measure on the homogeneous space G/H if there exists a continuous function $\lambda : G \times (G/H) \to (0, \infty)$ such that $d\mu_x(E) = \lambda(x, E)d\mu(E)$ for all $x \in G$ and Borel subset E of G/H.

Let Δ_G and Δ_H be the modular functions of G and H, respectively. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that $\rho(xh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(x)$ for each $x \in G$ and $h \in H$. It has been shown that for any locally compact group G and closed subgroup H of G, the pair (G, H) admits a rho-function (cf. [1], Proposition 2.54). If m_G and m_H are the Haar measures G and H, respectively, then for any given rho-function ρ, the homogeneous space G/H has a strongly quasi-invariant Radon measure μ which satisfies in the Mackey-Bruhat formula; i.e.,

$$\int_{G/H} \int_H f(xh)(\rho(xh))^{-1} dm_H d\mu(xH) = \int_G f(x) dm_G \qquad (f \in L^1(G)),$$

(cf. [1]).

Throughout this paper, we suppose that G is a locally compact topological group with the left Haar measure m_G, H is a compact subgroup of G with the normalized Haar measure m_H and G/H is a homogeneous space which is equipped to a strongly quassi invariant measure μ. Also, the map $q : G \to G/H$ by $q(x) = xH$ is the canonical quotient map.

3. Main Results

In this section, we suppose that $1 \leq p < \infty$ and H is a compact subgroup of locally compact group G. When H is closed then the function space $C_c(G/H)$ consists of all functions $P_H(f)$, where $f \in C_c(G)$ and

$$P_H(f)(xH) = \int_H f(xh)(\rho(xh))^{-1} dm_H. \tag{3.1}$$

This equivalently means that the linear map $P_H : C_c(G) \to C_c(G/H)$ is a surjective bounded linear operator. The extension of the linear map P_H of $L^1(G)$ onto $L^1(G/H)$ is norm-decreasing, that is

$$\|P_H(f)\|_1 \leq \|f\|_1 \qquad (f \in L^1(G)),$$

(cf. [1, 11, 12]). Now, by assuming that H is a compact subgroup of G, we consider that the linear map $P_H : C_c(G) \to C_c(G/H)$ given by

$$P_H(f)(xH) = \int_H f(xh)(\rho(xh))^{-1/p} dm_H. \tag{3.2}$$

Then we show that P_H is extendable from $L^p(G)$ onto $L^p(G/H)$ and also it is norm-decreasing for $1 \leq p < \infty$. Note that the value of p in relation (3.2) is determined by value of p in $L^p(G)$.

Proposition 3.1. *Let H be a compact subgroup of locally compact group G, μ be a strongly quassi invariant measure on G/H associated to the Mackey-Brouhat formula, and $1 \leq p < \infty$. Then the linear map P_H introduced in (3.2) is extendable to a unique surjective, norm-decreasing and bounded linear map from $L^p(G)$ onto $L^p(G/H)$ which still will be denoted by P_H.*

Proof. Let $f \in C_c(G)$, H be a compact subgroup of G and $1 \leq p < \infty$. Then the compactness of H, using Minkowski's inequality and the Mackey-Brouhat formula allow us to write

$$\begin{aligned}
\|P_H f\|_p^p &= \int_{G/H} |P_H f(xH)|^p d\mu(xH) \\
&= \int_{G/H} |\int_H f(xh)(\rho(xh))^{-1/p} dm_H|^p d\mu(xH) \\
&\leq \int_{G/H} \int_H |f(xh)|^p \rho(xh)^{-1} dm_H d\mu(xH) \\
&= \int_G |f(x)|^p dm_G \\
&= \|f\|_p^p.
\end{aligned}$$

So, $\|P_H f\|_p \leq \|f\|_p$. Hence, P_H has a unique extension to a norm-decreasing linear map from $L^p(G)$ onto $L^p(G/H)$ and still will be denoted by P_H.

The map P_H is surjective. Because if $\varphi \in L^p(G/H)$, then by taking $f = \rho^{1/p}(\varphi \circ q)$ and using of the Mackey-Brouhat formula we have

$$\begin{aligned}
\|f\|_p^p &= \int_G \rho(x)|\varphi \circ q|^p(x) dm_G \\
&= \int_{G/H} \int_H \rho(xh)|\varphi \circ q|^p(xh)(\rho(xh))^{-1} dm_H d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|^p d\mu(xH) \\
&= \|P_H f\|_p^p,
\end{aligned}$$

therefore, $f \in L^p(G)$ and also it's obvious that $P_H(f) = \varphi$. ∎

Note that by this fact that a relative invariant Radon measure is a special case of a strongly quasi invariant Radon measure and in the above theorem the homogeneous space G/H has been equipped with a strongly quasi invariant Radon measure, so the above theorem can be considered as a generalization of Proposition 3.4 in [8].

Corollary 3.2. *Let H be a compact subgroup of a locally compact group G. Then for all $\varphi \in L^p(G/H)$, we have*

$$\|\varphi\|_p = \|\varphi_\rho\|_p,$$

where $\varphi_\rho = \rho^{1/p}(\varphi \circ q)$ and q is the canonical quotient map on G/H.

Proof. For all $\varphi \in L^p(G/H)$, by the compactness of H and using the Mackey-Brouhat formula, we can write

$$\begin{aligned}
\|\varphi\|_p^p &= \int_{G/H} |\varphi(xH)|^p d\mu(xH) \\
&= \int_{G/H} \int_H |\varphi_\rho(xh)|^p (\rho(xh))^{-1} dm_H d\mu(xH) \\
&= \int_G |\varphi_\rho(xh)|^p dm_G \\
&= \|\varphi_\rho\|_p^p,
\end{aligned}$$

which this completes the proof. ∎

For a compact subgroup H of G we set

$$C_c(G : H) = \{f \in C_c(G) : R_h f = f, h \in H\}$$

and $L^p(G : H)$ is clouser of $C_c(G : H)$ under the norm $\|\cdot\|_p$. At the following proposition we characterize $L^p(G : H)$.

Proposition 3.3. *Let H be a compact subgroup of G. Then the space $L^p(G : H)$ is specified as follow*

$$\begin{aligned}
L^p(G : H) &= \{f \in L^p(G) : R_h f = f, h \in H\} \\
&= \{\rho^{1/p}(\varphi \circ q) : \varphi \in L^p(G/H)\},
\end{aligned}$$

which is a closed subalgebra of $L^p(G)$.

Proof. For this end, it is enough to show that for all $f \in L^p(G:H)$, the equality $f = (P_H f)_\rho := \rho^{1/p}(P_H f \circ q)$ holds. Let $f \in L^p(G:H)$. Then there is a sequence $(f_n)_n$ in $C_c(G:H)$ such that $\|f_n - f\|_p \to 0$. Now, by the compactness of H and Corollary 3.2 and Proposition 3.1, we can write

$$\begin{aligned}\|f - (P_H f)_\rho\|_p &\leq \|f - f_n\|_p + \|(P_H(f_n - f))_\rho\|_p \\ &= \|f - f_n\|_p + \|P_H(f_n - f)\|_p \\ &\leq 2\|f - f_n\|_p,\end{aligned}$$

which guarantees that $f = (P_H f)_\rho$. ■

Corollary 3.4. *For a compact subgroup H of G, the normed space $L^p(G/H)$ can be considered as a closed subspace of $L^p(G)$ whenever $1 \leq p < \infty$.*

Proof. By restriction of the map P_H on $L^p(G:H)$ and using Propositions 3.2 and 3.3 for all $f \in L^p(G:H)$, we have

$$\|P_H f\|_p = \|(P_H f)_\rho)\|_p = \|f\|_p.$$

Now, let $\varphi \in L^p(G/H)$. Then the facts of $\varphi_\rho \in L^p(G:H)$ and $P_H(\varphi_\rho) = \varphi$ imply the surjectivity of P_H. Therefore, the map $P_H : L^p(G:H) \to L^p(G/H)$ is an isometry isomorphism. Hence $L^p(G/H)$ can be considered as a closed subspace of $L^p(G)$. ■

At the following, we fix $p = 1$ and focus on the Banach algebra $L^1(G/H)$. We need characterize $L^\infty(G/H)$ as dual of $L^1(G/H)$ and then the left and right module actions $L^\infty(G/H)$ on $L^1(G/H)$. Then we can study the weak amenability of the Banach algebra $L^1(G/H)$.

Theorem 3.5. *The Banach algebra $L^1(G/H)$ always possesses a bounded right approximate identity when H is a compact subgroup of G.*

Proof. We know that $L^1(G)$ always possesses a bounded approximate identity $\{u_\alpha\}_\alpha$(cf. [12], Proposition 3.7.7). First, we show that $\{(P_H u_\alpha)_\rho\}_\alpha$ is a right approximate identity for $L^1(G:H)$ specified in Proposition 3.3. Let $f \in L^1(G:H)$. Then we have

$$f * \left(P_H u_\alpha\right)_\rho(x) = \int_H f * \mathcal{R}_h u_\alpha(x) dm_H(x) \qquad (x \in G).$$

Thus by the compactness of H and Fubini's theorem, we can write

$$\begin{aligned}\|f * \left(P_H u_\alpha\right)_\rho - f\|_1 &= \int_G |\int_H (f * \mathcal{R}_h u_\alpha - f)(x) dm_H(h)| dm_G(x) \\ &\leq \int_G \int_H |\mathcal{R}_h(f * u_\alpha) - \mathcal{R}_h f)(x)| dm_H(h) dm_G(x) \\ &= \int_H \int_G |(\mathcal{R}_h f * u_\alpha - \mathcal{R}_h f)(x)| dm_G(x) dm_H(h) \\ &= \int_H \|\mathcal{R}_h(f * u_\alpha - f)\|_1 dm_H(h) \\ &= \int_H \|(f * u_\alpha - f)\|_1 dm_H(h) \\ &= \|(f * u_\alpha - f)\|_1,\end{aligned}$$

which this implies that $\{(P_H u_\alpha)_\rho\}_\alpha$ is a right approximate identity for $L^1(G:H)$. Now by applying this fact that P_H from $L^1(G:H)$ onto $L^1(G/H)$ is an isometry isomorphism and also multiplicative, hence by using Proposition 3.2, we conclude that $\{P_H u_\alpha\}_\alpha$ is a right approximate identity for $L^1(G/H)$. ∎

Remark 3.6. $L^1(G/H)$ possesses a left approximate identity if and only if H is normal in G (cf. [8]).

Corollary 3.7. *For compact subgroup H of G, The Banach algebra $L^1(G/H)$ is amenable if and only if H is normal in G and G is amenable.*

Proof. This fact that every amenable Banach algebra has a bounded approximate identity guarantees that amenability of $L^1(G/H)$ is equivalent to that H is normal in G. ∎

Suppose that H is a compact subgroup of G. Then we set

$$L^\infty(G:H) = \{f \in L^\infty(G); \mathcal{R}_h f = f, h \in H\}$$

and at the following we show that there is an isometric isomorphism between $L^\infty(G:H)$ and $L^\infty(G/H)$.

Theorem 3.8. *Let H be a compact subgroup of G. Then there is a surjective linear map $P_\infty : L^\infty(G) \mapsto L^\infty(G/H)$ such that for all $f \in L^\infty(G)$*

$$P_\infty(f)(xH) = \int_H f(xh)dm_H(h) \ (\mu\text{-locally almost every } xH \in G/H). \tag{3.3}$$

Proof. Let $f \in L^\infty(G)$. We assign to f a continuous linear map on $L^1(G/H)$ as

$$\varphi \mapsto \int_G \varphi_\rho(x) f(x) dm_G(x),$$

where $\varphi \in L^1(G/H)$ and $\varphi_\rho = \rho(\varphi \circ q)$. By the duality between $L^\infty(G/H)$ and $L^1(G/H)$ there is an element $\psi_f \in L^\infty(G/H)$ such that

$$\int_{G/H} \varphi(xH)\psi_f(xH)d\mu(xH) = \int_G \varphi_\rho(x) f(x) dm_G(x).$$

Hence for all $\varphi \in L^1(G/H)$ we can write

$$\begin{aligned}\int_{G/H} \varphi(xH)\psi_f(xH)d\mu(xH) &= \int_G \varphi_\rho(x) f(x) dm_G(x)\\ &= \int_{G/H}\int_H \varphi_\rho(x) f(xh)(\rho(xh))^{-1} dm_H(h) d\mu(xH)\\ &= \int_{G/H} \varphi(xH) \int_H f(xh) dm_H(h) d\mu(xH),\end{aligned}$$

and this implies that

$$\psi_f(xH) = \int_H f(xh) dm_H(h), \tag{3.4}$$

for μ-locally almost every $xH \in G/H$. So, if the map $P_\infty : L^\infty(G) \mapsto L^\infty(G/H)$ is given by

$$P_\infty(f)(xH) = \int_H f(xh)dm_H(h) \qquad (\mu\text{-locally almost every } xH \in G/H),$$

then by (3.4) $P_\infty f = \psi_f$. So, $P_\infty f \in L^\infty(G/H)$, i.e. the map P_∞ is well-defined. Also, if $\varphi \in L^\infty(G/H)$, then $\varphi \circ q \in L^\infty(G)$ and $P_\infty(\varphi \circ q) = \varphi$. Hence P_∞ is surjective. ∎

Now we can easily show that:

Corollary 3.9. *The map $P_\infty : L^\infty(G : H) \to L^\infty(G/H)$ is an isometry isomorphism.*

Using the map P_H and P_∞, we may express the left and the right dual $L^1(G/H)$-module actions of $L^\infty(G/H)$ via corresponding the left and the right $L^1(G)$-module actions of $L^\infty(G)$. In detail, for all $\psi \in L^\infty(G : H)$ and $g \in L^1(G : H)$, we have

$$P_\infty(\psi \cdot g) = P_\infty(\psi) \cdot P_H(g) \text{ and } P_\infty(g \cdot \psi) = P_H(g) \cdot P_\infty(\psi),$$

which in general case we express this in the following theorem.

Theorem 3.10. *Let H be a compact subgroup of G. Then the left and right module actions $L^\infty(G/H)$ on $L^1(G/H)$ are given respectively by*

$$\varphi \cdot f = P_\infty(\varphi_\rho \cdot f_\rho) \text{ and } f \cdot \varphi = P_\infty(f_\rho \cdot \varphi_\rho),$$

where $f \in L^1(G/H)$, $\varphi \in L^\infty(G/H)$, $\varphi_\rho = \varphi \circ q$ and $f_\rho = \rho(f \circ q)$.

Proof. Let $f \in L^1(G/H)$, $\varphi \in L^\infty(G/H)$. We know that $L^1(G) * L^\infty(G) \subseteq L^\infty(G)$, hence $(f_\rho)\tilde{} * \varphi_\rho \in L^\infty(G)$. So by using Theorem 3.8 $P_\infty((f_\rho)\tilde{} * \varphi_\rho) \in L^\infty(G/H)$. Therefore by using the Mackey-Brouhat formula and the compactness of H for each $g \in L^1(G/H)$ we can write

$$\begin{aligned}
P_\infty((f_\rho)\tilde{} * \varphi_\rho)(g) &= \int_{G/H} T_\infty((f_\rho)\tilde{} * \varphi \circ q)(xH) g(xH) d\mu(xH) \\
&= \int_{G/H} \int_H \frac{((f_\rho)\tilde{} * \varphi \circ q)(x\xi)\rho(x\xi) g \circ q(x\xi)}{\rho(x\xi)} d\xi d\mu(xH) \\
&= \int_G ((f_\rho)\tilde{} * \varphi \circ q)(x)\rho(x) g \circ q(x) dx \\
&= \int_G \int_G \frac{f_\rho(y^{-1})}{\Delta(y)} \varphi \circ q(y^{-1}x) dy \rho(x) g \circ q(x) dx \\
&= \int_G f_\rho(y) \int_G \varphi \circ q(x) \rho(y^{-1}x) g \circ q(y^{-1}x) dx dy \\
&= \int_G \varphi \circ q(x) \int_G f_\rho(y) g_\rho(y^{-1}x) dy dx \\
&= \int_G \varphi \circ q(x) f_\rho * g_\rho(x) dx \\
&= \int_{G/H} \int_H \frac{\varphi \circ q(x\xi) f_\rho * g_\rho(x\xi)}{\rho(x\xi)} d\xi d\mu(xH) \\
&= \int_{G/H} \int_H \frac{\varphi \circ q(x\xi)\rho(x\xi)(f * g) \circ q(x\xi)}{\rho(x\xi)} d\xi d\mu(xH) \\
&= \int_{G/H} \varphi(xH) f * g(xH) d\mu(xH) \\
&= \varphi(f * g) = (\varphi \cdot f)(g).
\end{aligned}$$

Also by these facts that $L^\infty(G)*(L^1(G))\check{} \subseteq L^\infty(G)$ and $\varphi_\rho*(f_\rho)\check{} \in L^\infty(G)$, so $P_\infty(\varphi_r ho*(f_\rho)\check{}) \in L^\infty(G/H)$. Hence by using the Mackey-Brouhat formula and the compactness of H for each $g \in L^1(G/H)$ we can write

$$\begin{aligned}
P_\infty(\varphi_\rho*(f_\rho)\check{})(g) &= \int_{G/H} P_\infty(\varphi_\rho*(f_\rho)\check{})(xH)g(xH)d\mu(xH)\\
&= \int_{G/H}\int_H \frac{(\varphi\circ q*(f_\rho)\check{})(x\xi)\rho(x\xi)g\circ q(x\xi)}{\rho(x\xi)}d\xi d\mu(xH)\\
&= \int_G (\varphi\circ q*(f_\rho)\check{})(x)\rho(x)g\circ q(x)dx\\
&= \int_G \rho(x)g\circ q(x)\int_G \varphi\circ q(y)(f_\rho)\check{}(y^{-1}x)dydx\\
&= \int_G \rho(x)g\circ q(x)\int_G \varphi\circ q(y)\rho(x^{-1}y)(f\circ q)(x^{-1}y)dydx\\
&= \int_G \varphi\circ q(y)\int_G \rho(x^{-1}y)(f\circ q)(x^{-1}y)\rho(x)g\circ q(x)dxdy\\
&= \int_G \varphi\circ q(y)\int_G g_\rho(x)f_\rho(x^{-1}y)dxdy\\
&= \int_G \varphi\circ q(y)g_\rho*f_\rho(y)dy\\
&= \int_{G/H}\int_H \frac{\varphi\circ q(y\xi)(g*f)_\rho(y\xi)}{\rho(y\xi)}d\xi d\mu(xH)\\
&= \int_{G/H} \varphi(yH)(g*f)(yH)d\mu(yH)\\
&= \varphi(g*f) = (f\cdot\varphi)(g).
\end{aligned}$$

■

Corollary 3.11. *For a compact subgroup H of G, the left and right module actions $L^\infty(G/H)$ on $L^1(G/H)$ are given respectively by*

$$P_\infty(\varphi_\rho)\cdot P_H(f_\rho) = P_\infty(\varphi_\rho\cdot f_\rho) \text{ and } P_H(f_\rho)\cdot P_\infty(\varphi_\rho) = P_\infty(f_\rho\cdot\varphi_\rho),$$

where $f \in L^1(G/H)$, $\varphi \in L^\infty(G/H)$, $\varphi_\rho = \varphi\circ q$ and $f_\rho = \rho(f\circ q)$.

Now we can find a necessary and sufficient condition for that the Banch algebra $L^1(G/H)$ is weakly amenable.

Theorem 3.12. *Let $\tilde{D} : L^1(G/H) \mapsto L^\infty(G/H)$ be a continuous derivation. Then $\tilde{D}$ is an inner derivation if and only if there is a continuous derivation $D : L^1(G) \mapsto L^\infty(G)$ such that $\tilde{D}\circ P_H = P_\infty\circ D$.*

Proof. Let $\tilde{D} : L^1(G/H) \mapsto L^\infty(G/H)$ be a continuous derivation and there is a continuous derivation $D : L^1(G) \mapsto L^\infty(G)$ such that $\tilde{D}\circ P_H = P_\infty\circ D$. Then by Proposition 3.1 and Theorem 3.8 we have the following diagram:

$$\begin{array}{ccc}
L^1(G) & \xrightarrow{D} & L^\infty(G)\\
{\scriptstyle P_H}\big\downarrow & & \big\downarrow{\scriptstyle P_\infty}\\
L^1(G/H) & \xrightarrow{\tilde{D}} & L^\infty(G/H)
\end{array}$$

So, for each $f \in L^1(G/H)$ and $\varphi \in L^\infty(G/H)$ we can write

$$\begin{aligned}\tilde{D}(f) &= \tilde{D}(P_H(f_\rho)) = \tilde{D} \circ P_H(f_\rho) \\ &= P_\infty \circ D(f_\rho) = P_\infty(D(f_\rho)),\end{aligned} \tag{3.5}$$

in which $f = \rho(f \circ q)$. On the other hand, we know that $L^1(G)$ is weakly amenable, hence there is $\psi_0 \in L^\infty(G)$ such that

$$D(f_\rho) = f_\rho \cdot \psi_0 - \psi_0 \cdot f_\rho, \tag{3.6}$$

so, by (3.6) and (3.5) and Corollary 3.11 we can write

$$\begin{aligned}\tilde{D}(f) &= P_\infty(f_\rho \cdot \psi_0 - \psi_0 \cdot f_\rho) = P_\infty(f_\rho \cdot \psi_0) - P_\infty(\psi_0 \cdot f_\rho) \\ &= P_H(f_\rho) \cdot P_\infty(\psi_0) - P_\infty(\psi_0) \cdot P_H(f_\rho) \\ &= f \cdot P_\infty(\psi_0) - P_\infty(\psi_0) \cdot f \\ &= d_{P_\infty(\psi_0)}(f).\end{aligned}$$

So, for arbitrary continuous derivation $\tilde{D} : L^1(G/H) \mapsto L^\infty(G/H)$ there is $P_\infty(\psi_0) \in L^\infty(G/H)$ such that $\tilde{D} = d_{P_\infty(\psi_0)}$, i.e, $\tilde{D}$ is inner and hence $L^1(G/H)$ is weakly amenable.

For the revers, let $\tilde{D} : L^1(G/H) \mapsto L^\infty(G/H)$ is an inner derivation. We know that the restriction of P_∞ to $L^\infty(G : H)$ is an isometry isomorphism. So, it is enough to show that ${P_\infty}^{-1} \circ \tilde{D} \circ P_H : L^1(G) \mapsto L^\infty(G : H)$ is an inner derivation. For this, let $f \in L^1(G)$. Since $\tilde{D}$ is an inner derivation, so there is $\varphi_0 \in L^\infty(G/H)$ such that

$$\begin{aligned}(P_\infty^{-1} \circ \tilde{D} \circ P_H)(f) &= ({P_\infty}^{-1})(\tilde{D}(P_H f)) \\ &= ({P_\infty}^{-1})\big((P_H f) \cdot \varphi_0 - \varphi_0 \cdot (P_H f)\big) \\ &= ({P_\infty}^{-1})\big(P_\infty((P_H f)_\rho \cdot (\varphi_0)_\rho) - P_\infty((\varphi_0)_\rho \cdot (P_H f)_\rho)\big) \\ &= f \cdot (\varphi_0)_\rho - (\varphi_0)_\rho \cdot f \\ &= d_{(\varphi_0)_\rho}(f)\end{aligned}$$

which this implies that $P_\infty^{-1} \circ \tilde{D} \circ P_H$ is an inner derivation. ■

At the end, the following corollary be straightly derived from Theorem 3.12.

Corollary 3.13. *The Banach algebra $L^1(G/H)$ is weakly amenable if and only if for each continuous derivation $\tilde{D} : L^1(G/H) \to L^\infty(G/H)$ there is a continuous derivation $D : L^1(G) \to L^\infty(G)$ such that $\tilde{D} \circ P_H = P_\infty \circ D$.*

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