Thai Journal of Mathematics
Volume 21 Number 1 (2023)
Pages 153–169

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Strong Convergence Theorem for Generalized Mixed Equilibrium Problem and Bregman Totally Quasi-asymptotically Nonexpansive Mappings in Reflexive Banach Spaces

Kittisak Jantakarn and Anchalee Kaewcharoen*

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail : anchaleeka@nu.ac.th (A. Kaewcharoen); kittisakj61@nu.ac.th (K. Jantakarn)

Abstract In this paper, we propose a new iterative algorithm for finding common solutions of generalized mixed equilibrium problems and fixed point problems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Moreover, the strong convergence theorem under suitable control conditions is proven and the numerical result for supporting our main result is presented.

MSC: 47H10; 54H25

Keywords: generalized mixed equilibrium problems; Bregman totally quasi-asymptotically nonexpansive mappings; reflexive Banach spaces

Submission date: 10.03.2020 / Acceptance date: 16.09.2022

1. Introduction

We begin by recalling the interesting problems. Let E be a real reflexive Banach space, E^* be a dual space of E and C be a nonempty closed convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction, $\phi : C \to \mathbb{R}$ be a real-valued function and $\psi : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem:

$$\text{finding} \quad x \in C \quad \text{such that} \quad g(x, y) + \langle \psi(x), y - x\rangle + \phi(y) - \phi(x) \geq 0, \forall y \in C. \tag{1.1}$$

The solution set of problem (1.1) is denoted by $GMEP(g, \phi, \psi)$. If in problem (1.1), $\phi(x) = 0$ for each $x \in C$, then we obtain the generalized equilibrium problem:

$$\text{finding} \quad x \in C \quad \text{such that} \quad g(x, y) + \langle \psi(x), y - x\rangle \geq 0, \forall y \in C. \tag{1.2}$$

*Corresponding author.

Published by The Mathematical Association of Thailand.

The solution set of problem (1.2) is denoted by $GEP(g,\psi)$. If in problem (1.1), $\psi(x)=0$ for each $x\in C$, then we obtain the mixed equilibrium problem:

$$\text{finding} \quad x\in C \quad \text{such that} \quad g(x,y)+\phi(y)-\phi(x)\geq 0, \forall y\in C. \tag{1.3}$$

The solution set of problem (1.3) is denoted by $MEP(g,\phi)$. If in problem (1.1), $g(x,y)=0$ and $\phi(x)=0$ for each $x,y\in C$, then we obtain the variational inequality:

$$\text{finding} \quad x\in C \quad \text{such that} \quad \langle\psi(x),y-x\rangle\geq 0, \forall y\in C. \tag{1.4}$$

The solution set of problem (1.4) is denoted by $VI(\psi)$. If in problem (1.1), $\psi(x)=0$ and $\phi(x)=0$ for each $x\in C$, then we obtain the equilibrium problem:

$$\text{finding} \quad x\in C \quad \text{such that} \quad g(x,y)\geq 0, \forall y\in C. \tag{1.5}$$

The solution set of problem (1.5) is denoted by $EP(g)$.

An equilibrium problem was studied by Blum and Oettli [1], by mention above, we can see that the generalized mixed equilibrium problem can be reduced to many other problems such as mixed variational inequality, variational inequality, Nash equilibrium problems and equilibrium problems, (see, for instance, [2, 3]). Therefore, we are interested in studying this problem for developing the research.

It is apparent that the fixed point theory of nonexpansive mappings can be applied for solving solutions of certain evolution equations and solving convex feasibility, variational inequality and equilibrium problems. There are many papers that deal with methods for finding fixed points of nonexpansive and quasi-nonexpansive mappings in Hilbert, uniformly convex and uniformly smooth Banach spaces, (see, for instance, [4–6]).

When we try to extend this theory to general Banach spaces we discover some difficulties, and there are several ways to overpower these difficulties. One of them is to use the Bregman distance in place of the norm, Bregman (quasi-) nonexpansive mappings instead of the (quasi-) nonexpansive mappings and the Bregman projection instead of the metric projection.

In 1967, Bregman [7] introduced a Bregman technique using the distance function $D_f(\cdot,\cdot)$ in designing and analyzing optimization and feasibility algorithms. Bregman's technique can be applied in various ways.

In 2011, Reich and Sabach [8] introduced the concept of Bregman strongly nonexpansive mappings and studied the convergence theorems of two iterative methods for solving solutions of common fixed points of finitely many Bregman strongly nonexpansive mappings in reflexive Banach spaces.

In 2015, Darvish [9] established a new algorithm for solving the solutions of mixed equilibrium problems and fixed point problems for Bregman strongly nonexpansive mappings in Banach spaces and proved the strong convergence theorems under suitable control conditions.

In 2016, Zhu and Huang [10] created a new iterative method for solving solutions of equilibrium problems and fixed point problems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Let $T : C \to C$ be a Bregman totally quasi-asymptotically nonexpansive mapping. They introduced the iteration as follows:

$$\begin{cases} x_1 = u \in C, \text{ chosen arbitrarily}, \\ u_n : g(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n x_n), y - u_n \rangle \geq 0, \forall y \in C, \\ C_n = \{z \in C : D_f(z, u_n) \leq D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^n C_i, \\ x_{n+1} = proj_{D_n}^f u, \end{cases} \tag{1.6}$$

where $\xi_n = \upsilon_n \sup_{\upsilon \in F(T) \cap EP(g)} \zeta(D_f(\upsilon, x_n)) + \mu_n$. Then they obtained the strong convergence theorems. In the same year, Darvish [11] introduced the iterative method for finding the solutions of generalized mixed equilibrium problems and fixed point problems for Bregman strongly nonexpansive mappings and proved strong convergence theorems under suitable control conditions.

Motivated by works mentioned above, in this paper, we introduce an iterative method for solving solutions of generalized mixed equilibrium problems and fixed point problems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces and we prove the strong convergence theorems for the sequence generated by this iteration. The results in this work generalize and extend the results proved by Zhu and Huang [10] to the generalized mixed equilibrium problem.

2. Preliminaries

In this section, we begin by recalling some definitions and properties which will be used for proving our main results.

In this paper, we let E be a real reflexive Banach space, E^* be the dual space of E, $f : E \to (-\infty, +\infty]$ be a proper function and $f^* : E^* \to (-\infty, +\infty]$ be the Fenchel conjugate of f defined by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}, \forall x^* \in E^*.$$

We denote $domf$ by the set of domain of f and $int(domf)$ by the set of interior points of $domf$.

Definition 2.1. Let $x \in int(domf)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called to be

(i) Gâteaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$ exists for any y. In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x;
(ii) Gâteaux differentiable if it is Gâteaux differentiable for any $x \in int(domf)$;
(iii) Frêchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$;
(iv) uniformly Frêchet differentiable on $C \subseteq E$ if the above limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Lemma 2.2. [8] *Let $f : E \to (-\infty, +\infty]$ be uniformly Frêchet differentiable and bounded on bounded subsets of E. Then f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2.3. [12] A function $f : E \to (-\infty, +\infty]$ is said to be a Legendre function if the following conditions are satisfied:

(L1) The interior of the domain of f, $int(domf)$ is nonempty, f is Gâteaux differentiable on $int(domf)$ and $dom\nabla f = int(domf)$;

(L2) The interior of the domain of f^*, $int(domf^*)$ is nonempty, f^* is Gâteaux differentiable on $int(domf^*)$ and $dom\nabla f^* = int(domf^*)$.

Remark 2.4. If E is a real reflexive Banach space and f is the Legendre function, then the following conditions hold:

(a) f is the Legendre function if and only if f^* is the Legendre function;
(b) $(\partial f)^{-1} = \partial f^*$;
(c) $\nabla f = (\nabla f^*)^{-1}, ran\nabla f = dom\nabla f^* = int(domf^*), ran\nabla f^* = dom\nabla f = int(domf)$;
(d) the functions f and f^* are strictly convex on the interior of respective domains.

Definition 2.5. [13] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle$$

is called the Bregman distance with respect to f.

We can observe that the Bregman distance is not a distance in the usual sense. In general, $D_f(\cdot, \cdot)$ is not symmetric and does not satisfy the triangle inequality. By the definition of the Bregman distance, we obtain that the Bregman distance has the following important properties:

(1) (the two point identity) for any $x, y \in int(domf)$,

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y\rangle;$$

(2) (the three point identity) for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle;$$

(3) (the four point identity) for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w\rangle.$$

Remark 2.6. [9] Let E be a smooth and strictly convex Banach space. If the Legendre function $f : E \to (-\infty, +\infty]$ defined by $f(x) = \frac{1}{p}\|x\|^p, (1 < p < \infty)$, then the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p, (1 < p < \infty)$. Moreover, $\nabla f = I$, the identity mapping in Hilbert spaces.

Definition 2.7. [7] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of x in $int(domf)$ onto the nonempty closed convex set $C \subset domf$ is the necessarily unique vector $proj^f_C(x) \in C$ satisfying the following:

$$D_f(proj^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Definition 2.8. [14] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. A function f is called to be

(a) totally convex at a point $x \in int(domf)$, if its modulus of total convexity at $x, v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$, defined by $v_f(x,t) := \inf\{D_f(y,x) : y \in domf, \|y - x\| = t\}$ is positive whenever $t > 0$;
(b) totally convex if it is totally convex at every point $x \in int(domf)$;
(c) totally convex on bounded sets if $v_f(B,t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by $v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap domf\}$.

Lemma 2.9. [8] *If $x \in int(domf)$, then the following statements are equivalent:*

(1) the function f is totally convex at x;
(2) for any sequence $\{y_n\} \subset domf$, $\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x\| = 0$;
(3) for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $y \in domf$ and $D_f(y,x) \le \delta$, then $\|x - y\| \le \varepsilon$.

Lemma 2.10. [14] *The function f is totally convex on bounded sets if and only if it is sequentially consistent, i.e., for any two sequences $\{x_n\}$ and $\{y_n\}$ in int(domf) and domf, respectively, and $\{x_n\}$ is bounded, then*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.11. [15] *Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.12. [16] *Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function on int(domf). Let $x \in$ int(domf) and $C \subset$ int(domf) be a nonempty closed convex set. If $x \in C$, then the following statments are equivalent:*

(1) $z \in C$ is the Bregman projection of x onto C with respect to f, i.e., $z = proj_C^f(x)$;
(2) the vector z is the unique solution of the variational inequality:

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \ge 0, \forall y \in C;$$

(3) the vector z is the unique solution of the inequality:

$$D_f(y,z) + D_f(z,x) \le D_f(y,x), \forall y \in C.$$

Definition 2.13. Let C be a subset of E and $T : C \to C$ be a mapping. Denoted $F(T) = \{x \in C : Tx = x\}$ by the set of fixed points of T. A mapping T is called to be

(a) closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$, then $Tx = y$;
(b) uniformly asymptotically regular on C if $\lim_{n\to\infty} \sup_{x\in C} \|T^{n+1}x - T^n x\| = 0$;
(c) Bregman firmly nonexpansive if

$$\begin{aligned} D_f(Tx,Ty) + D_f(Ty,Tx) &+ D_f(Tx,x) + D_f(Ty,y) \\ &\le D_f(Tx,y) + D_f(Ty,x),\ \forall x,y \in C; \end{aligned}$$

(d) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if $D_f(p,Tx) \le D_f(p,x), \forall x \in C, p \in \hat{F}(T)$. A point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote $\hat{F}(T)$ the set of asymptotic fixed points of T;

(e) Bregman relatively nonexpansive if $F(T) \neq \emptyset, F(T) = \hat{F}(T)$ and $D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T)$;

(f) Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and $D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T)$;

(g) Bregman quasi-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$D_f(p, T^n x) \leq k_n D_f(p, x), \forall x \in C, p \in F(T); \tag{2.1}$$

(h) Bregman totally quasi-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}$ with $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$D_f(p, T^n x) \leq D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \forall n \geq 1, \forall x \in C, p \in F(T). \tag{2.2}$$

Remark 2.14. According to the definitions it is obvious that

(1) each Bregman relatively nonexpansive mapping is a Bregman quasi-nonexpansive mapping;

(2) each Bregman quasi-nonexpansive mapping is a Bregman quasi-asymptotically nonexpansive mapping. Indeed, if we take $k_n = 1$, then we have

$$D_f(p, T^n x) \leq k_n D_f(p, Tx) \leq k_n D_f(p, x), \forall x \in C, p \in F(T);$$

(3) each Bregman quasi-asymptotically nonexpansive mapping is Bregman totally quasi-asymptotically nonexpansive mapping, but the converse may be not true. Indeed, if we take $\zeta(t) = t, t \geq 0, v_n = k_n - 1$ and $\mu_n = 0$, then equation (2.1) can be rewritten as

$$D_f(p, T^n x) \leq D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \forall x \in C, p \in F(T).$$

This implies that each Bregman relatively nonexpansive mapping must be a Bregman totally quasi-asymptotically nonexpansive mapping, but the converse is not true.

Lemma 2.15. [17] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is uniformly Frêchet differentiable and bounded on bounded subsets of E. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be a Bregman firmly nonexpansive mapping with respect to f. Then $F(T) = \hat{F}(T)$.*

Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(C1) $g(x, x) = 0, \forall x \in C$;

(C2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0, \forall x, y \in C$;

(C3) $\forall x, y, z \in C, \limsup_{t\to 0^+} g(tz + (1-t)x, y) \leq g(x, y)$;

(C4) $\forall x \in C, g(x, \cdot)$ is convex and lower semicontinuous.

Definition 2.16. Let $f : E \to (-\infty, +\infty]$. We say that f is a coercive function if $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty$.

Lemma 2.17. [11] *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function, and C be a nonempty closed convex subset of $int(domf)$. Let $\phi : C \to \mathbb{R}$ be a proper lower semi-continuous convex function and $\psi : C \to E^*$ be a continuous monotone mapping. Assume that $g : C \times C \to \mathbb{R}$ satisfies conditions (C1)-(C4). For $x \in E$, define a mapping $Res^f_{g,\phi,\psi} : E \to 2^C$ as follows:*

$$Res^f_{g,\phi,\psi}(x) = \{z \in C : g(z,y)+\langle\psi(x), y-x\rangle+\phi(y)-\phi(z)+\langle\nabla f(z)-\nabla f(x), y-z\rangle \geq 0, \forall y \in C\}.$$

Then the following results hold:

(1) $Res^f_{g,\phi,\psi}$ is single-valued and $dom(Res^f_{g,\phi,\psi}) = E$;
(2) $Res^f_{g,\phi,\psi}$ is Bregman firmly nonexpansive;
(3) $GMEP(g,\phi,\psi)$ is a closed convex subset of C and $GMEP(g,\phi,\psi) = F(Res^f_{g,\phi,\psi})$;
(4) for all $x \in E, u \in F(Res^f_{g,\phi,\psi})$,

$$D_f(u, Res^f_{g,\phi,\psi}x) + D_f(Res^f_{g,\phi,\psi}x, x) \leq D_f(u,x).$$

Lemma 2.18. [18] *Let $f : E \to \mathbb{R}$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $int(domf)$ and let $x \in E$. If $\{D_f(x,x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.19. [19] *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E and $f : E \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of E. Let $T : C \to C$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Then the fixed point set $F(T)$ of T is a closed convex subset of C.*

Lemma 2.20. [15] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and C be a nonempty convex closed subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(proj^f_C(x_0), x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $proj^f_C(x_0)$.*

3. Main Results

In this section, we introduce a new iterative method for solving solutions of generalized mixed equilibrium problems and fixed point problems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces and we prove the strong convergence theorems for the sequence generated by this iteration.

Theorem 3.1. *Let E be a real reflexive Banach space and C be a nonempty convex closed subset of $int(domf)$. Let $f : E \to \mathbb{R}$ be a totally convex on bounded subsets of E, coercive Legendre function which is bounded, uniformly Frêchet differentiable, $T : C \to C$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $\phi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, $\psi : C \to E^*$ be a continuous monotone mapping and bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (C1)-(C4). Assume that T is uniformly asymptotically* *regular and* $\Omega := F(T) \cap GMEP(g,\phi,\psi) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = u \in C, \textit{chosen arbitrarily}, \\ u_n = Res^f_{g,\phi,\psi}(T^n(x_n)), \\ Y_n = \{z \in C : D_f(z,u_n) \leq D_f(z,x_n) + \xi_n\}, \\ Z_n = \bigcap\limits_{i=1}^{n} Y_i, \\ x_{n+1} = proj^f_{Z_n}(u), \end{cases} \tag{3.1}$$

where $\xi_n = \upsilon_n \sup\limits_{v\in\Omega} \zeta(D_f(v,x_n)) + \mu_n$, *and* $proj^f_{Z_n}$ *is the Bregman projection of* E *onto* Z_n. *If* $\Omega := F(T) \cap GMEP(g,\phi,\psi)$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to* $\overline{x} = proj^f_{\Omega}(u)$.

Proof. We split the concept of proving into five steps as follows:

Step 1: We show that Ω and Z_n are closed convex subsets of E. By Lemma 2.17, we obtain that $GMEP(g,\phi,\psi)$ is closed and convex. Using Lemma 2.19, we have $F(T)$ is also closed and convex. It follows that Ω is also closed and convex. Let $v \in \Omega$ be given. Since $Res^f_{g,\phi,\psi}$ is a single-valued mapping, for each $n \in \mathbb{N}$, we have $u_n = Res^f_{g,\phi,\psi}(T^n(x_n))$. By Lemma 2.17, we obtain that

$$D_f(v, Res^f_{g,\phi,\psi}(T^n(x_n))) + D_f(Res^f_{g,\phi,\psi}(T^n(x_n)), T^n(x_n)) \leq D_f(v, T^n(x_n)). \tag{3.2}$$

It follows that

$$\begin{aligned} D_f(v, Res^f_{g,\phi,\psi}(T^n(x_n))) &\leq D_f(v, T^n(x_n)) - D_f(Res^f_{g,\phi,\psi}(T^n(x_n)), T^n(x_n)) \\ &\leq D_f(v, T^n(x_n)). \end{aligned}$$

This implies that

$$\begin{aligned} D_f(v,u_n) &= D_f(v, Res^f_{g,\phi,\psi}(T^n(x_n))) \\ &\leq D_f(v, T^n(x_n)) \\ &\leq D_f(v,x_n) + \upsilon_n \zeta(D_f(v,x_n)) + \mu_n. \end{aligned}$$

Therefore

$$D_f(v,u_n) \leq D_f(v,x_n) + \xi_n,$$

where $\xi_n = \upsilon_n \sup_{v\in\Omega} \zeta(D_f(v,x_n)) + \mu_n$. It follows that $v \in Y_n$ for any $n \geq 1$. Hence $\Omega \subset Y_n$. Moreover, we have $\Omega \subset Z_n$. We now prove that Z_n is a convex set begin by proving Y_n is convex. Let $p,q \in Y_n$ and $t \in (0,1)$. Suppose that $w = tp + (1-t)q$. We will prove that $w \in Y_n$. By definition of Y_n, we have

$$D_f(p,u_n) \leq D_f(p,x_n) + \xi_n \quad \text{and} \quad D_f(q,u_n) \leq D_f(q,x_n) + \xi_n.$$

Since $D_f(z,u_n) \leq D_f(z,x_n) + \xi_n$ and definition of $D_f(\cdot,\cdot)$,

$$f(z) - f(u_n) - \langle \nabla f(u_n), z - u_n \rangle \leq f(z) - f(x_n) - \langle \nabla f(x_n), z - x_n \rangle + \xi_n.$$

It follows that

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), z - u_n \rangle - \langle \nabla f(x_n), z - x_n \rangle + \xi_n. \tag{3.3}$$

This implies that

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), p - u_n \rangle - \langle \nabla f(x_n), p - x_n \rangle + \xi_n \tag{3.4}$$

and

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), q - u_n \rangle - \langle \nabla f(x_n), q - x_n \rangle + \xi_n. \tag{3.5}$$

Therefore

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), tp + (1-t)q - u_n \rangle + \langle \nabla f(x_n), tp + (1-t)q - x_n \rangle + \xi_n.$$

This implies that

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), w - u_n \rangle - \langle \nabla f(x_n), w - x_n \rangle + \xi_n.$$

It follows that $w \in Y_n$ and so Y_n is convex. This yields Z_n is also convex. We next show that Z_n is a closed set. Let $\{z_m\} \subset Y_n$ and $z_m \to z$ (as $m \to \infty$). For each $m \in \mathbb{N}$, by definition of Y_n and (3.3), we obtain that

$$\begin{aligned} f(x_n) - f(u_n) &\leq \langle \nabla f(u_n), z_m - u_n \rangle - \langle \nabla f(x_n), z_m - x_n \rangle + \xi_n \\ &= \langle \nabla f(u_n), z_m - z + z - u_n \rangle - \langle \nabla f(x_n), z_m - z + z - x_n \rangle + \xi_n \\ &= \langle \nabla f(u_n), z_m - z \rangle + \langle \nabla f(u_n), z - u_n \rangle - \langle \nabla f(x_n), z_m - z \rangle \\ &\qquad - \langle \nabla f(x_n), z - x_n \rangle + \xi_n. \end{aligned} \tag{3.6}$$

Letting $m \to \infty$, we can obtain that

$$f(x_n) - f(u_n) \leq \langle \nabla f(u_n), z - u_n \rangle - \langle \nabla f(x_n), z - x_n \rangle + \xi_n.$$

This implies that $z \in Y_n$ and so Y_n is closed. This yields Y_n is closed and convex for any $n \geq 1$. Hence Z_n is also closed and convex. Therefore the sequence $\{x_n\}$ is well-defined.

Step 2: We prove that $\{x_n\}$ is bounded. Since $x_{n+1} = proj^f_{Z_n} u$, by Lemma 2.12, we have

$$\begin{aligned} D_f(x_{n+1}, u) &= D_f(proj^f_{Z_n} u, u) \\ &\leq D_f(v, u) - D_f(v, proj^f_{Z_n} u) \\ &\leq D_f(v, u), \forall v \in \Omega. \end{aligned} \tag{3.7}$$

Therefore the sequence $\{D_f(x_{n+1}, u)\}$ is bounded. Using Lemma 2.12, this yields the sequence $\{x_n\}$ is also bounded.

Step 3: We prove that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} = proj^f_{Z_n}(u)$ and $x_{n+2} = proj^f_{Z_{n+1}}(u) \in Z_{n+1} \subset Z_n$, by using Lemma 2.12, we have

$$D_f(x_{n+2}, proj^f_{Z_n}(u)) + D_f(proj^f_{Z_n}(u), u) \leq D_f(x_{n+2}, u). \tag{3.8}$$

It follows that

$$D_f(x_{n+2}, x_{n+1}) + D_f(x_{n+1}, u) \leq D_f(x_{n+2}, u). \tag{3.9}$$

Therefore the sequence $\{D_f(x_n, u)\}$ is increasing. Since $\{D_f(x_n, u)\}$ is bounded, $\lim_{n\to\infty} D_f(x_n, u)$ exists. By definition of Z_n, for any positive integer $m \geq n$, we have

$Z_m \subset Z_n$ and $x_m = proj^f_{Z_{m-1}}(u) \in Z_{m-1} \subset Z_{n-1}$. It follows that

$$\begin{aligned} D_f(x_m, x_n) &= D_f(x_m, proj^f_{Z_{n-1}}(u)) \\ &\leq D_f(x_m, u) - D_f(proj^f_{Z_{n-1}}(u), u) \\ &= D_f(x_m, u) - D_f(x_n, u). \end{aligned} \tag{3.10}$$

By taking $m, n \to \infty$, we obtain that

$$D_f(x_m, x_n) \to 0.$$

It follows from Lemma 2.10 that

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0. \tag{3.11}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence.

Step 4: We show that $\{x_n\}$ converges to a point in $\Omega := F(T) \cap GMEP(g, \phi, \psi)$. Since E is a reflexive Banach space and $\{x_n\}$ is a Cauchy sequence, without loss of generality we can assume that

$$\lim_{n\to\infty} x_n = x^* \in C.$$

We now prove that $x^* \in F(T)$. Taking $m = n+1$, we can obtain that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$, by Lemma 2.10, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Since $x_{n+1} = proj^f_{Z_n}(u) \in Z_n \subset Y_n$, we have

$$D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, x_n) + \xi_n. \tag{3.13}$$

It follows from $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0, v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and boundedness of $\{D_f(v_n, x_n)\}$, we obtain that

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{3.14}$$

Moreover, since $D_f(v, u_n) \leq D_f(v, x_n) + \xi_n$ and f is lower semicontinuous, we get that $\{D_f(v, x_n) + \xi_n\}$ is bounded, so is $\{D_f(v, u_n)\}$. By Lemma 2.18, we obtain that $\{u_n\}$ bounded. It follows from Lemma 2.10, we get that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.15}$$

Since

$$\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|,$$

we obtain that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.16}$$

Since f is uniformly Frêchet differentiable, it follows from Lemma 2.2 that ∇f is uniformly continuous. Therefore

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{3.17}$$

Since f is uniformly Frêchet differentiable, it is also uniformly continuous, we obtain that

$$\lim_{n\to\infty} |f(x_n) - f(u_n)| = 0. \tag{3.18}$$

Furthermore, we have

$$\begin{aligned}
&D_f(v, x_n) - D_f(v, u_n)\\
&\quad = f(v) - f(x_n) - \langle \nabla f(x_n), v - x_n\rangle - [f(v) - f(u_n) - \langle \nabla f(u_n), v - u_n\rangle]\\
&\quad = f(u_n) - f(x_n) + \langle \nabla f(u_n), v - u_n\rangle - \langle \nabla f(x_n), v - x_n\rangle\\
&\quad = f(u_n) - f(x_n) + \langle \nabla f(u_n), x_n - u_n + v - x_n\rangle - \langle \nabla f(x_n), v - x_n\rangle\\
&\quad = f(u_n) - f(x_n) + \langle \nabla f(u_n), x_n - u_n\rangle + \langle \nabla f(u_n), v - x_n\rangle - \langle \nabla f(x_n), v - x_n\rangle\\
&\quad = f(u_n) - f(x_n) + \langle \nabla f(u_n), x_n - u_n\rangle + \langle \nabla f(u_n) - \nabla f(x_n), v - x_n\rangle. \qquad (3.19)
\end{aligned}$$

Since $\{u_n\}$ is bounded, we obtain that $\{\nabla f(u_n)\}$ is also bounded. Therefore

$$\lim_{n\to\infty} (D_f(v, x_n) - D_f(v, u_n)) = 0. \qquad (3.20)$$

Since $u_n = Res^f_{g,\phi,\psi}(T^n(x_n))$, by Lemma 2.17 and the assumption of T, we have

$$\begin{aligned}
D_f(u_n, T^n(x_n)) &= D_f(Res^f_{g,\phi,\psi}(T^n(x_n)), T^n(x_n))\\
&\leq D_f(v, T^n(x_n)) - D_f(v, Res^f_{g,\phi,\psi}(T^n(x_n)))\\
&\leq D_f(v, x_n) + v_n\zeta(D_f(v, x_n)) + \mu_n - D_f(v, u_n). \qquad (3.21)
\end{aligned}$$

Since $\{D_f(v, x_n)\}$ is bounded, $\{\zeta(D_f(v, x_n))\}$ is also bounded and $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$), we obtain that

$$\lim_{n\to\infty} D_f(u_n, T^n(x_n)) = 0.$$

By Lemma 2.10, we have

$$\lim_{n\to\infty} \|u_n - T^n(x_n)\| = 0. \qquad (3.22)$$

Since

$$\|x_n - T^n(x_n)\| \leq \|x_n - u_n\| + \|u_n - T^n(x_n)\|,$$

we obtain that

$$\lim_{n\to\infty} \|x_n - T^n(x_n)\| = 0. \qquad (3.23)$$

Since

$$\|x^* - T^n(x_n)\| \leq \|x^* - x_n\| + \|x_n - T^n(x_n)\|,$$

it follows that

$$\lim_{n\to\infty} \|x^* - T^n(x_n)\| = 0. \qquad (3.24)$$

Moreover, we have

$$\|x^* - T^{n+1}(x_n)\| \leq \|x^* - T^n(x_n)\| + \|T^n(x_n) - T^{n+1}(x_n)\|.$$

Since T is uniformly asymptotically regular, we obtain that

$$\lim_{n\to\infty} \|x^* - T^{n+1}(x_n)\| = 0. \qquad (3.25)$$

This implies that $TT^n(x_n) \to x^*$ (as $n \to \infty$). From the closedness of T, we obtain that $T(x^*) = x^*$. Therefore $x^* \in F(T)$. Next, we prove that $x^* \in GMEP(g, \phi, \psi)$. Since f is uniformly Frêchet differentiable, we obtain that ∇f is uniformly continuous on bounded sets, it follows from (3.22), we obtain that

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(T^n(x_n))\| = 0. \tag{3.26}$$

Since $u_n = Res^f_{g,\phi,\psi}(T^n(x_n))$, we get that

$$\begin{aligned} g(u_n, y) + \langle \psi(T^n(x_n)), y - u_n\rangle + \phi(y) - \phi(u_n) \\ + \langle \nabla f(u_n) - \nabla f(T^n(x_n)), y - u_n\rangle \geq 0, \forall y \in C. \end{aligned} \tag{3.27}$$

We have from (C2) that

$$\begin{aligned} &\langle \psi(T^n(x_n)), y - u_n\rangle + \phi(y) - \phi(u_n) \\ &\quad + \langle \nabla f(u_n) - \nabla f(T^n(x_n)), y - u_n\rangle \geq -g(u_n, y) \geq g(y, u_n), \forall y \in C. \end{aligned} \tag{3.28}$$

Since $T^n(x_n) \to x^*$, ψ is continuous and g is lower semicontinuous,

$$\begin{aligned} \liminf_{n\to\infty} g(y, u_n) &\leq \liminf_{n\to\infty}(\langle \psi(T^n(x_n)), y - u_n\rangle + \phi(y) - \phi(u_n) \\ &\qquad + \langle \nabla f(u_n) - \nabla f(T^n(x_n)), y - u_n\rangle) \\ &\leq \limsup_{n\to\infty}(\langle \psi(T^n(x_n)), y - u_n\rangle + \phi(y) - \phi(u_n) \\ &\qquad + \langle \nabla f(u_n) - \nabla f(T^n(x_n)), y - u_n\rangle). \end{aligned} \tag{3.29}$$

This implies that

$$g(y, x^*) \leq \langle \psi(x^*), y - x^*\rangle + \phi(y) - \phi(x^*). \tag{3.30}$$

For any $y \in C$ and $t \in (0, 1]$, let $y_t = ty + (1-t)x^* \in C$. So, we have

$$g(y_t, x^*) \leq \langle \psi(x^*), y_t - x^*\rangle + \phi(y_t) - \phi(x^*). \tag{3.31}$$

This yields

$$g(y_t, x^*) + \langle \psi(x^*), x^* - y_t\rangle + \phi(x^*) - \phi(y_t) \leq 0. \tag{3.32}$$

Hence,

$$\begin{aligned} 0 &= g(y_t, y_t) + \langle \psi(x^*), y_t - y_t\rangle + \phi(y_t) - \phi(y_t) \\ &= g(y_t, ty + (1-t)x^*) + \langle \psi(x^*), ty + (1-t)x^* - ty_t - (1-t)y_t\rangle \\ &\quad + \phi(ty + (1-t)x^*) - t\phi(y_t) - (1-t)\phi(y_t) \\ &\leq tg(y_t, y) + (1-t)g(y_t, x^*) + t\langle \psi(x^*), y - y_t\rangle + (1-t)\langle \psi(x^*), x^* - y_t\rangle \\ &\quad + t\phi(y) + (1-t)\phi(x^*) - t\phi(y_t) - (1-t)\phi(y_t) \\ &= t\left[g(y_t, y) + \langle \psi(x^*), y - y_t\rangle + \phi(y) - \phi(y_t)\right] \\ &\quad + (1-t)\left[g(y_t, x^*) + \langle \psi(x^*), x^* - y_t\rangle + \phi(x^*) - \phi(y_t)\right] \\ &\leq t\left[g(y_t, y) + \langle \psi(x^*), y - y_t\rangle + \phi(y) - \phi(y_t)\right]. \end{aligned} \tag{3.33}$$

Since $t > 0$, it follows from (3.33), we obtain that

$$g(y_t, y) + \langle \psi(x^*), y - y_t\rangle + \phi(y) - \phi(x^*) \geq 0. \tag{3.34}$$

From (C3), we have

$$\begin{aligned}0 &\le \limsup_{t\to 0^+}\left(g(y_t,y)+\langle\psi(x^*),y-y_t\rangle+\phi(y)-\phi(y_t)\right)\\ &= \limsup_{t\to 0^+}\left(g(ty+(1-t)x^*,y)+\langle\psi(x^*),y-y_t\rangle+\phi(y)-\phi(y_t)\right)\\ &\le g(x^*,y)+\langle\psi(x^*),y-x^*\rangle+\phi(y)-\phi(x^*).\end{aligned} \tag{3.35}$$

This shows that $x^*\in GMEP(g,\phi,\psi)$. To sum up, we have

$$x^*\in\Omega := F(T)\cap GMEP(g,\phi,\psi).$$

Step 5: We show that $\{x_n\}$ converges strongly to $\overline{x}=proj^f_\Omega(u)$. By assumption, Lemma 2.17 and Lemma 2.19, we know that $F(T)\cap GMEP(g,\phi,\psi)$ is a nonempty closed convex subset of E. Therefore the $proj^f_\Omega(u)$ is well-defined. Since $proj^f_\Omega(u)\in\Omega\subset Y_n\subset Z_n$ and $x_{n+1}=proj^f_{Z_n}(u)$, it follows that

$$D_f(x_{n+1},u)\le D_f(proj^f_\Omega(u),u).$$

By Lemma 2.20, we can obtain that $x_n\to proj^f_\Omega(u)$ (as $n\to\infty$). Therefore, the sequence $\{x_n\}$ converges strongly to $\overline{x}=proj^f_\Omega(u)$. This completes the proof. ∎

If in Theorem 3.1, we take $\phi(x)=0$ and $\psi(x)=0$ for all $x\in C$, then we obtain the following corollary.

Corollary 3.2. [10] *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of $int(domf)$. Let $f:E\to\mathbb{R}$ be a totally convex on bounded subsets of E, coercive Legendre function which is bounded, uniformly Frêchet differentiable, $T:C\to C$ be a closed Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\upsilon_n\},\{\mu_n\}$ and a strictly increasing continuous function $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ such that $\upsilon_n\to 0,\mu_n\to 0$ (as $n\to\infty$) and $\zeta(0)=0$. Let $g:C\times C\to\mathbb{R}$ be a bifunction satisfying conditions (C1)-(C4). Assume that T is uniformly asymptotically regular and $F(T)\cap EP(g)\ne\emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1=u\in C, \text{ chosen arbitrarily},\\ u_n=Res^f_g(T^n(x_n)),\\ Y_n=\{z\in C: D_f(z,u_n)\le D_f(z,x_n)+\xi_n\},\\ Z_n=\bigcap_{i=1}^n Y_i,\\ x_{n+1}=proj^f_{Z_n}(u),\end{cases} \tag{3.36}$$

where $\xi_n=\upsilon_n\sup_{v\in F(T)\cap EP(g)}\zeta(D_f(v,x_n))+\mu_n$ and $proj^f_{Z_n}$ is the Bregman projection of E onto Z_n. If $F(T)\cap EP(g)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\overline{x}=proj^f_{F(T)\cap EP(g)}(u)$.

If in Theorem 3.1, we take $\upsilon_n=0,\mu_n=0$ and $T^n(x)=T(x)$ for each $n\in\mathbb{N}$, then T reduces to a Bregman strongly nonexpansive mapping and we have the following corollary.

Corollary 3.3. [11] *Let E be a real reflexive Banach space, C be a nonempty convex closed subset of E. Let $f:E\to\mathbb{R}$ be a totally convex on bounded subsets of E, coercive Legendre function which is bounded, uniformly Frêchet differentiable, T be a Bregman strongly* *nonexpansive mapping with respect to f such that $F(T) = \hat{F}(T)$ and T is uniformly continuous. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (C1)-(C4) and $\Omega := F(T) \cap GMEP(g, \phi, \psi) \neq \emptyset$ and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 &= x \in C, \textit{chosen arbitrarily}, \\ y_n &= Res^f_{g,\phi,\psi}(x_n), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{cases} \tag{3.37}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj^f_\Omega(x)$.

A Banach space E is uniformly convex if for any two sequences $\{x_n\}, \{y_n\}$ in E such that the conditions

$$\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + y_n\| = 2$$

imply

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

In addition, the Banach space E is said to be uniformly smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

is attained uniformly for all (x, y) in $S(E) \times S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$.

If in Theorem 3.1, we assume that E is a uniformly smooth and uniformly convex Banach space and $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$, then we obtain that $\nabla f = J_p$, where J_p is the generalization duality mapping from E onto E^*. Thus, we get the following corollary.

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space, $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ and C be a nonempty convex closed subset of $int(domf)$. Let $T : C \to C$ be a closed Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\upsilon_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\upsilon_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $\phi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, $\psi : C \to E^*$ be a continuous monotone mapping and $g : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (C1)-(C4). Assume that T is uniformly asymptotically regular and $\Omega := F(T) \cap GMEP(g, \phi, \psi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = u \in C, \textit{chosen arbitrarily}, \\ u_n := g(u_n, y) + \langle \psi T^n x_n, y - u_n \rangle + \phi(y) - \phi(u_n) \\ \qquad + \langle J_p(u_n) - J_p(T^n x_n), y - u_n \rangle \geq 0, \forall y \in C, \\ Y_n = \{z \in C : D_f(z, u_n) \leq D_f(z, x_n) + \xi_n\}, \\ Z_n = \bigcap_{i=1}^{n} Y_i, \\ x_{n+1} = proj^f_{Z_n}(u), \end{cases} \tag{3.38}$$

where $\xi_n = \upsilon_n \sup_{v\in\Omega} \zeta(D_f(v, x_n)) + \mu_n$, $proj^f_{Z_n}$ *is the Bregman projection of* E *onto* Z_n. *If* $\Omega := F(T) \cap GMEP(g, \phi, \psi)$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to* $\overline{x} = proj^f_\Omega(u)$.

4. Numerical Computational

In this section, we give example and numerical result to support Theorem 3.1. In addition, we compare the converging steps of introduced algorithm with Algorithm (1.6), which was presented in [10].

Let $\Pi_{i=1}^N[a, b]$ be the set of vectors $x \in \mathbb{R}^N$ where each component of x contained in the interval $[a, b]$.

We consider the bifunction $g : C \times C \to \mathbb{R}$ defined by $g(x, y) = x(y - x)$, $\phi : C \to \mathbb{R}$ defined by $\phi(x) = x^2$, $\psi : C \to \mathbb{R}$ defined as $\psi(x) = \cos(x)$ where $x \in C$. Let $\zeta : [0, +\infty) \to [0, +\infty)$ be a strictly increasing mapping with $\zeta(0) = 0$.

Example 4.1. Let $E = \mathbb{R}^N$ and $C = \Pi_{i=1}^N[-\pi, 0]$. Let $f : \mathbb{R}^N \to (-\infty, +\infty]$ defined by $f(x) = \frac{1}{2}x^2$. Let $T : C \to C$ defined by $T(x) = \frac{x}{3}$ with the nonnegative sequences $\{\upsilon_n\}$ and $\{\mu_n\}$ where $\upsilon_n = \frac{2}{n+2}$ and $\mu_n = \frac{1}{n^2}$ for all $n \geq 1$. We obtain that

$$\begin{aligned}
&D_f(0, T^n(x)) - D_f(0, x) - \upsilon_n\zeta(D_f(0, x)) - \mu_n \\
&= f(0) - f(T^n(x)) - \langle\nabla f(T^n(x)), 0 - T^n(x)\rangle - (f(0) - f(x) - \langle\nabla f(x), 0 - x\rangle) \\
&\quad - \upsilon_n\zeta(f(0) - f(x) - \langle\nabla f(x), 0 - x\rangle) - \mu_n \\
&= -(\frac{1}{2})(\frac{x}{3^n})^2 + (\frac{x}{3^n})^2 + \frac{1}{2}x^2 - x^2 - \upsilon_n\zeta(x^2 - \frac{1}{2}x^2) - \mu_n \\
&= (\frac{1}{2})(\frac{x}{3^n})^2 - \frac{1}{2}x^2 - \upsilon_n\zeta(\frac{1}{2}x^2) - \mu_n \leq 0.
\end{aligned}$$

Therefore, T is a Bregman totally quasi-asymptotically nonexpansive mapping. Furthermore, 0 is the unique solution in $\Omega = F(T)\cap GMEP(g, \phi, \psi)$. We randomly generated starting point $x_1 \in \mathbb{R}^N$ in the interval $[-\pi, 0]$, we get the following observation for different iterations and using the stopping criterion $\|x_n - \bar{x}\|^2 < 10^{-3}$. In Table 1, we randomly take 10 starting points and the presented results are in average.

Table 1. The numerical results for different size of $\mathbb{R}^N$

Size N	Average iterations	
	Algorithm (3.1)	Algorithm (1.6)
5	76	92
10	107	129
15	130	154
20	151	160
25	169	177
30	185	194

From Table 1, we see that the computational iterations of Algorithm (3.1) is less than Algorithm (1.6) in all cases of $\mathbb{R}^N$.

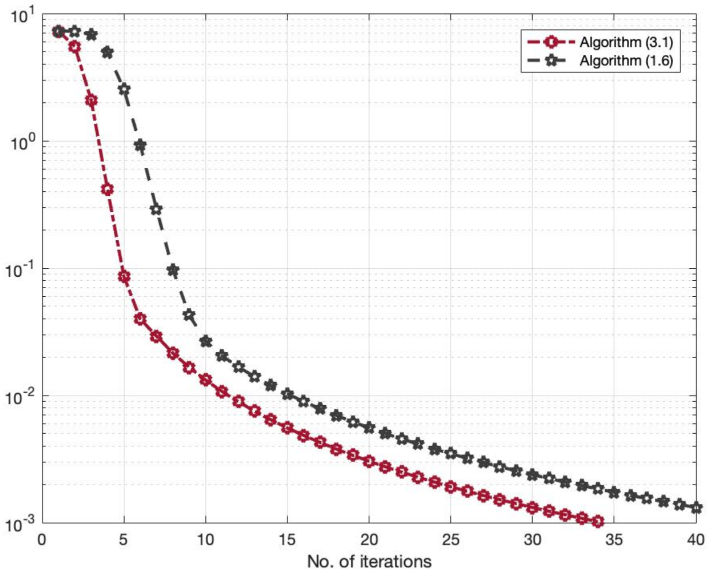

FIGURE 1. The convergence results of the Algorithm (3.1) and the Algorithm (1.6) for $N = 1$

ACKNOWLEDGEMENTS

The first author is thankful to the Science Achievement Scholarship of Thailand. We would like express our deep thank to Department of Mathematics, Faculty of Science, Naresuan University for the support.

REFERENCES

[1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.

[2] A. Farajzadeh, A. Shafie, On system of vector quasi-equilibrium problems for multi-valued mappings, Kragujevac J. Math. 42 (2018) 357–369.

[3] S. Jafari, A. Farajzadeh, S. Moradi, Locally densely defined equilibrium problems, J. Optim. Theory Appl. 170 (2016) 804–817.

[4] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008) 186–201.

[5] P. Chuasuk, A. Farajzadeh, A. Kaewcharoen, An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces, J. Comput. Anal. Appl. 28 (2020) 49–66.

[6] P. Lohawech, A. Kaewcharoen, A. Farajzadeh, Algorithms for the common solution of the split variational inequality problems and fixed point problems with applications, J. Inequal. Appl. (1) (2018) 1–17.

[7] L.M. Bregman, The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming, USSR Comput. Math. Math. Phys. 7 (1967) 200–217.

[8] S. Reich, S. Sabach, Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, Nonlinear Anal. 73 (2010) 122–135.

[9] V. Darvish, A new algorithm for mixed equilibrium problem and Bregman strongly nonexpansive mapping in Banach spaces, Math. FA. (2015).

[10] Z. Sheng, H. Jianhua, Strong convergence theorems for equilibrium problem and Bregman totally quasi-asymptotically nonexpansive mapping in Banach spaces, Acta Math. Sci. (5) (2016) 1433–1444.

[11] V. Darvish, Strong convergence theorem for generalized mixed equilibrium problems and Bregman nonexpansive mapping in Banach spaces, Math. Moravica 20 (2016) 69–87.

[12] H.H. Bauschke, J.M. Borwein, P.L. Combettes, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Comm. Contemp. Math. 3 (2001) 615–647.

[13] Y. Censor, A. Lent, An iterative row-action method for interval convex programing, J. Optim. Theory Appl. 34 (1981) 321–353.

[14] D. Butnariu, A.N. Iusem, Totally convex functions for fixed points computation and infinite dimensional optimization, Applied Optimization Vol. 40. Dordrecht: Kluwer Academic (2000).

[15] S. Reich, S. Sabach, Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim. 31 (2010) 22–44.

[16] D. Butnariu, E. Resmerita, Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, Abstr. Appl. Anal. (2006) Article ID 84919.

[17] S. Reich, S. Sabach, Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces, Fixed-Point Algorithms for Inverse Problemsin Science and Engineering, Springer, New York, (2011) 301–316.

[18] G. Kassay, S. Reich, S. Sabach, Iterative methods for solving systems of variational inequalities in reflexive Banach spaces, SIAM J. Optim. 21 (2011) 1319–1344.

[19] S.S. Chang, L. Wang, X.R. Wang, C.K. Chan, Strong convergence theorems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces, Appl. Math. Comput. 228 (2014) 38–48.