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Best Proximity Results for Multi-Valued $\alpha - F-$Proximal Contractive Mappings

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Abstract Based on the concepts of multi-valued F-contraction and α-proximal admissibility, we establish some best proximity point results for non-self multi-valued mappings. For such mappings, we study the existence of best proximity points. We show that many known results in literature are simple consequences of our obtained results. We also provide some concrete examples illustrating the obtained results.

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1. Introduction and Preliminaries

The existence and approximation of best proximity points is an interesting topic in optimization theory [1, 2]. On the other hand, the study of fixed point theorems for multi-valued operators is initiated by Markins [3] and Nadler [4]. Since then, some papers have been devoted to the treatment of multi-valued operators in variant (generalized) metric spaces. This paper is devoted to prove some best proximity points for multi-valued mappings. Recently, M. U. Ali, T. Kamran an N. Shahzad [5] introduced the notion of $\alpha - \psi$-proximal contractive multimaps and proved some best proximity points for such mappings. On the hand, the concept of F-contraction for single valued mappings was introduced by Wardowski [6]. Very recently, Altun et al [7] introduced the concept of multi-valued F-contraction and established some fixed point results. Later, Olgun et al [8] generalized the above concept and they obtained some nice fixed point resultsfor multi-valued contractive mappings. In this paper, using the concepts of multi-valued F-contraction and α-proximal admissible mapping, we give some best proximity points for non-self multi-valued mappings in complete metric spaces.

Let $C_b(X)$ be the family of all nonempty, closed and bounded subsets of the metric space (X, d). Also, $K(X)$ denotes the collection of nonempty compact subsets of X. For

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$A, B \in C_b(X)$ and $x \in X$, define

$$d(x,A) = \inf\{d(x,a) : \ a \in A\}, \ \delta(A,B) = \sup\{d(a,B) : a \in A\},$$
$$\delta(B,A) = \sup\{d(b,A) : b \in B\}, \ H(A,B) = \max\{\delta(A,B), \delta(B,A)\}.$$

The above H is the Pompeiu-Hausdorff metric induced by the metric d. For A and B two nonempty subsets of a metric space (X,d), define

$$d(A,B) = \inf\{d(a,b) : \ a \in A, \, b \in B\},$$
$$A_0 = \{a \in A : d(a,b) = d(A,B), \ \text{for some}\, b \in B\},$$
$$B_0 = \{b \in B : d(a,b) = d(A,B), \ \text{for some}\, a \in A\}.$$

Definition 1.1. Let (X,d) be a metric space. Consider A and B two nonempty subsets of X. Let $T : A \to B$ be a non-self mapping. We say $x^\star \in X$ is a best proximity point of T if

$$d(x^\star, Tx^\star) = d(A,B).$$

Best proximity point reduce to fixed point if $A = B$. In fact, $d(x^\star, Tx^\star) = d(A,B) = 0$ i.e., $x^\star = Tx^\star$. We say $x^\star \in A$ is best proximity point of multi-valued mapping $T : A \to 2^B$ if $d(x^\star, Tx^\star) = d(A,B)$. Now if $A = B$, then $d(x^\star, Tx^\star) = 0$. So we can say $x^\star \in Tx^\star$, that is best proximity point of a multi-valued mapping in our aspect can reduce to a fixed point(in general). Now we assume Tx is closed for every x. Since $d(x^\star, Tx^\star) = 0$ then there exists $\{y_n\} \subseteq Tx^\star$ such that $\lim_{n\to\infty} d(x^\star, y_n) = 0$. Since $Tx^\star$ is closed then $x^\star \in Tx^\star$. So this definition is suitable for closed(or compact) valued T.

For more results on best proximity points, we refer the reader to [9–14].

Definition 1.2. (See [14]) Let A and B be nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. The pair (A,B) is said to have the weak P-property if and only if

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \Rightarrow d(x_1,x_2) \leq d(y_1,y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Recently, M. U. Ali, T. Kamran and N. Shahzad [5] introduced a new type of contractive mappings called $\alpha - \psi$-proximal contractive Multimaps and using the concept of α-proximal admissible mapping, they [5] proved several best proximity points.

Definition 1.3. (See [5]) Let A and B be nonempty subsets of a metric space (X,d) and $\alpha : A \times A \to [0,\infty)$. A mapping $T : A \to 2^B \setminus \emptyset$ is named α-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 1 \\ d(u_1,y_1) = d(A,B), \\ d(u_2,y_2) = d(A,B) \end{cases} \Rightarrow \alpha(u_1,u_2) \geq 1,$$

where $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

Let $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions:

- (F_1) F is increasing, i.e., $F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \beta$;
- (F_2) $\forall \alpha_n > 0, \ \lim\limits_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) $\exists k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$;

(F_4) $F(\inf M) = \inf F(M)$ for all $M \subset (0, \infty)$ with $\inf M > 0$.

We denote by $\mathcal{F}$ the class of functions F satisfying the conditions $(F_1) - (F_3)$.

Remark 1.4. Note that if F is right-continuous and satisfies condition (F_1), then it satisfies (F_4).

Definition 1.5. Let (X, d) be a metric space and $T : A \to C_b(B)$ be a multi-valued mapping. We say that T is an $\alpha - F$-proximal contraction if there exist three functions $F \in \mathcal{F}$, $\alpha : A \times A \to [0, \infty)$ and $\tau : (0, \infty) \to (0, \infty)$ for which $\liminf_{t \to s^+} \tau(t) > 0$, for all $s \geq 0$, such that

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

for all $x, y \in A$ satisfying $H(Tx, Ty) > 0$ and $\alpha(x, y) \geq 1$.

2. Main Results

First, we need the following remark.

Remark 2.1. Let K be a compact subset of a metric space (X, d) and $x \in X$, then there exists $y \in K$ such that $d(x, K) = d(x, y)$.

Our first main result is

Theorem 2.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such tat $A_0 \neq \emptyset$. Let $T : A \to K(B)$ be an $\alpha - F$-proximal contraction. Suppose that*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$ and (A, B) satisfies the weak P-property;

(ii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) T is an α-proximal admissible;

(iv) T is continuous.

Then T has a best proximity point, that is, there exists an element $x^\star \in A_0$ such that $d(x^\star, Tx^\star) = d(A, B)$.

Proof. By assumption (ii), there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{2.1}$$

If $y_1 \in Tx_1$, then we have $d(A, B) \leq d(x_1, Tx_1) \leq d(x_1, y_1) = d(A, B)$, that is x_1 is a best proximity point of T and the proof is finished. Suppose now that $y_1 \notin Tx_1$. It follows that $H(Tx_0, Tx_1) > 0$. So, by (1.1), we get

$$\tau(d(x_0, x_1)) + F(H(Tx_0, Tx_1)) \leq F(d(x_0, x_1)). \tag{2.2}$$

On the other hand, from $0 < d(y_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (F_1)

$$\tau(d(x_0, x_1)) + F(d(y_1, Tx_1)) \leq F(d(x_0, x_1)).$$

Since Tx_1 is compact, there exists $y_2 \in Tx_1$ such that $d(y_1, Tx_1) = d(y_1, y_2)$. This leads to

$$F(d(y_1, y_2)) \leq F(d(x_0, x_1)) - \tau(d(x_0, x_1)). \tag{2.3}$$

From condition (i), we have $Tx_1 \subseteq B_0$, so there exists $x_2 \in A_0$ such that

$$d(x_2, y_2) = d(A, B). \tag{2.4}$$

By (2.1), (2.4) and the fact that T is α-proximal admissible, we have

$$\alpha(x_1, x_2) \geq 1.$$

From condition (i), the pair (A, B) satisfies the weak P-property , so

$$d(x_1, x_2) \leq d(y_1, y_2).$$

If $x_1 = x_2$, then we have x_1 is a best proximity point of T and also the proof is finished. Suppose that $x_1 \neq x_2$. From (2.3) and (F_1)

$$F(d(x_1, x_2)) \leq F(d(y_1, y_2)) \leq F(d(x_0, x_1)) - \tau(d(x_0, x_1)). \tag{2.5}$$

If $y_2 \in Tx_2$, then x_2 is a best proximity point of T. Suppose now that $y_2 \notin Tx_2$. Since $\alpha(x_1, x_2) \geq 1$, it follows from (1.1)

$$\tau(d(x_1, x_2)) + F(H(Tx_1, Tx_2)) \leq F(d(x_1, x_2)). \tag{2.6}$$

Then from $0 < d(y_2, Tx_2) \leq H(Tx_1, Tx_2)$ and (F_1)

$$F(d(y_2, Tx_2)) \leq F(d(x_1, x_2)) - \tau(d(x_1, x_2)).$$

Since Tx_2 is compact, there exists $y_3 \in Tx_2$ such that $d(y_2, Tx_2) = d(y_2, y_3)$. This leads to

$$F(d(y_2, y_3)) \leq F(d(x_1, x_2)) - \tau(d(x_1, x_2)). \tag{2.7}$$

Moreover, there exists $x_3 \in A_0$ such that

$$d(x_3, y_3) = d(A, B). \tag{2.8}$$

By (2.4), (2.8) and the fact that T is α-proximal admissible, we have

$$\alpha(x_2, x_3) \geq 1.$$

Since (A, B) satisfies the weak P-property , so

$$d(x_2, x_3) \leq d(y_2, y_3).$$

If $x_2 = x_3$, then we have x_2 is a best proximity point of T and also the proof is finished. Suppose that $x_2 \neq x_3$. From (2.7) and (F_1)

$$F(d(x_2, x_3)) \leq F(d(y_2, y_3)) \leq F(d(x_1, x_2)) - \tau(d(x_1, x_2)). \tag{2.9}$$

Repeating the above strategy, by induction, we arrive to construct two sequences $\{x_n\} \subseteq A_0$ and $\{y_n\} \subseteq B_0$ such that, for all $n = 1, 2, \ldots$

(i) $\alpha(x_n, x_{n+1}) \geq 1$, $x_n \neq x_{n+1}$;
(ii) $y_n \in Tx_{n-1}$, $y_n \notin Tx_n$;
(iii) $d(x_n, y_n) = d(A, B)$ and

$$F(d(x_n, x_{n+1})) \leq F(d(y_n, y_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)). \tag{2.10}$$

Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \ldots$

From (2.10), we have, for all $n = 1, 2, \ldots$

$$F(a_n) \leq F(a_{n-1}) - \tau(a_{n-1}) < F(a_{n-1}). \tag{2.11}$$

By (F_1), we get, for all $n = 1, 2, \ldots$

$$a_n < a_{n-1}, \tag{2.12}$$

which implies that $\{a_n\}$ is a decreasing sequence of positive real numbers and so it is convergent. Let $2\delta = \liminf_{n\to\infty} \tau(a_n) > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\delta < \tau(a_n), \quad \text{for all} \quad n \geq n_0. \tag{2.13}$$

It follows from (2.11) and (2.13), for all $n > n_0$

$$F(a_n) \leq F(a_{n-1}) - \delta \leq F(a_{n-2}) - 2\delta \leq \cdots \leq F(a_{n_0}) - (n - n_0)\delta. \tag{2.14}$$

It follows $\lim_{n\to\infty} F(a_n) = -\infty$, so that, by $(F2)$, $\lim_{n\to\infty} a_n = 0$. Thus from $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{2.15}$$

But then

$$\begin{aligned} a_n^k F(a_n) - a_n^k F(a_{n_0}) &\leq a_n^k (F(a_{n_0}) - (n - n_0)\delta) - a_n^k F(a_{n_0}) \\ &= -(n - n_0)\delta a_n^k \leq 0, \end{aligned} \tag{2.16}$$

which, for $n \to \infty$, yields $\lim_{n\to\infty} n a_n^k = 0$. It follows that there exists $n_1 \in \mathbb{N}$ such that

$$a_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad \text{for all } n \geq n_1. \tag{2.17}$$

Then for all $m > n \geq n_1$

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) = \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \to 0 \quad \text{as} \quad n \to \infty, \tag{2.18}$$

which shows that $\{x_n\}$ is a Cauchy sequence in A. From (2.10) and (F_1), we have, for all $n = 1, 2, \ldots$

$$d(y_n, y_{n+1}) < a_{n-1}. \tag{2.19}$$

Then, by a similar reasoning, we show that $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of the complete metric space (X, d), there exist $x^\star \in A$ and $y^\star \in B$ such that $x_n \to x^\star$ and $y_n \to y^\star$ as $n \to \infty$. Since $d(x_n, y_n) = d(A, B)$, for all n, by letting $n \to \infty$, we conclude that $d(x^\star, y^\star) = d(A, B)$.

The mapping T is continuous, so $\lim_{n\to\infty} H(Tx_n, Tx^\star) = 0$. On the other hand, since $y_{n+1} \in Tx_n$, we have

$$d(y^\star, Tx^\star) \leq d(y^\star, y_{n+1}) + d(y_{n+1}, Tx^\star) \leq d(y^\star, y_{n+1}) + H(Tx_n, Tx^\star).$$

Letting $n \to \infty$ in above inequalities, we get

$$d(y^\star, Tx^\star) \leq 0.$$

This leads to $y^\star \in \overline{Tx^\star} = Tx^\star$. Furthermore, one has

$$d(A, B) \leq d(x^\star, Tx^\star) \leq d(x^\star, y^\star) = d(A, B),$$

that is, $x^\star$ is a best proximity point of T. This ends the proof of Theorem 2.2.

■

Remark 2.3. If we take $C_b(B)$ instead of $K(B)$ in Theorem 2.2, we have the following problem: Does T has a best proximity point? In the following theorem, we give a partial answer for this problem by adding the condition (F_4) on F.

Theorem 2.4. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such tat $A_0 \neq \emptyset$. Let $T : A \to C_b(B)$ be an $\alpha - F$-proximal contraction. Suppose that*

(*i*) *for each* $x \in A_0$*, we have* $Tx \subseteq B_0$ *and* (A, B) *satisfies the weak* P*-property;*
(*ii*) *there exist elements* $x_0, x_1 \in A_0$ *and* $y_1 \in Tx_0$ *such that*

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(*iii*) T *is an* α*-proximal admissible;*
(*iv*) (F_4) *holds;*
(*v*) T *is continuous.*

Then T *has a best proximity point.*

Proof. By assumption (*ii*), there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{2.20}$$

To avoid the repetition, we may suppose that $y_1 \notin Tx_1$. It follows that $H(Tx_0, Tx_1) > 0$. So from (1.1), $0 < d(y_1, Tx_1) \leq H(Tx_0, Tx_1)$ and (F_1), we obtain

$$F(d(y_1, Tx_1)) \leq F(d(x_0, x_1)) - \tau(d(x_0, x_1)). \tag{2.21}$$

From (F_4), one can write $F(d(y_1, Tx_1)) = \inf\{F(d(y_1, z)) : z \in Tx_1\}$. This implies that from (2.21), there exists $y_2 \in Tx_1$ such that

$$F(d(y_1, y_2)) \leq F(d(x_0, x_1)) - \frac{\tau(d(x_0, x_1))}{2}. \tag{2.22}$$

Following the proof of Theorem 2.2, we can construct two sequences $\{x_n\} \subseteq A_0$ and $\{y_n\} \subseteq B_0$ such that, for all $n = 1, 2, \ldots$

(*i*) $\alpha(x_n, x_{n+1}) \geq 1$, $x_n \neq x_{n+1}$;
(*ii*) $y_n \in Tx_{n-1}$, $y_n \notin Tx_n$;
(*iii*) $d(x_n, y_n) = d(A, B)$ and

$$F(d(x_n, x_{n+1})) \leq F(d(y_n, y_{n+1})) \leq F(d(x_{n-1}, x_n)) - \frac{\tau(d(x_{n-1}, x_n))}{2}.$$

Also $\{x_n\}$ is a Cauchy sequence in A and $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of the complete metric space (X, d), there exist $x^\star \in A$ and $y^\star \in B$ such that $x_n \to x^\star$ and $y_n \to y^\star$ as $n \to \infty$. As in the proof of Theorem 2.2, we conclude that $x^\star$ is a best proximity point of T. ∎

In the next results, we replace the continuity hypothesis by the following condition in A:

(*H*)**:** if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 2.5. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such tat* $A_0 \neq \emptyset$*. Let* $T : A \to K(B)$ *be an* $\alpha - F$*-proximal contraction. Suppose that*

(*i*) *for each* $x \in A_0$*, we have* $Tx \subseteq B_0$ *and* (A, B) *satisfies the weak* P*-property;*
(*ii*) *there exist elements* $x_0, x_1 \in A_0$ *and* $y_1 \in Tx_0$ *such that*

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(*iii*) T *is an* α*-proximal admissible;*
(*iv*) (H) *holds.*

Then T *has a best proximity point.*

Proof. Following the proof of Theorem 2.2, there exists two sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0 such that

(i) $\alpha(x_n, x_{n+1}) \geq 1$, $x_n \neq x_{n+1}$;

(ii) $y_n \in Tx_{n-1}$, $y_n \notin Tx_n$;

(iii) $d(x_n, y_n) = d(A, B)$ and

$$F(d(x_n, x_{n+1})) \leq F(d(y_n, y_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)).$$

Also, there exist $x^\star \in A$ and $y^\star \in B$ such that $x_n \to x^\star$ and $y_n \to y^\star$ as $n \to \infty$ and $d(x^\star, y^\star) = d(A, B)$. We shall prove that $x^\star$ is a best proximity point of T. If there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} = Tx^\star$ for all k, then we have

$$d(A, B) \leq d(x_{n_k+1}, Tx_{n_k}) \leq d(x_{n_k+1}, y_{n_k+1}) = d(A, B),$$

yields to

$$d(A, B) \leq d(x_{n_k+1}, Tx^\star) \leq d(A, B), \ \forall k \geq 0.$$

Letting $k \to \infty$ in above inequalities, we get

$$d(A, B) \leq d(x^\star, Tx^\star) \leq d(A, B),$$

that is $x^\star$ is a best proximity point of T. So, without loss of generality, we may suppose that $Tx_n \neq Tu$ for all nonnegative integer n. By hypothesis (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^\star) \geq 1$ for all k. From (1.1) and as $\alpha(x_{n(k)}, x^\star) \geq 1$ for all $k \geq 1$, we get

$$F(H(Tx_{n(k)}, Tx^\star)) \leq F(d(x_{n(k)}, x^\star)) - \tau(d(x_{n(k)}, x^\star)) < F(d(x_{n(k)}, x^\star)).$$

From (F_1)

$$H(Tx_{n(k)}, Tx^\star) < d(x_{n(k)}, x^\star).$$

On the other hand, we have

$$d(y^\star, Tx^\star) \leq d(y^\star, y_{n(k)+1}) + d(y_{n(k)+1}, Tx^\star) \leq d(y^\star, y_{n(k)+1}) + H(Tx_{n(k)}, Tx^\star).$$

Then

$$d(y^\star, Tx^\star) \leq d(y^\star, y_{n(k)+1}) + d(x_{n(k)}, x^\star).$$

Passing to limit as $k \to \infty$, we obtain $d(y^\star, Tx^\star) = 0$. Hence, $d(A, B) \leq d(x^\star, Tx^\star) \leq d(x^\star, y^\star) = d(A, B)$. Therefore, $x^\star$ is a best proximity point of T.

■

Theorem 2.6. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such tat $A_0 \neq \emptyset$. Let $T : A \to C_b(B)$ be an $\alpha - F$-proximal contraction. Suppose that*

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$ and (A, B) satisfies the weak P-property;*

(ii) *there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is an α-proximal admissible;*

(iv) *(F_4) holds;*

(v) *(H) holds.*

Then T has a best proximity point.

Proof. The proof is similar to that of Theorem 2.5. ■

Example 2.7. Let $X = [0,\infty) \times [0,\infty)$ endowed with the usual metric d. Take $A = \{1\} \times [0,\infty)$ and $B = \{0\} \times [0,\infty)$. We mention that $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. Consider the mapping $T : A \to C_b(B)$ as

$$T(1,x) = \begin{cases} \{(0,0),(0,\frac{x^2}{2})\} & \text{if } 0 \le x \le 1 \\ \{(0,0),(0,\frac{x}{x+1})\} & \text{if } x > 1. \end{cases}$$

We have $T(1,x) \subseteq B_0$ for each $(1,x) \in A_0$. Take $\tau(t) = \ln 2$ and $F(t) = \ln t$ for all $t > 0$. Define $\alpha : A \times A \to [0,\infty)$ as follows

$$\alpha((1,x),(1,y)) = \begin{cases} 1 & \text{if } \quad x = y = 0 \\ \frac{1}{x+y} & \text{if not.} \end{cases}$$

Let $(1,x_1),(1,x_2),(1,u_1),(1,u_2)$ in A and $(0,y_1) \in T(1,x_1),(0,y_2) \in T(1,2)$ such that

$$\begin{cases} \alpha((1,x_1),(1,x_2)) \ge 1 \\ d((1,u_1),(0,y_1)) = d(A,B) = 1, \\ d((1,u_2),(0,y_2)) = d(A,B) = 1. \end{cases}$$

Then, necessarily, $x_1 = x_2 = 0$ or $0 < x_1 + x_2 \le 1$. Also, we have $u_1 = y_1$ and $u_2 = y_2$. So $y_1 \in \{0, \frac{x_1^2}{2}\}$ and $y_2 \in \{0, \frac{x_2^2}{2}\}$. Therefore, we have

$$\alpha((1,u_1),(1,u_2)) = \alpha((1,y_1),(1,y_2)) \ge 1,$$

that is, T is an α-proximal admissible. Let $(1,x),(1,y) \in A$ such that $x \neq y$ and $\alpha((1,x),(1,y)) \ge 1$. Then $0 < x + y \le 1$. In this case, we have

$$H(T(1,x),T(1,y)) = \max\{\delta(T(1,x),T(1,y)), \delta(T(1,y),T(1,x))\}.$$

We have

$$\begin{aligned} \delta(T(1,x),T(1,y)) &= \max\{d((0,0),\{(0,0),(0,\frac{y^2}{2})\}), d((0,\frac{x^2}{2}),\{(0,0),(0,\frac{y^2}{2})\})\} \\ &= \min\{\frac{x^2}{2}, \frac{|x^2-y^2|}{2}\} \le \frac{|x^2-y^2|}{2}. \end{aligned}$$

Similarly, we have

$$\delta(T(1,y),T(1,x)) \le \frac{|x^2-y^2|}{2}.$$

This yields that

$$H(T(1,x),T(1,y)) \le \frac{|x^2-y^2|}{2} = \frac{(x+y)|x-y|}{2} \le \frac{|x-y|}{2} = \frac{(d(1,x),(1,y))}{2}.$$

Then

$$\tau(d((1,x),(1,y))) + F(H(T(1,x),T(1,y))) \le F(d((1,x),(1,y))).$$

So the condition contraction (1.1) holds. Furthermore, T is continuous. Moreover, the condition (ii) of Theorem 2.4 is verified. Indeed, for $x_0 = (1,1)$, $x_1 = (1,0)$ and $y_1 = (0,0)$, we have

$$d(x_1,y_1) = d((1,0),(0,0)) = 1 = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) = 1.$$

Hence all hypotheses of Theorem 2.4 are verified. So T has a best proximity point which is $u = (1,0)$.

Example 2.8. Let $X = [0,\infty) \times [0,\infty)$ endowed with the usual metric d. Take $A = \{1\} \times [0,\infty)$ and $B = \{0\} \times [0,\infty)$. We mention that $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. Consider the mapping $T : A \to C_b(B)$ as

$$T(1,x) = \begin{cases} \{(0,0),(0,\frac{x^2+1}{4})\} & \text{if } 0 \le x \le 1 \\ \{(0,0),(0,\frac{1}{x})\} & \text{if } x > 1. \end{cases}$$

We have $T(1,x) \subseteq B_0$ for each $(1,x) \in A_0$. Take $\tau(t) = e^{-t}\ln 2$ and $F(t) = \ln t$ for all $t > 0$. Define $\alpha : A \times A \to [0,\infty)$ as follows

$$\alpha((1,x),(1,y)) = \begin{cases} 1 & \text{if} \quad x,y \in [0,1] \\ 0 & \text{if not.} \end{cases}$$

Let $(1,x_1),(1,x_2),(1,u_1),(1,u_2)$ in A and $(0,y_1) \in T(1,x_1),(0,y_2) \in T(1,2)$ such that

$$\begin{cases} \alpha((1,x_1),(1,x_2)) \ge 1 \\ d((1,u_1),(0,y_1)) = d(A,B) = 1, \\ d((1,u_2),(0,y_2)) = d(A,B) = 1. \end{cases}$$

Then, necessarily, $x_1, x_2 \in [0,1]$. Also, we have $u_1 = y_1$ and $u_2 = y_2$. So $y_1 \in \{0, \frac{x_1^2+1}{4}\}$ and $y_2 \in \{0, \frac{x_2^2+1}{4}\}$. Therefore, we have

$$\alpha((1,u_1),(1,u_2)) = \alpha((1,y_1),(1,y_2)) \ge 1,$$

that is, T is an α-proximal admissible. Let $(1,x),(1,y) \in A$ such that $x \neq y$ and $\alpha((1,x),(1,y)) \ge 1$. Then $x,y \in [0,1]$. In this case, we have

$$H(T(1,x),T(1,y)) = \max\{\delta(T(1,x),T(1,y)),\delta(T(1,y),T(1,x))\}.$$

We have

$$\begin{aligned}\delta(T(1,x),T(1,y)) &= \max\{d((0,0),\{(0,0),(0,\frac{y^2+1}{4})\}), d((0,\frac{x^2+1}{4}),\{(0,0),(0,\frac{y^2+1}{4})\})\} \\ &= \min\{\frac{x^2+1}{4},\frac{|x^2-y^2|}{4}\} \le \frac{|x^2-y^2|}{4}.\end{aligned}$$

Similarly, we have

$$\delta(T(1,y),T(1,x)) \le \frac{|x^2-y^2|}{4}.$$

This yields that

$$H(T(1,x),T(1,y)) \le \frac{|x^2-y^2|}{4} = \frac{(x+y)|x-y|}{2} \le \frac{|x-y|}{2} = \frac{(d(1,x),(1,y))}{2}.$$

Then

$$\ln 2 + F(H(T(1,x),T(1,y))) \le F(d((1,x),(1,y))).$$

This leads to

$$\tau(d((1,x),(1,y))) + F(H(T(1,x),T(1,y))) \le F(d((1,x),(1,y))).$$

So the condition contraction (1.1) holds. Furthermore, (H) holds. Indeed, let $\{(1,x_n)\}$ is a sequence in A such that $\alpha((1,x_n),(1,x_{n+1})) \ge 1$ for all n and $(1,x_n) \to (1,x) \in A$. Then, $x_n \in [0,1]$ for all n and $x_n \to x$. Thus, $x \in [0,1]$ and so $\alpha((1,x_n),(1,x)) \ge 1$ for all n.

Moreover, the condition (ii) of Theorem 2.6 is verified. In fact, for $x_0 = (1,0)$, $x_1 = (1,1)$ and $y_1 = (0,0)$, we have

$$d(x_1, y_1) = d((1,0),(0,0)) = 1 = d(A,B) \quad \text{and} \quad \alpha(x_0, x_1) = 1.$$

Hence all hypotheses of Theorem 2.6 are verified. So T has a best proximity point which is $u = (1,0)$.

3. Consequences

In this paragraph, we present some consequences on our obtained results.

3.1. Some Classical Best Proximity Point Results

We have the following results.

Corollary 3.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such tat $A_0 \neq \emptyset$. Let $T : A \to B$ be an $\alpha - F$-proximal contraction. Suppose that*

(i) $TA_0 \subseteq B_0$ and (A,B) satisfies the weak P-property;
(ii) there exist elements $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) T is an α-proximal admissible;
(iv) T is continuous or (H) holds.

Then T has a best proximity point.

Corollary 3.2. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be an $\alpha - F$-proximal contraction. Suppose that*

(i) there exist elements $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(ii) T is an α-admissible;
(iii) T is continuous or (H) holds.

Then T has a fixed point.

Corollary 3.3. *Let (X,d) be a complete metric space and $T : X \to C_b(X)$ be an $\alpha - F$-proximal contraction. Suppose that*

(i) there exist elements $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(ii) T is an α-admissible;
(iii) (F_4) holds;
(iv) T is continuous or (H) holds.

Then T has a fixed point.

3.2. Some Best Proximity Results on Metric Spaces Endowed with a Partial Order

Let (X,d) a partial metric space endowed with a partial order $\preceq$. We introduce the following definition.

Definition 3.4. Let A and B be nonempty subsets of a metric space (X, d) and $\preceq$ a partial order on X. $T : A \to B$ is named a proximal nondecresing mapping if

$$\begin{cases} x_1 \preceq x_2 \\ p(u_1, y_1) = d(A, B), \\ p(u_2, y_2) = d(A, B) \end{cases} \Rightarrow u_1 \preceq u_2$$

where $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$.

Wa also need the following hypothesis.

(H_1)**:** if $\{x_n\}$ is a sequence in A such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We state the following.

Corollary 3.5. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and $\preceq$ be a partial order on X. Let $T : A \to K(B)$ be a given multi-valued mapping. Suppose that there exists $\tau : (0, \infty) \to (0, \infty)$ for which $\liminf_{t\to s^+} \tau(t) > 0$, for all $s \geq 0$, such that*

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{3.1}$$

for all $x, y \in A$ satisfying $H(Tx, Ty) > 0$ and $x \preceq y$. Also, suppose that

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$ and (A, B) satisfies the weak P-property;
(ii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad x_0 \preceq x_1;$$

(iii) T is proximal nondecresing;
(iv) T is continuous or (H_1) holds.

Then T has a best proximity point.

Proof. It suffices to consider $\alpha : A \times A \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y \\ 0 & \text{if not.} \end{cases}$$

All hypotheses of Theorem 2.2 (resp. Theorem 2.5) are satisfied. This completes the proof. ∎

Corollary 3.6. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and $\preceq$ be a partial order on X. Let $T : A \to C_b(B)$ be a given multi-valued mapping. Suppose that there exists $\tau : (0, \infty) \to (0, \infty)$ for which $\liminf_{t\to s^+} \tau(t) > 0$, for all $s \geq 0$, such that*

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{3.2}$$

for all $x, y \in A$ satisfying $H(Tx, Ty) > 0$ and $x \preceq y$. Also, suppose that

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$ and (A, B) satisfies the weak P-property;
(ii) there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad x_0 \preceq x_1;$$

(iii) T is proximal nondecresing;
(iv) (F_4) holds;

(v) *T is continuous or* (H_1) *holds.*

Then T has a best proximity point.

3.3. Some Best Proximity Results on a Metric with a Graph

Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta := \{(x, x) : x \in X\} \subset E(G)$. We need in the sequel the following hypothesis:

(H_G)**:** if $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x) \in E(G)$ for all k.

Again, we introduce the following definition.

Definition 3.7. Let A and B be nonempty subsets of a metric-like space (X, d) endowed with a graph G. $T : A \to B$ is named a G-proximal mapping if

$$\begin{cases} (x_1, x_2) \in E(G) \\ d(u_1, y_1) = d(A, B), \\ d(u_2, y_2) = d(A, B) \end{cases} \Rightarrow (u_1, u_2) \in E(G)$$

where $x_1, x_2, u_1, u_2 \in A, y_1 \in Tx_1$ and $y_2 \in Tx_2$.

We have the following best proximity point results on a metric space endowed with a graph.

Corollary 3.8. *Let A and B be nonempty closed subsets of a complete metric space* (X, d) *endowed with a graph G such that* $A_0 \neq \emptyset$. *Let* $T : A \to K(B)$ *be a given multi-valued mapping. Suppose that there exists* $\tau : (0, \infty) \to (0, \infty)$ *for which* $\liminf_{t \to s^+} \tau(t) > 0$, *for all* $s \geq 0$, *such that*

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{3.3}$$

for all $x, y \in A$ *satisfying* $H(Tx, Ty) > 0$ *and* $(x, y) \in E(G)$. *Also, suppose that*

(i) *for each* $x \in A_0$, *we have* $Tx \subseteq B_0$ *and* (A, B) *satisfies the weak P-property;*

(ii) *there exist elements* $x_0, x_1 \in A_0$ *and* $y_1 \in Tx_0$ *such that*

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad (x_0, x_1) \in E(G);$$

(iii) *T is G-proximal;*

(iv) *T is continuous or* (H_G) *holds.*

Then T has a best proximity point.

Proof. It suffices to consider $\alpha : A \times A \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in E(G) \\ 0 & \text{if not.} \end{cases}$$

All hypotheses of Theorem 2.2 (resp. Theorem 2.5) are satisfied. This completes the proof. ∎

Corollary 3.9. *Let A and B be nonempty closed subsets of a complete metric space* (X, d) *endowed with a graph G such that* $A_0 \neq \emptyset$. *Let* $T : A \to C_b(B)$ *be a given multi-valued* *mapping. Suppose there exists* $\tau : (0, \infty) \to (0, \infty)$ *for which* $\liminf_{t \to s^+} \tau(t) > 0$, *for all* $s \geq 0$, *such that*

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)), \tag{3.4}$$

for all $x, y \in A$ *satisfying* $H(Tx, Ty) > 0$ *and* $(x, y) \in E(G)$. *Also, suppose that*

(*i*) *for each* $x \in A_0$, *we have* $Tx \subseteq B_0$ *and* (A, B) *satisfies the weak* P*-property;*

(*ii*) *there exist elements* $x_0, x_1 \in A_0$ *and* $y_1 \in Tx_0$ *such that*

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad (x_0, x_1) \in E(G);$$

(*iii*) T *is* G*-proximal;*

(*iv*) (F_4) *holds;*

(*v*) T *is continuous or* (H_G) *holds.*

Then T *has a best proximity point.*

4. Conclusion

We recall that we managed in this paper to propose some new best proximity points for multi-valued $\alpha - F$-proximal contractive mappings. This was achieved by introducing the notion α-proximal admissibility which is an extension and a generalization for the case of multi-valued contractive mappings. As applications, we have obtained many known best proximity results on metric spaces endowed with a partial order and many best proximity results on a metric with a graph. In this paper the hypothesis (A, B) satisfies the weak P-property is very important. Also If we take $C_b(B)$ instead of $K(B)$ in Theorem 2.2, the existence of best proximity point for T is not guaranteed. In fact by adding the condition (F_4) on F we have obtained a partial answer for this problem. We wonder is there the same result under other conditions.

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References

[1] F. Giannessi, A. Maugeri, P.M. Pardalos, Equilibrium problems: nonsmooth optimization and variational inequality models. Nonconvex Optim App 58 (2004).

[2] P.M. Pardalos, T.M. Rassias, A.A. Khan, Equilibrium problems: nonlinear analysis and variational inequality problems in honor of George Isac. Springer Optim Appl. 35 (2010).

[3] S.G. Matthews, Partial Metric Topology, Research Report 212, Department of Computer Science, University of Warwick, 1992.

[4] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30 (1969) 282–291.

[5] M.U. Ali, T. Kamran, N. Shahzad, Best proximity point for $\alpha - \psi$-proximal contractive multimaps, Abstract and Applied Analysis 2014 (2014) Article ID 181598.

[6] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. 2012 (2012) Article no. 94.

[7] I. Altun, G. Minak, H. Dag, Multivalued F-contractions on complete metric space, J. Nonlinear and Convex Anal. 16 (4) 659–666.

[8] M. Olgun, G. Minak, I. Altun, A new approach to Mizoguchi-Takahashi type fixed point theorem, J. Nonlinear and Convex Anal. 17 (3) 579–587.

[9] A. Abkar, M. Gabeleh, The existence of best proximity points for multivalued non-self-mappings, RACSAM 107 (2013) 319–325.

[10] M. Jleli, E. Karapinar, B. Samet, On best proximity points under the P-property on partially ordered metric spaces, Abstract and Applied Analysis 2013 (2013) Article ID 150970.

[11] S. Karpagam, S. Agrawal, Best proximity points theorems for cyclic Meir-Keeler contraction maps, Nonlinear Anal. 74 (2011) 1040–1046.

[12] A. Latif, M. Hezarjaribi, P. Salimi, N. Hussain, Best proximity point theorems for $\alpha-\psi$-proximal contractions in intuitionistic fuzzy metric spaces, J. Ineq. Appl. 2014 (2014) Article no. 352.

[13] V.S. Raj, A best proximity theorems for weakly contractive non-self mappings, Nonlinear Anal. 74 (2011) 4804–4808.

[14] J. Zhang, Y. Su, Q. Cheng, A note on "best proximity point theorem for Geraghty-contractions", Fixed Point Theory Appl. 2013 (2013) Article no. 99.