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In memoriam Professor Charles E. Chidume (1947–2021)

A New Iterative Method for a Finite Family of the Split Generalized Equilibrium Problem and Fixed Point Problem

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Abstract In this paper, we introduce a new iterative method for finding the common element of the set of solutions of a finite family of split generalized equilibrium problems, finite variational inequality problems, and the set of common fixed points of a countable family of a nonexpansive mapping in Hilbert spaces. Under appropriate conditions imposed on the parameters, strong convergence theorems are obtained. An example is given to demonstrate the main result of this paper.

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Keywords: split generalized equilibrium problem; fixed point problem; firmly nonexpansive mapping

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1. Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two infinite dimensional real Hilbert spaces with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C and Q be a nonempty closed convex subset of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Definition 1.1. An element $x \in C$ is said to be a *fixed point* of a mapping $S : C \to C$ if $Sx = x$.

We denote the set of solutions of fixed point problem by $\mathrm{Fix}(S)$, that is, $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$.

Definition 1.2. A mapping $S : C \to C$ is said to be

(i) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C;$$

(ii) *firmly nonexpansive* if

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It is well known that every nonexpansive operator $S : \mathcal{H}_1 \to \mathcal{H}_1$ satisfies the following inequality;

$\langle (I-S)x-(I-S)y, Sy-Sx\rangle \leq \frac{1}{2}\|(I-S)y-(I-S)x\|^2, \forall x,y \in \mathcal{H}_1$ where I is a Identity operator. Therefore, for all $x \in \mathcal{H}_1$ and $y \in \mathrm{Fix}(S)$, we have

$$\langle (I-S)x-y-Sy, y-Tx\rangle \leq \frac{1}{2}\|(S-I)x\|^2. \tag{1.1}$$

We also know that Fix(S) of nonexpansive mapping S is closed and convex. The fixed point problem for the mapping S is to find $x \in C$ such that $Sx = x$. Many iterative algorithms have been introduced for finding fixed points of nonexpansive mappings.

Definition 1.3. Let $B : C \to \mathcal{H}_1$ be a nonlinear mapping. B is said to be

(i) *monotone*, if

$$\langle Bx - By, x-y\rangle \geq 0,\ \forall x,y \in C.$$

(ii) *strongly monotone*, if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x-y\rangle \geq \beta\|x-y\|^2,\ \forall x,y \in C.$$

In such a case, B is said to be β-*strongly monotone.*

(iii) *inverse strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x-y\rangle \geq \beta\|Bx-By\|^2,\ \forall x,y \in C.$$

In such a case, B is said to be β-*inverse strongly monotone* (for short, $\beta-$ism).

Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Bx, y-x\rangle \geq 0, \forall y \in C. \tag{1.2}$$

We denote the set of solutions to the problem (1.2) by VI(C, B). One can easily see that the variational inequality problem is equivalent to a fixed point problem. It is well known that if B is strongly monotone and Lipschitz continuous mapping on C, then 1.2 has a unique solution. There are several different approaches to solving this problem in finite dimensional and infinite dimensional spaces, see, for example, [1–3] and the research in this direction is intensively continued.

On the other hand, an equilibrium problem for a bifunction $g : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$g(x,y) \geq 0, \forall y \in C. \tag{1.3}$$

The set of solutions of 1.3 is denoted by EP(g), that is,

$$\mathrm{EP}(g) = \{x \in C : g(x,y) \geq 0,\ \forall y \in C\}.$$

It is easy to see that $\mathrm{EP}(g) = \mathrm{VI}(C,B)$ when $g(x,y) = \langle Bx, y-x\rangle \geq 0$, for all $x,y \in C$. Let $h : C \times C \to \mathbb{R}$ be a nonlinear bifunction, then the generalized equilibrium problem (for short, GEP) is to find $x^* \in C$ such that

$$g(x^*,x) + h(x^*,x) \geq 0, \forall y \in C. \tag{1.4}$$

We denote the solution set of generalized equilibrium problem 1.4 by GEP(g,h). Note that this problem reduces to the equilibrium problem when the bifunction h is a zero mapping; this problem reduces to the mixed equilibrium problem when the bifunction $h(x^*,x) = \varphi(x) - \varphi(x^*)$, where $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions. Next, let Q be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}_2$, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear and bounded operator. Kazmi and Rizvi [1] proposed the split generalized equilibrium problem (SGEP, for short): SGEP is to find $x^* \in C$ such that

$$g(x^*, x) + h(x^*, x) \geq 0, \forall x \in C, \tag{1.5}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } G(y^*, y) + H(y^*, y) \geq 0, \forall y \in Q, \tag{1.6}$$

where $g, h : C \times C \to \mathbb{R}$ and $G, H : Q \times Q \to \mathbb{R}$ are four nonlinear bifunctions. We denote the solution set of SGEP (1.5) and (1.6) by $\mathrm{GEP}(C, g, h)$ and $\mathrm{GEP}(Q, G, H)$, respectively. The solution set of SGEP is denoted by

$$\Gamma = \{z \in C : z \in \mathrm{GEP}(C, g, h) \text{ such that } Az \in \mathrm{GEP}(Q, g, H)\}.$$

Notice that (i) If $H = 0$ and $G = 0$, then the split generalized equilibrium problem reduces to the generalized equilibrium problem considered by Cianciaruso et al. [2].

(ii) If $h = 0$ and $H = 0$, then the split generalized equilibrium problem reduces to the split equilibrium problem introduced in 2011 by Moudafi [3].

(iii) If $h = \varphi(\cdot, \cdot)$ and $H = \phi(\cdot, \cdot)$, where $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions, then the split generalized equilibrium problem reduces to the split mixed equilibrium problem.

In this paper, we are interested in finding the common solution for a finite family of the split generalized equilibrium problems, that is, find a $x^* \in C$, such that

$$g_i(x^*, x) + h_i(x^*, x) \geq 0, \forall x \in C, \tag{1.7}$$

and such that

$$y^* = A_i x^* \in Q \text{ solves } G_i(y^*, y) + H_i(y^*, y) \geq 0, \forall y \in Q, \tag{1.8}$$

where $g_i, h_i : C \times C \to \mathbb{R}$ and $G_i, H_i : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A_i : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, for $1 \leq i \leq N_1$.

In 2016 Wang et al. [4] proposed iterative algorithm for a family of split equilibrium problems and fixed point problems in Hilbert spaces with applications and in 2019, Qingging Cheng [5] proposed a new parallel hybrid viscosity method for fixed point problem, variational inequality problems and split generalized equilibrium problems in Hilbert spaces.

Motivated by the work of Wang et al. [4], Qingging Cheng [5] and through the ongoing research in this direction, we propose a new iterative method for finding a common element of the set of solutions of a finite family of split generalized equilibrium problems, finite variational inequality problems and the set of common fixed points of a countable family of a nonexpansive mapping in Hilbert spaces. Moreover, strong convergence of the iterative method is obtained in the framework of Hilbert space.

2. Preliminaries

Throughout the paper, we denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively.

Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Then for each point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C.$$

P_C is called the (nearest point or metric) projection of $\mathcal{H}$ onto C.

It is well known that P_C is a firmly nonexpansive mapping of $\mathcal{H}$ onto C and satisfies

$$\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle, \forall x, y \in \mathcal{H}. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \forall x \in \mathcal{H}, y \in C. \tag{2.2}$$

Let $S : C \to C$ be a mapping. It is well known that S is nonexpansive if and only if the complement $I - S$ is $\frac{1}{2}$-inverse strongly monotone. Assume that $\mathrm{Fix}(S) \ne \emptyset$. Then we have

$$\|Sx - x\|^2 \le 2\langle x - Sx, x - p\rangle \tag{2.3}$$

for all $x \in C$ and $p \in \mathrm{Fix}(S)$.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. ([4]) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to \mathcal{H}$ be a nonexpansive mapping with $\mathrm{Fix}(S) \ne \emptyset$. Then $\mathrm{Fix}(P_C S) = \mathrm{Fix}(S) = \mathrm{Fix}(SP_C)$.

Lemma 2.2. ([4]) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and let $\{B_j\}_{j=1}^N$ be a finite family of inverse strongly monotone mappings from C to H with the constants $\{\beta_j\}_{j=1}^N$and assume that $\cap_{j=1}^N VI(C, B_j) \ne \emptyset$. Let $B = \sum_{j=1}^N \gamma_j B_j$ with $\{\gamma_j\}_{j=1}^N \subset (0, 1)$ and $\sum_{i=1}^N \gamma_j = 1$. Then $B : C \to H$ is a β-inverse strongly monotone mapping with $\beta = \min\{\beta_1, ..., \beta_N\}$ and $VI(C, B) = \cap_{j=1}^N VI(C, B_j)$.

Lemma 2.3. ([6]) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\alpha > 0$ and let $A : C \to \mathcal{H}$ be $\alpha-$inverse strongly monotone. If $0 < \lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into $\mathcal{H}$.

Lemma 2.4. ([6]) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and let A be a mapping of C into $\mathcal{H}$. Let $u \in C$. Then for $\lambda > 0$,

$$u \in VI(C, A) \Longleftrightarrow u = P_C(I - \lambda A)u.$$

Lemma 2.5. ([7]) Assume A is a strongly positive linear bounded operator on a Hilbert space $\mathcal{H}$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.

Lemma 2.6. ([7]) Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and let $f : \mathcal{H} \to \mathcal{H}$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \ge (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \ \ x, y \in \mathcal{H}.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.7. [8] Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $g(x, x) \ge 0$ for all $x \in C$;

(ii) g is monotone, that is, $g(x, y) + g(y, x) \le 0$ for all $x, y \in C$;

(iii) g is upper hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t\to 0} g(tz + (1 - t)x, y) \le g(x, y);$$

(iv) for each $x \in C$ fixed, the funtion $y \mapsto g(x, y)$ is convex and lower semicontinuous.

Suppose that $h : C \times C \to \mathbb{R}$ is a bifunction satisfying the following assumptions:

(i) $h(x,x) \geq 0$, for all $x \in C$;
(ii) for each $y \in C$ fixed, the function $x \longmapsto h(x,y)$ is upper semicontinuous,;
(iii) for each $x \in C$ fixed, the function $y \longmapsto h(x,y)$ is convex and lower semicontinuous.
Then, for fixed $r > 0$ and $z \in C$,there exists a nonempty compact convex subset K of $\mathcal{H}_1$ and $x \in C \cap K$ such that

$$g(x,y) + h(y,x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \forall y \in C \backslash K.$$

Lemma 2.8. [8] Assume that $g, h : C \times C \to \mathbb{R}$ satisfying Lemma 2.7. Let $r > 0$ and $u \in \mathcal{H}_1$, then there exists $w \in C$ such that

$$g(w,v) + h(w,v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \forall v \in C.$$

Lemma 2.9. [8] Assume that the bifunctions $g, h : C \times C \to \mathbb{R}$ satisfying Lemma 2.7 and h is monotone. For r and $x \in \mathcal{H}_1$, define the mapping $T_r^{(g,h)} : \mathcal{H}_1 \to C$ as follows:

$$T_r^{(g,h)}(x) = \Big\{z \in C : g(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\Big\}.$$

Then, the following hold:
(i) $T_r^{(g,h)}(x) \neq \emptyset$.
(ii) $T_r^{(g,h)}$ is single-valued.
(iii) $T_r^{(g,h)}$ is firmly nonexpansive, i.e., for any $x, y \in \mathcal{H}_1$,

$$\|T_r^{(g,h)}x - T_r^{(g,h)}y\|^2 \leq \langle T_r^{(g,h)}x - T_r^{(g,h)}y, x - y\rangle.$$

(iv) $\mathrm{Fix}(T_r^{(g,h)}) = \mathrm{GEP}(C, g, h)$.
(v) $\mathrm{GEP}(C, g, h)$ is compact and convex.

Let $G, H : Q \times Q \to \mathbb{R}$ satisfying Lemma 2.7. From the previous lemma, we can define a mapping $T_s^{G,H} : H_2 \to Q$ as follows:

$$T_s^{G,H}(w) := \left\{d \in Q : F(d,e) + H(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \forall e \in Q\right\},$$

where $s > 0$ and $w \in \mathcal{H}_2$. Then $T_s^{G,H} : \mathcal{H}_2 \to Q$ also satisfies the same properties in Lemma 2.9.

Lemma 2.10. ([9]) Let $g, h : C \times C \to \mathbb{R}$ satisfying Lemma *2.7* and h is monotone. Let $T_r^{(g,h)}$ and $T_s^{(g,h)}$ be defined as in Lemma *2.9* with $r, s > 0$. Then, for any $x, y \in \mathcal{H}_1$, one has

$$\|T_r^{(g,h)}x - T_s^{(g,h)}y\| \leq |x - y| + |1 - \frac{s}{r}|\|T_r^{(g,h)}x - x\|.$$

Lemma 2.11. ([9]) Let $g, h : C \times C \to \mathbb{R}$ satisfying Lemma *2.7* and h is monotone. Let $T_r^{(g,h)}$ and $T_s^{(g,h)}$ be defined as in Lemma *2.9* with $r, s > 0$. Then the following holds:

$$\left\|T_r^{(g,h)}x - T_s^{(g,h)}x\right\|^2 \leq \frac{r - s}{r}\langle T_r^{(g,h)}x - T_r^{(g,h)}x, T_r^{(g,h)}x - x\rangle,$$

for all $x \in \mathcal{H}_1$.

Lemma 2.12. (Demiclosedness principle) Let S be a nonexpansive mapping on a closed convex subset C of a real Hilbert space $\mathcal{H}$. Then $I - S$ is demiclosed at any point $y \in \mathcal{H}$, that is, if $x_n \rightharpoonup x$ and $x_n - Sx_n \to y \in \mathcal{H}$, that $x - Sx = y$; inparticular, if $y = 0$, then $x \in \mathrm{Fix}(S)$.

Lemma 2.13. ([10]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\sigma_n)a_n + b_n,$$

for each $n \geq 0$, where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(2) $\limsup_{n\to\infty} \dfrac{b_n}{\sigma_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Now, we give the main results of this paper.

Theorem 3.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be two real Hilbert spaces and $C \subset \mathcal{H}_1, Q \subset \mathcal{H}_2$ be nonempty closed convex subsets. For each $i = 1, \ldots, N_1$ with $N_1 \in \mathbb{N}$, let $A_i : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $A_i^* : \mathcal{H}_2 \to \mathcal{H}_1$ be the adjoint of A_i. Assume that $g_i, h_i : C \times C \to \mathbb{R}$ and $G_i, H_i : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Lemma 2.7; h_i, H_i is monotone and G_i is upper semicontinuous for $1 \leq i \leq N_1$ with $N_1 \in \mathbb{N}$ and $B_j : C \to \mathcal{H}_1$ be a β_j-inverse strongly monotone operator for each $j = 1, \ldots, N_2$ with $N_2 \in \mathbb{N}$. Let $\{S_k\}$ be a countable family of nonexpansive mappings from C into C. Assume that $\Omega = \Theta \cap \Gamma \cap \Lambda \neq \emptyset$, where $\Theta = \bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k), \Gamma = \{z \in C : z \in \mathrm{GEP}(C, g_i, h_i)$ such that $A_i z \in \mathrm{GEP}(Q, G_i, H_i), i = 1, ..., N_1\}$ and $A_i z \in EP(G_i)\}$ and $\Lambda = \bigcap_{j=1}^{N_2} VI(C, B_j)$. Let $\{\gamma_1, ..., \gamma_{N_2}\} \subset (0,1)$ with $\sum_{j=1}^{N_2} \gamma_j = 1$. Let $\{x_n\}$ be a sequence generated from an arbitrary $\nu, x_1 \in C$ by the following algorithm:

$$\begin{cases} u_{n,i} = T_{r_{n,i}}^{g_i,h_i}\big(I - \gamma A_i^*\big(I - T_{r_{n,i}}^{G_i,H_i}\big)A_i\big)x_n, \\ u_n = u_{n,i_n}, i_n = \arg\max_{1\leq i\leq N_1}\{\|u_{n,i} - x_n\|\}, \\ y_n = P_C(I - \lambda_n(\sum_{j=1}^{N_2} \gamma_j B_j))u_n \\ x_{n+1} = \alpha_n \nu + \sum_{k=1}^{n}(\alpha_{k-1} - \alpha_k)S_k y_n \end{cases} \tag{3.1}$$

for each $i = 1, ..., N_1$ and $n \in \mathbb{N}$ where $\{r_{n,i}\} \subset (0,\infty), \gamma \in \big(0, \frac{1}{L^2}\big], L = \max\{L_1, ..., L_{N_1}\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, ..., N_1\}, \{\lambda_n\} \subset (0, 2\beta)$ with $\beta = \min\{\beta_1, ..., \beta_{N_2}\}$ and $\{\alpha_n\} \subset (0,1)$ is a strictly decreasing sequence. Let $\alpha_0 = 1$ and assume that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\beta$;

(C4) $\liminf_{n\to\infty} r_{n,i} > 0$.

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $z = P_\Omega \nu$.

Proof. First we show that, for each $i \in \{1, 2, \ldots, N_1\}$ and $n \in \mathbb{N}$, $A_i^*\big(I - T_{r_{n,i}}^{G_i,H_i}\big)A_i$ is a $\dfrac{1}{2L_i^2}$ -inverse strongly monotone mapping. In fact, since $T_{r_{n,i}}^{G_i,H_i}$ is firmly nonexpansive and $I - T_{r_{n,i}}^{G_i,H_i}$ is $\dfrac{1}{2}$-inverse strongly monotone, we have

$$
\begin{aligned}
&\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i y\|^2 \\
&\quad = \langle A_i^*(I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y), A_i^*(I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y)\rangle \\
&\quad = \langle (I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y), A_i A_i^*(I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y)\rangle \\
&\quad \leq L_i^2 \langle (I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y), (I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y)\rangle \\
&\quad = L_i^2 \|(I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y)\|^2 \\
&\quad \leq 2L_i^2 \langle A_i x - A_i y, (I - T_{r_{n,i}}^{G_i,H_i})(A_i x - A_i y)\rangle \\
&\quad = 2L_i^2 \langle x - y, A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i y\rangle,
\end{aligned}
$$

for all $x, y \in H_1$, which implies that $A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i$ is a $\frac{1}{2L_i^2}$ - inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L_i^2}]$. Then $I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i$ is nonexpansive for each $i = 1, ..., N_1$ and $n \in \mathbb{N}$. We devide the proof into five steps as follows.

Step 1. We first show that the sequences $\{x_n\}$ is bounded. Let $p \in \Omega$. Then for each $i \in \{1, 2, \ldots, N_1\}$, we have $p = T_{r_{n,i}}^{g_i,h_i} p$ and $(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)p = p$. Therefore we have

$$
\begin{aligned}
\|u_{n,i} - p\| &= \left\| T_{r_{n,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n - T_{r_{n,i}}^{G_i,H_i}(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)p \right\| \\
&\leq \left\| (I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n - (I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)p \right\| \\
&\leq \|x_n - p\|, \qquad (3.2)
\end{aligned}
$$

for each $i \in \{1, 2, \ldots, N_1\}$.
From (3.1) and (3.2), we obtain

$$
\|u_n - p\| = \|u_{n,i_n} - p\| \leq \|x_n - p\|. \qquad (3.3)
$$

Let $B = \sum_{j=1}^{N_2} \gamma_j B_j$, by Lemma 2.2, we know that B is $\beta-$ism, and from the condition $0 < \lambda_n < 2\beta$, we see that $I - \lambda_n B$ is nonexpansive, and $P_C(I - \lambda_n B)$ is also nonexpansive. We have $p \in \Omega$, that is, $p \in \bigcap_{j=1}^{N_2} VI(C, B_j)$. Then from 3.3 we have

$$
\begin{aligned}
\|y_n - p\| &= \|P_C(I - \lambda_n B)u_n - p\| \\
&= \|P_C(I - \lambda_n B)u_n - P_C(I - \lambda_n B)p\| \\
&\leq \|u_n - p\| \\
&\leq \|x_n - p\|. \qquad (3.4)
\end{aligned}
$$

It follows from 3.4 that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \left\| \alpha_n \nu + \sum_{k=1}^{n} (\alpha_{k-1} - \alpha_i) S_k y_n - p \right\| \\
&= \|\alpha_n \nu - \alpha_n p + \sum_{k=1}^{n} (\alpha_{k-1} - \alpha_k)(S_k y_n - S_k p)\| \\
&\leq \alpha_n \|\nu - p\| + \sum_{k=1}^{n} (\alpha_{k-1} - \alpha_k)\|y_n - p\| \\
&\leq \alpha_n \|\nu - p\| + \sum_{k=1}^{n} (\alpha_{k-1} - \alpha_k)\|x_n - p\| \\
&= \alpha_n \|\nu - p\| + (1 - \alpha_n)\|x_n - p\|
\end{aligned}
$$

$$\leq \max\{\|\nu - p\|, \|x_n - p\|\},$$

for all $n \in \mathbb{N}$, which implies that $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded. So from 3.3 and 3.4, we get $\{u_{n,i}\}$ and $\{y_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|u_{i,n+1} - u_{i,n}\| = 0$ for each $i = 1, \dots, N_1$. Since the mapping $I - \gamma A^*(I - T_{r_{n,i}}^{G_i,H_i})A$ is nonexpensive. Then for each $i = 1, \dots, N_1$ by Lemmas 2.10 and 2.11, we have

$$\begin{aligned}
&\|u_{i,n+1} - u_{i,n}\| \\
= &\left\|T_{r_{n+1,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1} - T_{r_{n,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n\right\| \\
\leq &\left\|(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1} - (I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n\right\| \\
&+\frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\left\|T_{r_{n+1,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1} \right.\\
&\left. -(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1}\right\| \\
\leq &\|x_{n+1} - x_n\| + \left\|(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_n - (I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n\right\| \\
&+\frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\delta_{n+1,i} \\
= &\|x_{n+1} - x_n\| + \left\|\gamma A_i^*(T_{r_{n+1,i}}^{G_i,H_i}A_ix_n - T_{r_{n,i}}^{G_i,H_i}A_ix_n)\right\| + \frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\delta_{n+1,i} \\
= &\|x_{n+1} - x_n\| + \|\gamma A_i^*\|\left\|T_{r_{n+1,i}}^{G_i,H_i}A_ix_n - T_{r_{n,i}}^{G_i,H_i}A_ix_n\right\| + \frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\delta_{n+1,i} \\
\leq &\|x_{n+1} - x_n\| + \gamma\left\|A_i^*\right\|\left[\frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\left|\langle T_{r_{n+1,i}}^{G_i,H_i}A_ix_n\right.\right. \\
&\left.\left. -T_{r_{n,i}}^{G_i,H_i}A_ix_n, T_{r_{n+1,i}}^{G_i,H_i}A_ix_n - A_ix_n\rangle\right|\right]^{\frac{1}{2}} + \frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\delta_{n+1,i} \\
\leq &\|x_{n+1} - x_n\| + \gamma\left\|A_i^*\right\|\left[\frac{|r_{n+1,i} - r_{n,i}|}{r}\sigma_{n+1}\right]^{\frac{1}{2}} + \frac{|r_{n+1,i} - r_{n,i}|}{r_{n+1,i}}\delta_{n+1,i} \\
\leq &\|x_{n+1} - x_n\| + \eta_{n+1,i},
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
\sigma_{n+1,i} &= \sup_{n\in\mathbb{N}}\left|\langle T_{r_{n+1,i}}^{G_i,H_i}A_ix_n - T_{r_{n,i}}^{G_i,H_i}A_ix_n, T_{r_{n+1,i}}^{G_i,H_i}A_ix_n - A_ix_n\rangle\right|, \\
\delta_{n+1,i} &= \sup_{n\in\mathbb{N}}\left\|T_{r_{n+1,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1} - (I - \gamma A_i^*(I - T_{r_{n+1,i}}^{G_i,H_i})A_i)x_{n+1}\right\|
\end{aligned}$$

and

$$\eta_{n+1,i} = \gamma\left\|A_i^*\right\|\left[\tfrac{|r_{n+1,i} - r_{n,i}|}{r}\sigma_{n+1,i}\right]^{\frac{1}{2}} + \tfrac{|r_{n+1,i} - r_{n,i}|}{r}\delta_{n+1,i}.$$

Note that for each $i = 1, \dots, N_1$, we obtain

$$\begin{aligned}
&\big\|(I-\lambda_{n+1}B)u_{n+1,i}-(I-\lambda_nB)u_{n,i}\big\| \\
&\qquad = \big\|(I-\lambda_{n+1}B)u_{n+1,i}-(I-\lambda_{n+1}B)u_{n,i}+(\lambda_n-\lambda_{n+1})Bu_{n,i}\big\| \\
&\qquad \le \big\|(I-\lambda_{n+1}B)u_{n+1,i}-(I-\lambda_{n+1}B)u_{i,n}\big\|+\big\|(\lambda_n-\lambda_{n+1})Bu_{n,i}\big\| \\
&\qquad = \big\|(I-\lambda_{n+1}B)u_{n+1,i}-(I-\lambda_{n+1}B)u_{i,n}\big\|+|\lambda_n-\lambda_{n+1}|\big\|Bu_{n,i}\big\| \\
&\qquad \le \|u_{n+1,i}-u_{n,i}\|+|\lambda_n-\lambda_{n+1}|\big\|Bu_{n,i}\big\|. \qquad (3.6)
\end{aligned}$$

Now for each $i=1,\dots,N_1$, let $M_i=\sup_{n\in\mathbb{N}}\|Bu_{n,i}\|$ by (3.1), (3.5) and (3.6), we have

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \big\|P_C(I-\lambda_{n+1}B)u_{n+1,i}-P_C(I-\lambda_nB)u_{n,i}\big\| \\
&\le \big\|(I-\lambda_{n+1}B)u_{n+1,i}-(I-\lambda_nB)u_{n,i}\big\| \\
&\le \|u_{n+1,i}-u_{n,i}\|+|\lambda_n-\lambda_{n+1}|\big\|Bu_{n,i}\big\| \\
&= \|x_{n+1}-x_n\|+\eta_{i,n+1}+|\lambda_n-\lambda_{n+1}|\|Bw_n\| \\
&\le \|x_{n+1}-x_n\|+\eta_{i,n+1}+|\lambda_n-\lambda_{n+1}|M_1. \qquad (3.7)
\end{aligned}$$

Since $\{\alpha_n\}$ strictly decreasing, by using (3.7), we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \Bigg\|\Big(\alpha_n\nu+\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)S_ky_n+(\alpha_{n-1}-\alpha_n)S_ny_n\Big) \\
&\qquad -\Big(\alpha_{n-1}\nu+\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)S_ky_{n-1}\Big)\Bigg\| \\
&= \Bigg\|\alpha_n\nu-\alpha_{n-1}\nu+\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)S_ky_n\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)S_ky_{n-1}+(\alpha_{n-1}-\alpha_n)S_ny_n\Bigg\| \\
&= \Bigg\|(\alpha_n-\alpha_{n-1})\nu+\sum_{i=k}^{n-1}(\alpha_{k-1}-\alpha_k)(S_ky_n-S_ky_{n-1})+(\alpha_{n-1}-\alpha_n)S_ny_n\Bigg\| \\
&\le \|(\alpha_n-\alpha_{n-1})\nu\|+\Bigg\|\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)(S_ky_n-S_ky_{n-1})\Bigg\|+\|(\alpha_{n-1}-\alpha_n)S_ny_n\| \\
&\le (\alpha_{n-1}-\alpha_n)\|\nu\|+\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)\|S_ky_n-S_ky_{n-1}\|+(\alpha_{n-1}-\alpha_n)\|S_ny_n\| \\
&\le (\alpha_{n-1}-\alpha_n)\|\nu\|+\sum_{k=1}^{n-1}(\alpha_{k-1}-\alpha_k)\|y_n-y_{n-1}\|+(\alpha_{n-1}-\alpha_n)\|S_ny_n\| \\
&= (\alpha_{n-1}-\alpha_n)\|\nu\|+(1-\alpha_{n-1})\|y_n-y_{n-1}\|+(\alpha_{n-1}-\alpha_n)\|S_ny_n\|
\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_{n-1})\big(\|x_n-x_{n-1}\|+\eta_{i,n}+|\lambda_{n-1}-\lambda_n|M_1\big) \\
&\quad +(\alpha_{n-1}-\alpha_n)(\|S_ny_n\|+\|\nu\|) \\
&\le (1-\alpha_{n-1})\big(\|x_n-x_{n-1}\|+\eta_{i,n}+|\lambda_{n-1}-\lambda_n|M_1\big)+(\alpha_{n-1}-\alpha_n)M_2 \\
&\le (1-\alpha_{n-1})\|x_n-x_{n-1}\|+\eta_{i,n}+|\lambda_{n-1}-\lambda_n|M_1+(\alpha_{n-1}-\alpha_n)M_2,
\end{aligned}$$

where $M_2 = \sup\{\|S_n y_n\| + \|\nu\| : n \in \mathbb{N}\}$ by (C1) and (C2) in Lemma 2.13 we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Moreover, by (3.5) and (3.7), we have

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0, \quad \lim_{n\to\infty} \|u_{i,n+1} - u_{i,n}\| = 0, \quad i \in \{1, ..., N_1\}. \tag{3.9}$$

Step 3. We show that $\lim_{n\to\infty} \|S_k x_n - x_n\| \to 0$ for each $k \in \mathbb{N}$.
First we will show that $\lim_{n\to\infty} \|u_{i,n} - x_n\| = 0$ for each $i \in \{1, ..., N_1\}$
Since for each $A_i^*(I - T_{r_n}^{G_i,H_i})A_i$ is $\frac{1}{2L_i^2}$-inverse strongly monotone,
by (3.1), we have

$$\begin{aligned}
\|u_{n,i} - p\|^2 &= \Big\|T_{r_{n,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n - T_{r_{n,i}}^{g_i,h_i}(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)p\Big\|^2 \\
&\leq \Big\|(I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)x_n - (I - \gamma A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i)p\Big\|^2 \\
&= \Big\|(x_n - p) - \gamma(A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p)\Big\|^2 \\
&= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p\rangle \\
&\quad + \gamma^2\Big\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p\Big\|^2 \\
&\leq \|x_n - p\|^2 - \frac{\gamma}{L_i^2}\Big\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p\Big\|^2 \\
&\quad + \gamma^2\Big\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p\Big\|^2 \\
&= \|x_n - p\|^2 + \gamma\Big(\gamma - \frac{1}{L_i^2}\Big)\Big\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n - A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i p\Big\|^2 \\
&= \|x_n - p\|^2 + \gamma\Big(\gamma - \frac{1}{L_i^2}\Big)\Big\|A_i^*(I - T_{r_{n,i}}^{G_i,H_i})A_i x_n\Big\|^2.
\end{aligned} \tag{3.10}$$

Now, from (3.1) and (3.10) it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \Big\|\alpha_n(\nu - p) + \sum_{k-1}^{n}(\alpha_{k-1} - \alpha_k)(S_k y_n - p)\Big\|^2 \\
&\leq \Big\|\alpha_n(\nu - p)\Big\|^2 + \Big\|\sum_{k-1}^{n}(\alpha_{k-1} - \alpha_i)(S_k y_n - p)\Big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|\nu - p\|^2 + \sum_{k-1}^{n}(\alpha_{k-1} - \alpha_i)\|(S_k y_n - p)\|^2 \\
&\le \alpha_n\|\nu - p\|^2 + \sum_{k-1}^{n}(\alpha_{k-1} - \alpha_k)\|y_n - p\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|u_{n,i_n} - p\|^2 \\
&\le \alpha_n\|\nu - p\|^2 \\
&\quad +(1-\alpha_n)\bigg[\|x_n - p\|^2 + \gamma\Big(\gamma - \frac{1}{L_{i_n}^2}\Big)\Big\|A_{i_n}^*(I - T_{r_{n,i_n}}^{G_{i_n},H_{i_n}})A_{i_n}x_n\Big\|^2\bigg],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\alpha_n)\gamma\Big(\gamma - \frac{1}{L_{i_n}^2}\Big)\Big\|A_{i_n}^*(I - T_{r_{n,i_n}}^{G_{i_n},H_{i_n}})A_{i_n}x_n\Big\|^2 \\
&\qquad \le \alpha_n\|\nu - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad \le \alpha_n\|\nu - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, as $n \to \infty$, we obtain

$$
\lim_{n\to\infty}\Big\|A_{i_n}^*(I - T_{r_{n,i_n}}^{G_{i_n},H_{i_n}})A_{i_n}x_n\Big\| = 0 \text{ for each } i \in \{1, ..., N_1\}. \tag{3.11}
$$

Therefore

$$
\lim_{n\to\infty}\Big\|(I - T_{r_{n,i_n}}^{G_{i_n},H_{i_n}})A_{i_n}x_n\Big\| = 0 \text{ for each } i \in \{1, ..., N_1\}. \tag{3.12}
$$

Since $T_{r_{n,i}}^{g_i,h_i}$ is firmly nonexpansive and $p = T_{r_{n,i}}^{g_i,h_i}p$, we have

$$
\begin{aligned}
\|u_{n,i} - p\|^2 &= \Big\|T_{r_{n+1,i}}^{g_i,h_i}\big(I - \gamma A_i^*\big(I - T_{r_{n+1,i}}^{G_i,H_i}\big)A_i\big)x_n - p\Big\|^2 \\
&= \Big\|T_{r_{n,i}}^{g_i,h_i}(x_n + \gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n) - T_{r_{n,i}}^{g_i,h_i}p\Big\|^2 \\
&\le \big\langle u_{n,i} - p, x_n + \gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n - p\big\rangle \\
&= \frac{1}{2}\Big\{\|u_{n,i} - p\|^2 + \|x_n + \gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n - p\|^2 \\
&\quad -\|u_{n,i} - x_n - \gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n\|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big\{\|u_{n,i}-p\|^2+\|x_n-p\|^2+\gamma^2\|A_i^*(T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\|^2 \\
&\quad +2\gamma\langle x_n-p, A_i^*(T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\rangle \\
&\quad -\big[\|u_{n,i}-x_n\|^2+\gamma^2\|A_i^*(T_{r_{n,i}}^{F_i,H_i}-I)A_ix_n\|^2 \\
&\quad -2\gamma\langle u_{n,i}-x_n, A_i^*(T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\rangle\big]\Big\} \\
&= \frac{1}{2}\Big\{\|u_{n,i}-p\|^2+\|x_n-p\|^2-\|u_{n,i}-x_n\|^2 \\
&\quad +2\gamma\langle A_iu_{n,i}-A_ip, (T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\rangle\Big\} \\
&\leq \frac{1}{2}\Big\{\|u_{n,i}-p\|^2+\|x_n-p\|^2-\|u_{n,i}-x_n\|^2 \\
&\quad +2\gamma\|A_iu_{n,i}-A_ip\|\|(T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\|\Big\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_{n,i}-p\|^2 \leq\ & \|x_n-p\|^2-\|u_{n,i}-x_n\|^2 \\
& +2\gamma\|A_iu_{n,i}-A_ip\|\big\|(T_{r_{n,i}}^{G_i,H_i}-I)A_ix_n\big\|.
\end{aligned} \tag{3.13}
$$

Now, from (3.1), (3.4) and (3.13), it follows that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|\nu-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\leq \alpha_n\|\nu-p\|^2+(1-\alpha_n)\|u_n-p\|^2 \\
&\leq \alpha_n\|\nu-p\|^2+(1-\alpha_n)\Big[\|x_n-p\|^2-\|u_n-x_n\|^2 \\
&\quad +2\gamma\|A_{i_n}u_n-A_{i_n}p\|\big\|(T_{r_{n,i_n}}^{G_{i_n}H_{i_n}}-I)A_{i_n}x_n\big\|\Big].
\end{aligned} \tag{3.14}
$$

Thus

$$
\begin{aligned}
(1-\alpha_n)\|u_n-x_n\|^2 &\leq \alpha_n\|\nu-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +(1-\alpha_n)2\gamma\|A_{i_n}u_n-A_{i_n}p\|\big\|(T_{r_{n,i_n}}^{G_{i_n}H_{i_n}}-I)A_{i_n}x_n\big\| \\
&\leq \alpha_n\|\nu-p\|^2+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\quad +(1-\alpha_n)2\gamma\|A_{i_n}u_n-A_{i_n}p\|\big\|(T_{r_{n,i_n}}^{G_{i_n}H_{i_n}}-I)A_{i_n}x_n\big\|.
\end{aligned}
$$

Since $\alpha_n\to 0, \|x_n-x_{n+1}\|\to 0$ and $\|(T_{r_{n,i_n}}^{G_{i_n},H_{i_n}}-I)A_{i_n}x_n\|\to 0$ as $n\to\infty$, we have

$$
\|u_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.15}
$$

Next, we show that
$\lim_{n\to\infty}\|y_n-u_n\|=0$ where $u_n=u_{n,i_n}, i_n=\arg\max_{1\leq i\leq N_1}\{\|u_{n,i}-x_n\|\}$.

Since $p = P_C(I - \lambda_n B)p$. By (3.1) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|y_n - P_C(I-\lambda_n B)p\|^2 \\
&\leq \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|P_C(I-\lambda_n B)u_n - P_C(I-\lambda_n B)p\|^2 \\
&\leq \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|(I-\lambda_n B)u_n - (I-\lambda_n B)p\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|u_n - \lambda_n Bu_n - p - \lambda_n Bp\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|u_n - p - \lambda_n(Bu_n - Bp)\|^2 \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n) \\
&\quad \Big(\|u_n - p\|^2 - 2\lambda_n\langle u_n - p, Bu_n - Bp\rangle + \lambda_n^2\|Bu_n - Bp\|^2\Big) \\
&\leq \alpha_n\|\nu - p\|^2 \\
&\quad +(1-\alpha_n)\Big(\|u_n - p\|^2 - 2\lambda_n\beta\|Bu_n - Bp\|^2 + \lambda_n^2\|Bu_n - Bp\|^2\Big) \\
&\leq \alpha_n\|\nu - p\|^2 \\
&\quad +(1-\alpha_n)\Big(\|x_n - p\|^2 - 2\lambda_n\beta\|B_{u_n} - B_p\|^2 + \lambda_n^2\|B_{u_n} - B_p\|^2\Big) \\
&= \alpha_n\|\nu - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad +(1-\alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bp\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\alpha_n)\lambda_n(2\beta - \lambda_n)\|Bu_n - Bp\|^2 \\
&\qquad \leq \alpha_n\|\nu - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

Since $\alpha_n \to 0$ and $0 < \lim_{n\to\infty}\lambda_n = \lambda < 2\beta$, by (3.8) we have

$$\lim_{n\to\infty}\|Bu_n - Bp\| = 0. \tag{3.16}$$

Since P_C is firmly nonexpansive and $(I - \lambda_n B)$ is nonexpansive, by (3.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Bu_n) - P_C(p - \lambda_n Bp)\|^2 \\
&\leq \langle y_n - p, u_n - \lambda_n Bu_n - (p - \lambda_n Bp)\rangle \\
&= \frac{1}{2}\Big(\|y_n - p\|^2 + \|(I-\lambda_n B)u_n - (I-\lambda_n B)p\|^2 \\
&\quad -\|y_n - u_n + \lambda_n(Bu_n - Bp)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n + \lambda_n(Bu_n - Bp)\|^2\Big) \\
&= \frac{1}{2}\Big(\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 - \lambda_n^2\|Bu_n - Bp\|^2 \\
&\quad -2\lambda_n\langle y_n - u_n, Bu_n - Bp\rangle\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 - \lambda_n^2\|Bu_n - Bp\|^2 \\
&\quad +2\lambda_n\|y_n - u_n\|\|Bu_n - Bp\|\Big),
\end{aligned}$$

which implies that

$$\begin{aligned} 2\|y_n-p\|^2 &\leq \|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Bu_n-Bp\|^2 \\ &\quad +2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\| \\ \|y_n-p\|^2 &\leq \|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Bu_n-Bp\|^2 \\ &\quad +2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\| \\ &\leq \|x_n-p\|^2-\|y_n-u_n\|^2 \\ &\quad +2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|. \end{aligned} \tag{3.17}$$

From (3.1) and (3.17), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \alpha_n\|\nu-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\ &\leq \alpha_n\|\nu-p\|^2+(1-\alpha_n) \\ &\quad \Big(\|x_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|\Big) \\ &\leq \alpha_n\|\nu-p\|^2+\|x_n-p\|^2-(1-\alpha_n)\|y_n-u_n\|^2 \\ &\quad +2(1-\alpha_n)\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} (1-\alpha_n)\|y_n-u_n\|^2 &\leq \alpha_n\|\nu-p\|^2+\|x_n-x_{n+1}\|\Big(\|x_{n+1}-p\|+\|x_n-p\|\Big) \\ &\quad +2(1-\alpha_n)\lambda_n\Big(\|y_n\|-\|u_n\|\Big)\|Bu_n-Bp\|. \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\{y_n\}$, $\{u_n\}$ are bounded, by (3.8) and (3.16), we have

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.18}$$

Moreover, it follows that

$$\|x_{n+1}-y_n\| \leq \|x_{n+1}-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|,$$

by (3.8), (3.15) and (3.18), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.19}$$

Next, from (3.1), we have

$$\sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)(S_ky_n-y_n)=x_{n+1}-y_n-\alpha_n(\nu-y_n). \tag{3.20}$$

Since $\{\alpha_n\}$ is strictly decreasing, for each $k\in\mathbb{N}$, by (2.3) and (3.20), we have

$$\begin{aligned} (\alpha_{k-1}-\alpha_k)\|S_ky_n-y_n\|^2 &\leq \sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)\|S_ky_n-y_n\|^2 \\ &\leq 2\sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)\langle S_ky_n-y_n,p-y_n\rangle \\ &= 2\langle x_{n+1}-y_n,y_n-p\rangle-2\alpha_n\langle v-y_n,p-y_n\rangle \\ &\leq 2\|x_{n+1}-y_n\|\|y_n-p\|+2\alpha_n\|v-y_n\|\|y_n-p\|. \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{y_n\}$ are bounded by (3.19), we have

$$\lim_{n\to\infty}\|S_k y_n - y_n\| = 0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.21}$$

Moreover, it follows that

$$\begin{aligned}
\|S_k x_n - x_n\| &\leq \|S_k x_n - S_k y_n\| + \|S_k y_n - y_n\| + \|y_n - x_n\| \\
&\leq \|x_n - y_n\| + \|S_k y_n - y_n\| + \|y_n - x_n\| \\
&= 2\|y_n - x_n\| + \|S_k y_n - y_n\| \\
&\leq 2(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) + \|S_k y_n - y_n\| \\
&= 2\|y_n - x_{n+1}\| + 2\|x_{n+1} - x_n\| + \|S_k y_n - y_n\|,
\end{aligned}$$

by (3.8), (3.19) and (3.21), we have

$$\lim_{n\to\infty}\|S_k x_n - x_n\| = 0, \forall k \in \mathbb{N}. \tag{3.22}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle \nu - z, x_n - z\rangle \leq 0$.

Let $z = P_\Omega \nu$. Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\sup\langle \nu - z, x_n - z\rangle = \lim_{j\to\infty}\langle \nu - z, x_{n_j} - z\rangle. \tag{3.23}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ converging weakly to a point $w \in C$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$.

Now, we will show that $w \in \Omega$. First of all, we show that $w \in \Theta = \bigcap_{k=1}^{\infty}\mathrm{Fix}(S_k)$. From the fact that $x_n - S_k x_n \to 0$ for each $k \in \mathbb{N}$ and $x_{n_j} \rightharpoonup w$, therefore by Lemma 2.12, we obtain $w \in \bigcap_{k=1}^{\infty}\mathrm{Fix}(S_k) = \Theta$.

Next, we show that $w \in \Gamma, i.e., w \in \mathrm{GEP}(g_i, h_i)$ and $A_i w \in \mathrm{GEP}(G_i, H_i)$ for all $i \in \{1, ..., N_1\}$.

From (3.1) and (3.15), we have

$$\|u_{n,i} - x_n\| \to 0 \text{ as } n \to \infty, 1 \leq i \leq N_1$$

and from (3.12), we obtain

$$\left\|\left(T_{r_{n,i}}^{G_i,H_i} - I\right)A_i x_n\right\| \to 0 \text{ as } n \to \infty, 1 \leq i \leq N_1.$$

Let $u_{n,i} = T_{r_{n,i}}^{g_i,h_i}\mu_{n,i}$ where $\mu_{n,i} = x_n + \gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n$, and we have

$$\begin{aligned}
\|\mu_{n,i} - x_n\| &= \left\|\gamma A_i^*(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n\right\| \\
&\leq \gamma\|A_i\|\left\|(T_{r_{n,i}}^{G_i,H_i} - I)A_i x_n\right\| \to 0, (n \to \infty).
\end{aligned}$$

Since $\|u_{n,i} - \mu_{n,i}\| \leq \|u_{n,i} - x_n\| + \|\mu_{n,i} - x_n\|$. Then we have

$$\|u_{n,i} - \mu_{n,i}\| \to 0 \text{ as } n \to \infty, 1 \leq i \leq N_1.$$

Since $u_{n,i} = T_{r_{n,i}}^{g_i,h_i}\mu_{n,i}$, we get

$g_i(u_{n,i}, u) + h_i(u_{n,i}, u) + \frac{1}{r_{n,i}}\langle u - u_{n,i}, u_{n,i} - \mu_{n,i}\rangle \geq 0, \forall u \in C,$

which implies that

$h_i(u_{n,i}, u) + \frac{1}{r_{n,i}}\langle u - u_{n,i}, u_{n,i} - \mu_{n,i}\rangle \geq -g_i(u_{n,i}, u) \geq f_i(u, u_{n,i}), \forall u \in C.$

Since $\|u_{n,i} - \mu_{n,i}\| \to 0, u_{n,i} \rightharpoonup q, g_i$ is lower semicontinuous in the second argument and

h_i is upper semicontinuous in the first argument, we obtain
$h_i(w,u) \geq g_i(u,w),\ \forall u \in C.$
Then we have
$g_i(u,w) + h_i(u,w) \leq g_i(u,w) - h_i(w,u) \leq 0, \forall u \in C.$
Let $\hat{u} = tu + (1-t)w \in C$, we have $\hat{u} \in C$ and $g_i(\hat{u},w) + h_i(\hat{u},w) \leq 0$. Notice that

$$\begin{aligned} 0 &= g_i(\hat{u},\hat{u}) + h_i(\hat{u},\hat{u}) \\ &= t\big[g_i(\hat{u},u) + h_i(\hat{u},u)\big] + (1-t)\big[g_i(\hat{u},w) + h_i(\hat{u},w)\big] \\ &\leq t\big[g_i(\hat{u},u) + h_i(\hat{u},u)\big]. \end{aligned}$$

Hence $g_i(\hat{u},u) + h_i(\hat{u},u) \geq 0, \forall u \in C.$
Since g_i is upper hemicontinuous and h_i is upper semicontinuous in the first argument, we have
$g_i(w,u) + h_i(w,u) \geq 0, \forall u \in C.$
That is, $w \in \mathrm{GEP}(g_i,h_i)$ for all $i \in \{1,\ldots,N_1\}$.

Next, we show that $A_i w \in \mathrm{GEP}(G_i,H_i)$. Since $x_{n_l} \rightharpoonup q$ and continuity of A_i, we have $A_i x_{n_l} \rightharpoonup A_i w$. Let $\vartheta_{n,i} = A_i x_n - T^{G_i,H_i}_{r_{n,i}} A_i x_n$, from (3.15), we have $\lim_{n\to\infty} \vartheta_{n,i} = 0$ for all $i \in \{1,...,N_1\}$. And since $T^{G_i,H_i}_{r_{n,i}} A_i x_n = A_i x_n - \tau_{n,i}$ for all $\varepsilon \in Q$, we have
$G_i(A_i x_n - \vartheta_{n,i}, \varepsilon)$
$+H_i(A_i x_n - \vartheta_{n,i}, \varepsilon) + \frac{1}{r_{n,i}}\langle \varepsilon - (A_i x_n - \vartheta_{n,i}), (A_i x_n - \vartheta_{n,i}) - A_i x_n\rangle \geq 0.$
Since G_i and H_i are upper semicontinuous in the first argument, we have
$G_i(A_i w, \varepsilon) + H_i(A_i w, \varepsilon) \geq 0, \forall \varepsilon \in Q.$
Then we obtain $A_i w \in \mathrm{GEP}(G_i,H_i)$, for all $i \in \{1,...,N_1\}$. Therefore, $w \in \Gamma$.

Finally, we will show that $w \in \Lambda = \bigcap_{j=1}^{N_2} VI(C,B_j)$ by demiclosedness principle, that is, we only need to show that $w = P_C(w - \lambda B_i w)$, where $\lambda = \lim_{n\to\infty} \lambda_n$. By (3.1) and (3.18), one has $\|u_n - P_C(I-\lambda_n B)u_n\| \to 0$ where $u_n = u_{n,i_n}, i_n = \arg\max_{1\leq i\leq N_1}\{\|u_{n,i} - x_n\|\}$. Thus, we have

$$\begin{aligned} \|u_n - P_C(I-\lambda B)u_n\| &\leq \|u_n - P_C(I-\lambda_n B)u_n\| \\ &\quad + \|P_C(I-\lambda_n B)u_n - P_C(I-\lambda_n B)u_n\| \\ &\leq \|u_n - P_C(I-\lambda_n B)u_n\| + \|(I-\lambda_n B)u_n - (I-\lambda B)u_n\| \\ &\leq \|u_n - P_C(I-\lambda_n B)u_n\| + |\lambda - \lambda_n|\|Bu_n\|. \end{aligned} \tag{3.24}$$

Since $\lambda_n \to \lambda > 0, \{Bu_n\}$ are bounded and $\|u_n - P_C(I-\lambda B)u_n\| \to 0$, we have

$$\lim_{n\to\infty} \|u_n - P_C(I-\lambda B)u_n\| = 0. \tag{3.25}$$

On the other hand, since $\{\lambda_n\} \subset (0,2\beta)$, one has $\lambda \in (0,2\beta]$. Thus $I-\lambda B$ is nonexpansive and, further, we have $P_C(I-\lambda B)$ is nonexpansive. Noting that $u_{n_j} \rightharpoonup w$ as $j\to\infty$, by Lemma 2.12, we get $w = P_C(I-\lambda B)w$. By Lemma 2.1, we get $w \in \Lambda = \bigcap_{j=1}^{N_2} VI(C,B_j)$. Therefore, $w \in \Omega$. By the property on P_C (2.2), we have

$$\lim_{n\to\infty} \sup\langle \nu - z, x_n - z\rangle = \lim_{j\to\infty}\langle \nu - z, x_{n_j} - z\rangle = \langle \nu - z, w - z\rangle \leq 0. \tag{3.26}$$

Step 5. We show that $x_n \to z = P_\Omega \nu$ as $n \to \infty$. By (3.1), we get

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\alpha_n\nu + \sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)S_k y_n - z\right\|^2 \\
&= \alpha_n\langle \nu - z, x_{n+1}-z\rangle + \sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)\langle S_k y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n\langle \nu - z, x_{n+1}-z\rangle + \frac{\sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)}{2}(\|S_k y_n - z\|^2 \\
&\quad +\|x_{n+1}-z\|^2) \\
&\le \alpha_n\langle \nu - z, x_{n+1}-z\rangle + \frac{\sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)}{2}(\|x_n - z\|^2 \\
&\quad +\|x_{n+1}-z\|^2) \\
&= \alpha_n\langle \nu - z, x_{n+1}-z\rangle + \frac{1-\alpha_n}{2}(\|x_n - z\|^2 + \|x_{n+1}-z\|^2) \\
&\le \alpha_n\langle \nu - z, x_{n+1}-z\rangle + \frac{1-\alpha_n}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2,
\end{aligned}$$

which implies that

$$\|x_{n+1}-z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle \nu - z, x_{n+1}-z\rangle.$$

By Lemma 2.13 and (3.26), we can conclude that $\lim_{n\to\infty}\|x_n - z\| = 0$. Hence $\{x_n\}$ converges strongly to $z = P_\Omega \nu$. This completes the proof. ∎

Remark 3.2. We present several corollaries of Theorem 3.1, that is, we can think out the following cases:

(i) $h_i = 0$ and $H_i = 0$, for all $i \in \{1, \dots, N_1\}$;
(ii) $H_i = 0$ and $G_i = 0$, for all $i \in \{1, \dots, N_1\}$;
(iii) $N_1 = N_2 = 1$.

Next, we give an example to demonstrate Theorem 3.1 as follows.

Example 3.3. We consider the case that $N_1 = 1$ and $N_2 = 2$.

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}, C = Q = [-5, 5]$. Let $A_1 : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by $A_1(x) = x$ for each $x \in \mathcal{H}_1$. Then, we have $A_1^* y = y$ for each $y \in \mathcal{H}_2$. For each $x, y \in C$, define the bifunction $g_1, h_1 \colon C \times C \to \mathbb{R}$ by $f_1(z, y) = y^2 + 3zy - 4z^2$, and $h_1(z, y) = y^2 - z^2$ for all $x, y \in C$. For each $x, y \in Q$, define the bifunction $G_1, H_1 \colon C \times C \to \mathbb{R}$ by $F_1(z, y) = 3y^2 + 2zy - 5z^2$ and $H_1(z, y) = 0$ for all $x, y \in Q$. For j=1,2 , let $B_j \colon C \to \mathcal{H}_1$ be defined by $B_1(x) = 2x$ and $B_2(x) = 6x$ for each $x \in C$. Then it is easy to see that B_1 and B_2 are $\frac{1}{2}$ and $\frac{1}{6}$-inverse strongly monotone operator from C into $\mathcal{H}_1$, respectively. It follows that $\Lambda = \bigcap_{j=1}^{2} VI(C, B_j) = \{0\}$. For each $k \in \mathbb{N}$, let $S_k \colon C \to C$ defined by

$$S_k(x) = \begin{cases} x + \dfrac{1}{2k}, & x \in [-5, 0) \\ x, & x \in [0, 5]. \end{cases}$$

Then $\{S_k\}$ is a countable family of nonexpansive mappings from C into C and it easy to see that $\Theta = \bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k) = [0, 5]$. Put $\alpha_n = \frac{1}{3n}, \lambda_n = \frac{1}{4}$ and $\gamma = \gamma_1 = \gamma_2 = \frac{1}{2}$. It is easy to verify that $g_1, h_1, G_1, H_1, A_1, B_1, B_2, \alpha_n, \lambda_n, \gamma, \gamma_1$ and γ_2 satisfy all the conditions of Theorem 3.1. Therefore, by Lemma 2.9, we see that $T_r^{g_1,h_1}$ and $T_r^{G_1,H_1}$ single-value mappings on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Hence, for $r_n = r > 0, x \in \mathcal{H}_1$ and $x \in \mathcal{H}_2$, there exist $z_1 \in C$ and $z_2 \in Q$ such that

$$g_1(z_1, y) + h_1(z_1, y) + \frac{1}{r}\langle y - z_1, z_1 - x\rangle \geq 0, \ \ \forall y \in C,$$

and

$$G_1(z_2, y) + H_1(z_2, y) + \frac{1}{r}\langle y - z_2, z_2 - x\rangle \geq 0, \ \ \forall y \in Q.$$

We can reform the above inequalities to standard quadratic form in the variable y as follows:

$$L_1(y) = 2ry^2 + (3rz_1 + z_1 - x)y + (xz_1 - 5rz_1^2 - z_1^2) \geq 0, \ \forall y \in C,$$

and

$$L_2(y) = 3ry^2 + (2rz_2 + z_2 - x)y + (xz_2 - 5rz_2^2 - z_2^2) \geq 0, \ \forall y \in Q.$$

It is easy to verify that the discriminants of the above two quadratic inequalities are nonnegative. And since $L_1(y) \geq 0$ for all $y \in C$ and $L_2(y) \geq 0$ for all $y \in Q$, we see that the discriminant must be zero. Then we obtain $z_1 = T_r^{g_1,h_1}(x) = \frac{x}{1+7r}$ and $z_2 = T_r^{G_1,H_1}(x) = \frac{x}{1+8r}$. By Theorem 3.1, let $\Omega = \Theta \cap \Gamma \cap \Lambda \neq \emptyset$, where $\Theta = \bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k), \Gamma = \{z \in C : z \in \mathrm{GEP}(C, g_1, h_1)$ such that $A_1 z \in \mathrm{GEP}(Q, G_1, H_1)\}$ and $A_1 z \in EP(G_1)\}$ and $\Lambda = \bigcap_{j=1}^{2} VI(C, B_j)$. Then $\Omega = \{0\}$.

Now, take $\nu = \dfrac{1}{2}$ and $x_1 = 5$ and define the sequence $\{x_n\}$ by (3.1). We get

$$\begin{aligned} \big(I - \gamma A_1^*\big(I - T_{r_{n,1}}^{G_1,H_1}\big)A_1\big)x_n &= x_n - \gamma A_1^*\big(I - T_{r_{n,1}}^{G_1,H_1}\big)A_1 x_n \\ &= x_n - \gamma A_1^*\big(A_1 x_n - T_{r_{n,1}}^{G_1,H_1} A_1 x_n\big) \\ &= x_n - \gamma A_1^*\big(x_n - T_{r_{n,1}}^{G_1,H_1} x_n\big) \\ &= x_n - \gamma A_1^*\big(x_n - \frac{x_n}{1+8r}\big) \\ &= x_n - \gamma\big(x_n - \frac{x_n}{1+8r}\big). \end{aligned} \tag{3.27}$$

Let $r = \dfrac{1}{8}$. Then from (3.27), we obtain

$$\big(I - \gamma A_1^*\big(I - T_{r_{n,1}}^{G_1,H_1}\big)A_1\big)x_n = \frac{3x_n}{4}.$$

Therefore,

$$u_n = u_{n,1} = T_{r_{n,1}}^{g_1,h_1}\frac{3x_n}{4} = \frac{2x_n}{5}.$$

Next, we compute the sequence y_n. By the definition of y_n in (3.1), we obtain

$$y_n = P_C\left(I - \lambda_n \frac{B_1 + B_2}{2}\right)u_n = P_C(0) = 0$$

for all $n \in \mathbb{N}$.

Finally, we compute the sequence x_n. By the following iteration:

$$x_{n+1} = \alpha_n v + \sum_{k=1}^{n}(\alpha_{k-1} - \alpha_k)S_k y_n = \alpha_n \nu = \frac{1}{6n}. \tag{3.28}$$

Thus from (3.28), we obtain

$$x_{n+1} \to 0 = P_\Omega \nu = P_{\{0\}} \frac{1}{2}$$

as $n \to \infty$ as shown by Theorem 3.1.

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