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In memoriam Professor Charles E. Chidume (1947–2021)

Absolute C^*-Embedding of P'-Spaces

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Abstract In this paper, we first show that for any non-real compact P'-space X, X is C^*-embedded in each P'-space in which X is embedded if and only if $\mid \upsilon X - X \mid = 1$. Using this, for any non-realcompact P'-space X, we show that $\mid \upsilon X - X \mid = 1$ if and only if X is C^*-embedded in each compactification of X, equivalently, X is an almost Lindelöf space and that if $\mid \upsilon X - X \mid = 1$, then υX is a Lindelöf space.

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1. Introduction

All spaces in this paper are Tychonoff spaces and $\beta X(\upsilon X$, resp.) denotes the Stone-Čech compactification(Hewitt realcompactification, resp.) of a space X.

Hewitt([1]) proved that a space X is C^*-embedded in each space in which X is embedded if and only if X is an almost compact space, that is, $|\beta X - X| \leq 1$ and that if X is an almost compact space, then X is a pseudocompact space. Aull([2]) proved that a P-space X is C^*-embedded in each P-space in which X is embedded if and only if X is an almost Lindelöf space, that is, for any two disjoint zero-sets in X, at least one of them is Lindelöf. Moreover, Dow and Förster([3]) showed that an F-space X is C^*-embedded in each F-space in which X is embedded if and only if X has no P-cover or X is an almost compact space. Veksler([4]) introduced the concept of P'-spaces which is a generalization of the concept of P-spaces.

In this paper, we first show that for any non-realcompact P'-space X, X is C^*-embedded in each P'-space in which X is embedded if and only if $\mid \upsilon X - X \mid = 1$. Using this, for any non-realcompact P'-space X, we show that $\mid \upsilon X - X \mid = 1$ if and only if X is C^*-embedded in each compactification of X, equivalently, X is an almost Lindelöf space and that if $\mid \upsilon X - X \mid = 1$, then υX is a Lindelöf space.

For the terminology, we refer to [5] and [6].

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2. P'-Spaces

The ring of real valued continuous functions on a space X is denoted by $C(X)$ and $C^*(X)$ denotes the subring of bounded functions of $C(X)$. For any $f \in C(X)$, $f^{-1}(0)$ is called *a zero-set in* X and $X - f^{-1}(0)$ is called *a cozero-set in* X

Recall that a space X is called *a P-space* if every zero-set in X is open in X. Veksler([4]) introduced the concept of P'-spaces which is a generalization of the concept of P-spaces.

Definition 2.1. A space X is called *a P'-space* if every zero-set in X is a regular closed set in X.

Veksler([4]) showed that for any locally compact realcompact space X, $\beta X - X$ is a P'-space. Let $\mathbb{R}$ be the set of all real numbers with the usual topology. Then $\beta\mathbb{R} - \mathbb{R}$ is a P'-space but not P-space.

For any zero-set Z in a space X, $cl_{\upsilon X}(Z)$ is a zero-set in υX and for any non-empty zero-set A in υX, $A \cap X \neq \emptyset$([5]). Hence X is a P'-space if and only if υX is a P'-space.

A subspace S of a space X is called $C(C^*$,resp.)-embedded in X if for any $f \in C(X)(C^*(X)$, resp.), there is a $g \in C(X)(C^*(X)$, resp.) such that $g|_X = f$ and S is called *z-embedded in* X if for any zero-set Z in S, there is a zero-set A in X such that $Z = A \cap S$.

It is well-known that a space X is a P-space if and only if every cozero-set in X is C-embedded in X([5]). We can get the following results similar to P-spaces.

Proposition 2.2. *Let X be a space. Then the following are equivalent :*
(1) X is a P'-space.
(2) For any zero-set Z in X with $int_X(Z) = \emptyset$, $Z = \emptyset$.
(3) If $f \in C(X)$ and $pos(f) = \{x \in X | f(x) > 0\}$ contains a dense subset of X, then $pos(f) = X$.
(4) Every dense z-embedded subspace of X is C-embedded.
(5) Every dense cozero-set in X is C-embedded.
(6) If $f \in C(X)$ and $X - f^{-1}(0)$ is dense in X, then f has the inverse element in $C(X)$.

Proof. (1) ⇒ (2) is tirvial.

(2) ⇒ (3) Let $f \in C(X)$ such that $pos(f)$ contains a dense subset of X. Then $cl_X(pos(f)) = X$. Since $pos(f)$ is a cozero-set in X, $X - pos(f) = g^{-1}(0)$ for some $g \in C(X)$. Since $pos(f)$ is dense in X, $int_X(g^{-1}(0)) = \emptyset$. By the assumption, $g^{-1}(0) = \emptyset$ and so $pos(f) = X$.

(3) ⇒ (4) Let S be a dense z-embedded subspace of X. Take any disjoint zero-sets A, B in S. Then there are $f, g \in C(X)$ such that $A = f^{-1}(0) \cap S$ and $B = g^{-1}(0) \cap S$. Since S is dense in X, $int_X(f^{-1}(0) \cap g^{-1}(0)) = \emptyset$. Let $h = f^2 + g^2$. Then $h^{-1}(0) = f^{-1}(0) \cap g^{-1}(0)$ and so $pos(h)$ is dense in X. By the assumption, $h^{-1}(0) = \emptyset$ and by Urysohn's extension theorem, S is C^*-embedded in X.

Let $l \in C(X)$ with $S \cap l^{-1}(0) = \emptyset$ and $l(x) \geq 0$ for all $x \in X$. Since S is dense in X and $S \subseteq pos(l)$, $pos(l) = X$ and so $l^{-1}(0) = \emptyset$. Hence S and $l^{-1}(0)$ are completely separated in X. Thus S is C-embedded in X.

(4) ⇒ (5) is trivial.

(5) ⇒ (6) Let $f \in C(X)$ such that $X - f^{-1}(0)$ is dense in X. Then $X - f^{-1}(0)$ is C-embedded in X and since $(X - f^{-1}(0)) \cap f^{-1}(0) = \emptyset$, $X - f^{-1}(0)$ and $f^{-1}(0)$ are completely separated in X and hence $f^{-1}(0) = \emptyset$. Thus f has the inverse element in $C(X)$.

(6) $\Rightarrow$ (1) Let $f \in C(X)$. Suppose that $x \in X - cl_X(int_X(f^{-1}(0)))$. Then there is a $g \in C(X)$ such that $x \in int_X(g^{-1}(0))$ and $int_X(f^{-1}(0)) \cap int_X(g^{-1}(0)) = \emptyset$. Note that $int_X(f^{-1}(0)) \cap int_X(g^{-1}(0)) = int_X(f^{-1}(0) \cap g^{-1}(0)) = int_X((f^2+g^2)^{-1}(0)) = \emptyset$. By the assupmtion, f^2+g^2 has the inverse element in $C(X)$ and $(f^2+g^2)^{-1}(0) = f^{-1}(0) \cap g^{-1}(0) = \emptyset$. Since $x \notin f^{-1}(0)$, $f^{-1}(0) = cl_X(int_X(f^{-1}(0)))$ and so X is a P'-space. ■

A space X is called *a weakly Lindelöf space* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\mathcal{V}$ is dense in X. Every Lindelöf space is a weakly Lindelöf space.

Proposition 2.3. *Let X be a weakly Lindelöf P'-space. Then X is a Lindelöf space.*

Proof. Let $\mathcal{U}$ be an open cover of X such that for any $U \in \mathcal{U}$, U is a cozero-set in X. Then there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $cl_X(\cup\mathcal{V}) = X$. Since $\cup\mathcal{V}$ is a dense cozero-set in X and X is a P'-space, by Proposition 2.2, $\cup\mathcal{V} = X$. Hence $\mathcal{V}$ is a countable subcover of $\mathcal{U}$ and so X is a Lindelöf space. ■

It is well-known that a countable P-space is a discrete space([5]).

Proposition 2.4. *Let X be a countable P'-space. Then X is a discrete space.*

Proof. Let $x \in X$. For any $y \in X$ with $x \neq y$, there is a zero-set Z_y in X such that $y \notin Z_y$ and $x \in Z_y$. Let $Z = \cap\{Z_y \mid y \in X - \{x\}\}$. Since X is a countable set, Z is a zero-set in X and $Z = \{x\}$. Since X is a P'-space, $int_X(Z) \neq \emptyset$. Hence $int_X(Z) = \{x\}$ and so $\{x\}$ is open in X. ■

3. Absolute C^*-Embedding of P'-Spaces

In this section, for P'-spaces, we will show results similar to those proved by Hewitt([1]), Aull([2]), and Dow and Förster([3]).

We recall that a space X is called *an F-space* if every cozero-set in X is C^*-embedded in X. Let X be an F-space and S a subspace of βX such that $X \subseteq S \not\subseteq \upsilon X$. Let $t \in S - \upsilon X$. Then there is a zero-set Z in βX such that $t \in Z$ and $Z \cap \upsilon X = \emptyset$. Since X is a dense subspace of S, $int_S(Z \cap S) = \emptyset$ and since $Z \cap S \neq \emptyset$, S is not a P'-space.

Lemma 3.1. *Let X be a P'-subspace of a space Y. Suppose that there is an onto continuous map $f : \upsilon X \longrightarrow Y$ such that for any $x \in X$, $f(x) = x$. Then Y is a P'-space.*

Proof. Take any non-empty zero-set Z in Y. Then $f^{-1}(Z)$ is a non-empty zero-set in υX and hence $f^{-1}(Z) \cap X = Z \cap X$ is a zero-set in X. Since X is a P'-space, $int_X(Z \cap X) \neq \emptyset$ and since $cl_X(int_X(Z \cap X)) = cl_Y(int_Y(Z)) \cap X$, $int_Y(Z) \neq \emptyset$. Hence Y is a P'-space. ■

Let X be a space and $\mathcal{F}$ a z-filter on X. Then $\mathcal{F}$ is called *free*(*fixed*, resp.) if $\cap\{F \mid F \in \mathcal{F}\} = \emptyset(\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$, resp.) and $\mathcal{F}$ is called *real* if it is closed under the countable intersection property.

For any dense subspace X of a space T, X is C-embedded in T if and only if every point of T is the limit point of the unique real z-ultrafilter on X([5]). For a subspace X of a space Y, $\upsilon X \subseteq \upsilon Y$ means that there is an embedding $h : \upsilon X \longrightarrow \upsilon Y$ such that $h \circ \upsilon_X = \upsilon_Y \circ j$, where $j : X \hookrightarrow Y$ is the inclusion map.

Theorem 3.2. *Let X be a P'-space. Then $|\upsilon X - X| \leq 1$ if and only if for any P'-space Y in which X is embedded, $\upsilon X \subseteq \upsilon Y$.*

Proof. ($\Rightarrow$) Let Y be a P'-space in which X is embedded. If X is a realcompact space, then we have the result. Suppose that $\mid \upsilon X - X \mid= 1$. Then there is a unique free real z-ultrafilter $\mathcal{F}$ on X and there is an $y \in \upsilon Y$ such that $y \in \cap\{cl_{\upsilon Y}(F) \mid F \in \mathcal{F}\}$. Let $T = X \cup \{y\}$. Since $\mathcal{F}$ is the unique real z-ultrafilter on X which converges to y in T, every point of T is a limit point of a unique real z-ultrafilter on X. Hence X is C-embedded in T and $\upsilon X = \upsilon T$([5]). Since υX is a P'-space, T is also a P'-space. Let $\mathcal{G}$ be a real z-ultrafilter on T. Since X is C-embedded in T, $\mathcal{G}_X = \{G \cap X \mid G \in \mathcal{G}\}$ is a real z-ultrafilter on X. By the assumption, $\mathcal{G}_X = \mathcal{F}$ or $\mathcal{G}_X$ is fixed and so $\cap\{cl_T(G \cap X)|G \in \mathcal{G}\} = \cap\{G \mid G \in \mathcal{G}\} \neq \emptyset$. Hence T is a realcompact space and thus $\upsilon X \subseteq \upsilon Y$.

($\Leftarrow$) Suppose that $\mid \upsilon X - X \mid\geq 2$. Pick $a, b \in \upsilon X - X$ with $a \neq b$. Let $R = \{(x, x) \mid x \in \upsilon X\} \cup \{(a, b), (b, a)\}$, $Y = \upsilon X/R$ be the quotient space and $q : \upsilon X \longrightarrow Y$ the quotient map.

Take any open set U in X. Then there is an open set V in υX such that $U = V \cap X$ and $\{a, b\} \cap V = \emptyset$. Since V is open in Y, X is a dense subspace of Y. By Lemma 3.1, Y is a P'-space. By the assumption, there is an embedding $h : \upsilon X \longrightarrow \upsilon Y$ such that $h \circ \upsilon_X = \upsilon_Y \circ j$, where $j : X \hookrightarrow Y$ is a dense embedding. Since $\upsilon_X : X \longrightarrow \upsilon X$ is a dense embedding, $h = \upsilon_Y \circ q$ and $h(a) = h(b)$. This is a contradiction, because h is an one-to-one map. ∎

Corollary 3.3. *Let X be a non-realcompact P'-space. Then the following are equivalent:*
(1) $|\upsilon X - X| = 1$.
(2) For any P'-space Y in which X is embedded, X is C^*-embedded in Y.
(3) For any P'-space Y in which X is dense embedded, X is C^*-embedded in Y.

Proof. (1) $\Rightarrow$ (2) Let $\upsilon X = X \cup \{p\}$ and Y a P'-space in which X is embedded. By the above theorem, $\upsilon X \subseteq \upsilon Y$. Suppose that X is not C^*-embedded in Y. Then there are disjoint zero-sets A, B in X such that $cl_Y(A) \cap cl_Y(B) \neq \emptyset$. Pick $k \in cl_Y(A) \cap cl_Y(B)$. Let $R = \{(x, x) \mid x \in X\} \cup \{(k, k), (p, p), (k, p), (p, k)\}$ and $T = (X \cup \{k, p\})/R$ be the quotient space. Since there is an onto continuous map from υX to T, by Lemma 3.1, T is a P'-space and by Theorem 3.2, there is an embedding $h : \upsilon X \longrightarrow \upsilon T$ such that $\upsilon_T \circ j = h \circ \upsilon_X$, where $q : X \cup \{k, p\} \longrightarrow T$ is the quotient map and $j : X \hookrightarrow T$ is the inclusion map.

Let $\mathcal{G}$ be a real z-ultrafilter on T. Let $Z \in \mathcal{G}$. Since $h^{-1}(cl_{\upsilon T}(Z))$ is a non-empty zero-set in υX, $h^{-1}(cl_{\upsilon T}(Z)) \cap X = Z \cap X \neq \emptyset$. Hence $\mathcal{G}_X = \{G \cap X \mid G \in \mathcal{G}\}$ is a real z-ultrafilter on X and by (1), $\cap\{G \cap X \mid G \in \mathcal{G}\} \neq \emptyset$ or $p \in \cap\{cl_{\upsilon X}(G \cap X) \mid G \in \mathcal{G}\}$. For any $G \in \mathcal{G}$, $h(cl_{\upsilon X}(G \cap X)) \subseteq cl_T(h(G \cap X)) = cl_T(G \cap X) \subseteq G$. Hence $\cap\{G \mid G \in \mathcal{G}\} \neq \emptyset$ and T is a realcompact space. Moreover, $T = \upsilon X$ and X is C^*-embedded in T. Since A, B are disjoint zero-sets in X, $cl_T(A) \cap cl_T(B) = \emptyset$. This is a contradiction, because $q(p) = q(k) \in cl_T(A) \cap cl_T(B)$. Hence X is C^*-embedded in Y.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Take any P'-space Y in which X is embedded. Then there is a continuous map $f : \upsilon X \longrightarrow \upsilon Y$ such that $f \circ \upsilon_X = \upsilon_Y \circ j$, where $j : X \hookrightarrow Y$ is the inclusion map. Let $T = f(\upsilon X)$. Note that $cl_T(X) = cl_{\upsilon Y}(X) \cap T = cl_{\upsilon Y}(f(X)) \cap T \supseteq f(\upsilon X) \cap T = T$. Hence X is dense in T. By Lemma 3.1, T is a P'-space and by (3), X is C^*-embedded in T. Take any zero-set Z in T such that $X \cap Z = \emptyset$. Then $f^{-1}(Z)$ is a zero-set in υX and $X \cap f^{-1}(Z) = \emptyset$. Since X is C-embedded in υX, $f^{-1}(Z) = \emptyset$ and so $Z = \emptyset$. Hence X is C-embedded in T.

Let $\mathcal{G}$ be a real z-ultrafilter on T. Since X is a dense C-embedded in T and T is a P'-space, $\mathcal{G}_X$ is a real z-ultrafilter on X. Hence $\cap\{cl_{\upsilon X}(G \cap X) \mid G \in \mathcal{G}\} \neq \emptyset$. Pick $p \in \cap\{cl_{\upsilon X}(G \cap X) \mid G \in \mathcal{G}\}$. For any $G \in \mathcal{G}$, $f(p) \in G$ and so $\cap\{G \mid G \in \mathcal{G}\} \neq \emptyset$. Hence T is a realcompact space and $T = \upsilon X$. Thus $f : \upsilon X \longrightarrow \upsilon Y$ is an embedding and $\upsilon X \subseteq \upsilon Y$. ■

By Proposition 2.2 and Corollary 3.3, we have the following :

Corollary 3.4. *Let X be a non-realcompact space. Then $|\upsilon X - X| = 1$ if and only if for any P'-space Y in which X is embedded, X is C-embedded in Y.*

Proposition 3.5. *Let X be a non-realcompact P'-space. Then $|\upsilon X - X| = 1$ if and only if for any compactification K of X, X is C^*-embedded in K.*

Proof. ($\Rightarrow$) Let (K, j) be a compactification of X and $\upsilon X = X \cup \{p\}$. Suppose that X is not C^*-embedded in K. Then there are disjoint zero-sets A, B in X such that $cl_K(A) \cap cl_K(B) \neq \emptyset$. Pick $k \in cl_K(A) \cap cl_K(B)$. Note that there is a continuous map $f : \upsilon X \longrightarrow K$ such that $f \circ \upsilon_X = j$.

Suppsoe that $f(p) = k$. By Lemma 3.1, $X \cup \{k\}$ is a P'-subspace of K. Since $|\upsilon X - X| = 1$, by Corollary 3.3, X is C^*-embedded in $X \cup \{k\}$ and so $\emptyset = cl_{X\cup\{k\}}(A) \cap cl_{X\cup\{k\}}(B) = cl_K(A) \cap cl_K(B) \cap (X \cup \{k\})$. Hence $k \notin cl_K(A) \cap cl_K(B)$ which is a contradiction. Thus $f(p) \neq k$.

Let $T = X \cup \{f(p), k\}$ and $Y = T/R$ the quotient space, where $R = \{(x, x) \mid x \in X\} \cup \{(f(p), k), (k, f(p)), (f(p), f(p)), (k, k)\}$. Similar to the proof of Corollary 3.3, we have a contradiction. Thus one have the result.

($\Leftarrow$) Let Y be a P'-space in which X is dense embedded. Since $cl_{\beta Y}(X)$ is a compactification of X, X is C^*-embedded in $cl_{\beta Y}(X)$. Since $cl_{\beta Y}(X)$ is C^*-embedded in βY, X is C^*-embedded in βY. Hence X is C^*-embedded in Y. ■

A space X is C^*-embedded in each compactification of X if and only if X is an almost Lindelöf space([7]). Hence we have the following :

Corollary 3.6. *Let X be a non-realcompact P'-space. Then $|\upsilon X - X| = 1$ if and only if X is an almost Lindelöf space.*

If $|\beta X - X| \leq 1$, then X is a pseudo-compact space, that is, υX is a compact space([5]). A space X with a dense weakly Lindelöf subspace is a weakly Lindelöf sapce.

Proposition 3.7. *Let X be a P'-space with $|\upsilon X - X| = 1$. Then υX is a Lindelöf sapce.*

Proof. Suppose that υX is not a Lindelöf space. Then there is a z-filter $\mathcal{F}$ on υX such that it has the countable intersection property and $\cap\{F \mid F \in \mathcal{F}\} = \emptyset$. Let $\upsilon X = X \cup \{p\}$. Then there is a $B \in \mathcal{F}$ such that $p \notin B$ and so there is a zero-set A in X such that $p \in cl_{\upsilon X}(A)$ and $A \cap B = \emptyset$. Since $|\upsilon X - X| = 1$, by Corollary 3.6, A or $B \cap X$ is Lindelöf.

Suppose that A is Lindelöf. Let $\mathcal{G}$ be a free real z-ultrafilter on X such that $p \in \cap\{cl_{\upsilon X}(G) \mid G \in \mathcal{G}\}$. Then $A \in \mathcal{G}$ and $\mathcal{G}_A = \{Z \cap A \mid Z \in \mathcal{G}\}$ is a z-filter on A with the countable intersection property. Since A is Lindelöf, $\cap\{Z \cap A \mid Z \in \mathcal{G}\} \neq \emptyset$ and hence $\cap\{Z \mid Z \in \mathcal{G}\} \neq \emptyset$. This is a contradiction.

Suppose that $B \cap X$ is Lindelöf. Since X is a P'-space, υX is a P'-space. Since $B = cl_{\upsilon X}(B \cap X)$, B is weakly Lindelöf. Since $\cap\{F \mid F \in \mathcal{F}\} = \emptyset$, $\cap\{F \cap B \mid F \in \mathcal{F}\} = \emptyset$ and since B is weakly Lindelöf, there is a sequence (F_n) in $\mathcal{F}$ such that $cl_B(int_B(\cap\{F_n \cap B \mid$

$n \in N\})) = \emptyset$. Since $cl_B(int_{\upsilon X}((\cap\{F_n \mid n \in N\}) \cap B)) \subseteq cl_B(int_B(\cap\{F_n \cap B \mid n \in N\}))$, $cl_{\upsilon X}(int_{\upsilon X}((\cap\{F_n \mid n \in N\}) \cap B)) = \emptyset$. Since υX is a P'-space and $(\cap\{F_n \mid n \in N\}) \cap B$ is a zero-set υX, $(\cap\{F_n \mid n \in N\}) \cap B = \emptyset$. Since $\mathcal{F}$ has the countable intersection property, it is a contradiction. ■

We give an example of a non-realcompact P'-space X with $|\upsilon X - X| = 1$.

Example 3.8. Let W be the space of all countable ordinals and $W^* = W \cup \{w_1\}$ the one-point compactification of W([5]), where w_1 is the first uncountable ordinal. Let D be a discrete space of cardinality $\aleph_1$ and $p \notin D$. Let $S = D \cup \{p\}$, topologized as follows: Each point of D is isolated and a subset G of S that contains p is open in S if and only if $\mid S - G \mid \leq \aleph_0$. Let $K = W^* \times S - \{(w_1, p)\}$ and $K^* = W^* \times S$. Then clearly, K^* is a Lindelöf space.

Take any $f \in C(K)$. Note that the subspace $A = \{m \mid m \text{ is a finite ordinal}\} \cup \{w_0\}$ of W^* is homeomorphic to $\mathbb{N}^*$, where w_0 is the fist countable ordinal and $\mathbb{N}^*$ is the one-point compactification of the discrete space $\mathbb{N}$. Hence there is a subset U of S such that $\mid S - U \mid \leq \aleph_0$ and for any $m \in A$, f is constant on $\{m\} \times U$([6]). For any $s \in U$, the sequence $\{(m, s)\}$ converges to (w_1, s) in K^* and $\{f((m, s))\}$ is conevrgent to $f((w_1, s))$ in $\mathbb{R}$ with the usual topology. For any $s, t \in U$ and any $m \in A$, $f((m, s)) = f((m, t))$ and hence $f((w_1, s)) = f((w_1, t))$. Define a map $h : K^* \longrightarrow \mathbb{R}$ by $h(x) = f(x)$ if $x \in K$ and $h((w_1, p)) = f((w_1, \alpha))$ for some $\alpha \in U$. Then h is a continuous extension of f to K^*. Hence $\upsilon K = K^*$.

Now, we will show that υK is a P'-space. Take any non-empty zero-set $Z = Z(g)$ in υK and $(x, y) \in Z$. Then there is a finite ordinal n such that g is constant on $\{\alpha \mid n \leq \alpha\} \times \{y\}$([5]).

<u>Case.1</u> $x \leq n$:
If $p \neq y$, then $\{(x, y)\}$ is open in υK and $int_{\upsilon K}(Z) \neq \emptyset$. Suppose that $p = y$. Then there is a subset G of S such that $p \in G$, $\mid S - G \mid \leq \aleph_0$, and for any $m \in A$, g is constant on $\{m\} \times G$ and so $\{x\} \times G \subseteq Z$. Since $\{x\} \times G$ is open υK, $int_{\upsilon K}(Z) \neq \emptyset$.

<u>Case.2</u> $x > n$:
If $p \neq y$, then $\{\alpha \mid n \leq \alpha\} \times \{y\}$ is open in υK, $\{\alpha \mid n \leq \alpha\} \times \{y\} \subseteq Z(g)$, and so $int_{\upsilon K}(Z) \neq \emptyset$. Suppose that $p = y$. Then there is a subset V of S such that $p \in V$, $\mid S - V \mid \leq \aleph_0$, and for any $m \in A$, g is constant on $\{m\} \times V$. If x is not a limit ordinal, then $\{x\} \times V \subseteq int_{\upsilon K}(Z)$ and hence $int_{\upsilon K}(Z) \neq \emptyset$. Suppose that x is a limit ordinal. Then $w_0 \leq x$ and

$$\{\alpha \mid n \leq \alpha\} \times \{p\} \subseteq Z(g), \ \{(m, s) \mid s \in V \ and \ n \leq m < w_0\} \subseteq Z(g).$$

By the property of W^*, $\{(\alpha, s) \mid s \in V \ and \ n \leq \alpha\} \subseteq Z(g)$ and so $int_{\upsilon K}(Z) \neq \emptyset$. Thus υK is a P'-space.

Author Contributions

All authors contributed equally to the writing of this paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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