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Common Fixed Point Theorems for Generalized Rational $F_{\mathcal{R}}$-Contractive Pairs of Mappings

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Abstract In this paper, we prove some common fixed point theorems for a pair of mappings satisfying a condition of $F_{\mathcal{R}}$-contraction type on a complete metric space endowed with a binary relation, and we will give examples to support our result.

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1. Introduction

In [1], D. Wardowski introduced a new type of contraction; called F-contractions that generalize the famous Banach contraction principle and prove a fixed point theorem. Later, many authors give fixed point theorems for maps satisfying such contraction type. In [2], K. Sawangsup and al. introduced the notion of $F_{\mathcal{R}}-$contraction where $\mathcal{R}$ is a binary relation on a complete metric space and established some fixed point results. Recently M.B. Zada and al. [3], modified the definition of $F_{\mathcal{R}}-$ contraction introduced by Sawangsup for two maps, namely rational $F_{\mathcal{R}}$-contractive pairs of mappings and prove a common fixed point theorem for such pairs of maps.
In this work, inspired by M.B. Zada and al. [3], we prove some common fixed point theorems for a pair of mappings satisfying a condition of rational $F_{\mathcal{R}}$-contraction type and we give examples to support our results.

Preliminaries

Throughout this paper, $\mathbb{R}_+$ denotes the set of nonnegative real numbers and $\mathbb{R}_+^*$ the set of positive real numbers.
We start with some definitions and properties that we need in our results.

Definition 1.1. ([1]) we call $\mathcal{F}$ the following set,

$$\mathcal{F} = \{F : [0, +\infty[\to \mathbb{R} \ \text{ satisfying } \ (F_1), (F_2), (F_3)\}$$

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where

- (F_1) F is strictly increasing i.e. $\forall s; t \in \mathbb{R}_+$ with $s < t$, we have $F(s) < F(t)$.
- (F_2) for each sequence $(s_n)_n \subset \mathbb{R}_+^*$: $\lim_{n\to+\infty} s_n = 0 \Longleftrightarrow F(s_n) \xrightarrow{n\to+\infty} -\infty$.
- (F_3) There exists $\mu \in]0,1[$ such that: $\lim_{s\to 0^+} s^\mu F(s) = 0$.

Example 1.2. ([2]) examples of functions belonging to $\mathcal{F}$.
$F_1(s) = lns, \quad s > 0$.
$F_2(s) = -\frac{1}{\sqrt{s}}, \quad s > 0$.

Definition 1.3. ([1]) Let (E, d) be a metric space. A mapping $T : E \to E$ is said to be an F-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\forall x, y \in E : \quad d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 1.4. ([4]) Given a mapping $T : E \to E$, a binary relation $\mathcal{R}$ defined on E is called $T-$closed if

$$\forall x, y \in E : \quad (x, y) \in \mathcal{R} \Longrightarrow (Tx, Ty) \in \mathcal{R}.$$

Remark 1.5. The previous property is equivalent to say that T is nondecreasing

$$(x\mathcal{R}y \Longrightarrow Tx\mathcal{R}Ty).$$

Definition 1.6. ([5]) Let $x_1, x_2 \in E$ and $\mathcal{R}$ be a binary relation on a nonempty set E. A path (of length $n \in \mathbb{N}$) in $\mathcal{R}$ from x_1 to x_2 is a sequence $\{t_0, t_1, t_2, \cdots, t_n\} \subset E$ such that:

$$t_0 = x_0, \cdots, t_n = x_2, \quad (t_j, t_{j+1}) \in \mathcal{R}, \quad \forall j = 0, 1, \cdots, n-1.$$

$\Gamma(x_1, x_2, \mathcal{R})$ represents the class of all paths from x_1 to x_2 in $\mathcal{R}$.

Remark 1.7. t_j are not necessarily distinct.

Definition 1.8. ([6]) A metric space (E, d) equipped with a binary relation $\mathcal{R}$ is $\mathcal{R}$-nondecreasing-regular if for all sequences $(x_n)_n$; we have :

$$((x_n, x_{n+1}) \in \mathcal{R}, \ \forall n \in \mathbb{N} \ \ and \ \ x_n \to x) \Longrightarrow (x_n, x) \in \mathcal{R}, \forall n \in \mathbb{N}.$$

Definition 1.9. ([2]) Let (E, d) be a metric space, $\mathcal{R}$ a binary relation on E and let $T : E \to E$ be a mapping. Let $\mathcal{W} = \{(x, y) \in \mathcal{R} : \quad d(Tx, Ty) > 0\}$.
T is said to be an $F_\mathcal{R}$-contraction if there exists $\delta > 0, F \in \mathcal{F}$ such that:

$$\forall (x, y) \in \mathcal{W} : \quad \delta + F(d(Tx, Ty)) \leq F(d(x, y)).$$

M.B. Zada and al. [3] modified the definition 1.4 ([4]) for two maps as follows.

Definition 1.10. Let $E \neq \emptyset$. Let T, S be two self mappings on E and $\mathcal{R}$ a binary relation on E. Then, $\mathcal{R}$ is $(T, S)-$closed if for all $x, y \in E$:

$$(x, y) \in \mathcal{R} \Longrightarrow (Tx, Sy) \in \mathcal{R}, (Sx, Ty) \in \mathcal{R}.$$

And they introduced the concept of rational $F_\mathcal{R}-$contractive pair of mappings.

Definition 1.11. Let (E, d) be a metric space and T, S be self-mappings on E and let $\mathcal{R}$ be a binary relation on E. Put

$$\mathcal{X} = \{(x, y) \in \mathcal{R} : \quad d(Tx, Sy) > 0\}.$$

We say that (T, S) is a rational $F_{\mathcal{R}}$-contractive pair of mappings if there exist $\delta > 0$ and $F \in \mathcal{F}$ such that:

$$\delta + F(d(Tx, Ty)) \leq F\left(d(x, y) + \frac{d(y, Tx) \cdot d(x, Sy)}{1 + d(x, y)}\right), \text{ for all } (x, y) \in \mathcal{X}.$$

Denote by $E((T, S), \mathcal{R})$ the set of all order pairs $(x, y) \in E \times E$ such that $(Tx, Sy) \in \mathcal{R}$.
M.B. Zada and al. [3] proved the following theorem:

Theorem 1.12. ((3.3) in [3]): *Let (E, d) be a complete metric space; $\mathcal{R}$ be a binary relation on E and $T, S : E \to E$. two mappings. Suppose that the following conditions hold:*

- (C_1) *$E((T, S), \mathcal{R})$ is nonempty.*
- (C_2) *$\mathcal{R}$ is $(T, S)-$closed.*
- (C_3) *T and S are continuous.*
- (C_4) *The pair (T, S) is rational $F_{\mathcal{R}}$-contractive.*

Then there is a common fixed point of T and S.

Our goal in this work is to introduce a new type of rational $F_{\mathcal{R}}$-contraction for a pair of mappings and we prove fixed point results.

Definition 1.13. Let (E, d) be a metric space and $T, S : E \to E$ two mappings and let $\mathcal{R}$ be a binary relation on E. We say that (T, S) is a generalized rational $F_{\mathcal{R}}$- contractive pair of mappings if there exist $\delta > 0$ and $F \in \mathcal{F}$ such that: $\forall (x, y) \in \mathcal{X}$

$$\delta + F(d(Tx, Sy)) \leq F(\max\{d(x, y) + \frac{d(y, Tx) \cdot d(x, Sy)}{1 + d(x, y)}; \alpha d(x, Tx); \beta d(y, Sy)\}) \tag{1.1}$$

where $\alpha, \beta \in [0, 1]$.

Let remember that $\mathcal{X} = \{(x, y) \in \mathcal{R}, \quad d(Tx, Sy) > 0\}$
and $E((T, S), \mathcal{R}) = \{(x, y) \in E^2 : \quad (Tx, Sy) \in \mathcal{R}\}$.

2. Main Results

Now we present our main result.

Theorem 2.1. *Let (E, d) be a complete metric space; $\mathcal{R}$ a binary relation on E and $T, S : E \to E$ two mappings. Suppose that the following conditions hold:*

1. *$E((T, S), \mathcal{R})$ is non empty.*
2. *$\mathcal{R}$ is $(T, S)-$closed.*
3. *T and S are continuous.*
4. *The pair (T, S) is generalized rational $F_{\mathcal{R}}$-contractive in the sense of (1.1).*

Then T and S have a common fixed point.

Proof. by (1): Let $(a, b) \in E((T, S), \mathcal{R})$, then $(Ta, Sb) \in \mathcal{R}$.
Put $x_1 = Ta, x_2 = Sb$ and define $(x_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} x_{2n+1} = Tx_{2n}, & n \geq 1 \\ x_{2n+2} = Sx_{2n+1} \end{cases}$$

if $x_{2N} = x_{2N+1}$ for some $N \in \mathbb{N}^*$, then we have necessarily $d(Tx_{2N}, Sx_{2N+1}) = 0$, otherwise $d(Tx_{2N}, Sx_{2N+1}) > 0$ which implies $(x_{2N}, x_{2N+1}) \in \mathcal{X}$. So, by (1.1)

$$\delta + F(d(Tx_{2N}, Sx_{2N+1})) \leq F[\max\{d(x_{2N}, x_{2N+1}) + \frac{d(x_{2N+1}, Tx_{2N}) \cdot d(x_{2N}, Sx_{2N+1})}{1 + d(x_{2N}, x_{2N+1})}; \alpha d(x_{2N}, Tx_{2N}); \beta d(x_{2N+1}, Sx_{2N+1})\}]$$
$$\delta + F(d(x_{2N+1}, x_{2N+2})) \leq F[\max\{0; 0; 0; \beta d(x_{2N+1}, x_{2N+2})\}]$$
$$\delta + F(d(x_{2N+1}, x_{2N+2})) \leq F(\beta d(x_{2N+1}, x_{2N+2})).$$

Since F is strictly increasing and $\beta \leq 1$, we get:
$\delta + F(d(x_{2N+1}, x_{2N+2})) \leq F(d(x_{2N+1}, x_{2N+2}))$
which implies $\delta \leq 0$,that is a contradiction.
Hence $d(Tx_{2N}, Sx_{2N+1}) = 0$ and so : $Tx_{2N} = Sx_{2N+1}$,
finally: $x_{2N} = x_{2N+1} = Tx_{2N} = Sx_{2N+1}$ and x_{2N} is a fixed point of T, x_{2N+1} is a fixed point of S. Then $x_{2N} = x_{2N+1}$ is a common fixed point of T and S.
That is we can assume $x_{2n} \neq x_{2n+1}, \forall n \in \mathbb{N}^*$ and so $d(Tx_{2n}, Sx_{2n+1}) > 0$.
Using assumption (2) we have:

$$\begin{array}{rcl} (x_1, x_2) & = & (Ta, Sb) \in \mathcal{R} \\ (x_2, x_3) & = & (Sx_1, Tx_2) \in \mathcal{R} \\ (x_3, x_4) & = & (Tx_2, Sx_3) \in \mathcal{R} \\ & \vdots & \end{array}$$

by induction $(x_{2n}, x_{2n+1}) = (Sx_{2n-1}, Tx_{2n}) \in \mathcal{R}$ thus $(x_{2n}, x_{2n+1}) \in \mathcal{X}, \forall n \in \mathbb{N}$.
Put $x = x_{2n}, y = x_{2n-1}$ in (1.1) we obtain:

$$F(d(x_{2n}, x_{2n+1})) = F(d(x_{2n+1}, x_{2n})) = F(d(Tx_{2n}, Sx_{2n-1})) \leq F[\max\{d(x_{2n}, x_{2n-1}) + \frac{d(x_{2n-1}, Tx_{2n})d(x_{2n}, Sx_{2n-1})}{1 + d(x_{2n}, x_{2n-1})}; \alpha d(x_{2n}, Tx_{2n}); \beta d(x_{2n-1}, Sx_{2n-1})\}] - \delta$$
$$\leq F[\max\{d(x_{2n}, x_{2n-1}); \alpha d(x_{2n}, x_{2n+1})\}] - \delta$$

If $\max\{d(x_{2n}, x_{2n-1}); \alpha d(x_{2n}, x_{2n+1})\} = \alpha d(x_{2n}, x_{2n+1})$, we get

$$F(d(x_{2n}, x_{2n+1})) \leq F(\alpha d(x_{2n}, x_{2n+1})) - \delta \leq F(d(x_{2n}, x_{2n+1})) - \delta$$

which is a contradiction (since $\delta > 0$). So

$$F(d(x_{2n}, x_{2n+1})) \leq F(d(x_{2n}, x_{2n-1})) - \delta, \quad \forall n \geq 1. \tag{2.1}$$

Similary, setting $x = x_{2n}, y = x_{2n+1}$ in (1.1), we obtain

$$F(d(x_{2n+1}, x_{2n+2})) = F(d(Tx_{2n}, Sx_{2n+1})) \leq F[\max\{d(x_{2n}, x_{2n+1}) + \frac{d(x_{2n+1}, Tx_{2n})d(x_{2n}, Sx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}; \alpha d(x_{2n}, Tx_{2n}); \beta d(x_{2n+1}, Sx_{2n+1})\}] - \delta$$
$$\leq F[\max\{d(x_{2n}, x_{2n+1}); \beta d(x_{2n+1}, x_{2n+2})\}] - \delta$$

If $\max\{d(x_{2n}, x_{2n+1}); \beta d(x_{2n+1}, x_{2n+2})\} = \beta d(x_{2n+1}, x_{2n+2})$, we have

$$F(d(x_{2n+1}, x_{2n+2})) \leq F(\beta d(x_{2n+1}, x_{2n+2})) - \delta \leq F(d(x_{2n+1}, x_{2n+2})) - \delta$$

which is a contradiction. Hence

$$F(d(x_{2n+1},x_{2n+2})) \leq F(d(x_{2n},x_{2n+1})) - \delta, \tag{2.2}$$

with (2.1) and (2.2) we deduce that

$$F(d(x_n,x_{n+1})) \leq F(d(x_{n-1},x_n)) - \delta, \quad \forall n \geq 1. \tag{2.3}$$

Using (2.3), we obtain

$$\begin{aligned} F(d(x_n,x_{n+1})) &\leq F(d(x_{n-1},x_n)) - \delta \\ &\leq F(d(x_{n-2},x_{n-1})) - 2\delta \\ &\leq F(d(x_{n-3},x_{n-2})) - 3\delta \\ \vdots &\leq F(d(x_1,x_2)) - (n-1)\delta \end{aligned} \tag{2.4}$$

(2.5)

Thus $\lim_{n\to+\infty} F(d(x_n,x_{n+1})) = -\infty$, by condition (F_2) in definition of $\mathcal{F}$, we get

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \tag{2.6}$$

From condition (F_3) in definition of $\mathcal{F}$, we can find $\mu \in]0,1[$ such that

$$\lim_{n\to+\infty} (d(x_n,x_{n+1}))^\mu F(d(x_n,x_{n+1})) = 0. \tag{2.7}$$

Using (2.4), we have

$$(d(x_n,x_{n+1})^\mu[F(d(x_n,x_{n+1})) - F(d(x_1,x_2))] \leq -(n-1)\delta(d(x_n,x_{n+1})^\mu \tag{2.8}$$

take $n \to +\infty$ in (2.8) and using (2.6),(2.7)

$$(d(x_n,x_{n+1})^\mu[F(d(x_n,x_{n+1})) - F(d(x_1,x_2))] \leq -(n-1)\delta(d(x_n,x_{n+1})^\mu$$

$n \to +\infty$, we get $\lim_{n\to+\infty}(n-1)(d(x_n,x_{n+1})^\mu = 0$ which implies, that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : (n-1)(d(x_n,x_{n+1}))^\mu \leq 1$ thus

$$d(x_n,x_{n+1}) \leq \frac{1}{(n-1)^{\frac{1}{\mu}}}, \quad \forall n \geq n_0 > 1$$

.

Now, we show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $n,m \in \mathbb{N}, \quad m > n \geq n_0$

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) = \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \frac{1}{(k-1)^{\frac{1}{\mu}}} = \sum_{k=n-1}^{m-2} \frac{1}{(k')^{\frac{1}{\mu}}} < \sum_{k=n-1}^{\infty} \frac{1}{(k')^{\frac{1}{\mu}}}. \end{aligned}$$

Since $\sum_{k=n-1}^{\infty} \frac{1}{(k')^{\frac{1}{\mu}}}$ is the remainder of the convergent series $\sum_{k\geq 1} \frac{1}{(k')^{\frac{1}{\mu}}}$, we obtain $R_{n-1} = \sum_{k=n-1}^{\infty} \frac{1}{(k')^{\frac{1}{\mu}}} \xrightarrow{n\to+\infty} 0$ that is $d(x_n,x_m) \xrightarrow[(m\to+\infty)]{n\to+\infty} 0$. And hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (E,d). Since E is complete, there exists $u \in E$ such that:

$x_n \xrightarrow{n\to+\infty} u.$

Next we show that $T(u) = S(u) = u$.
Since T and S are continuous and $x_{2n} \xrightarrow{n\to+\infty} u, \quad x_{2n-1} \xrightarrow{n\to+\infty} u.$

$$x_{2n+1} = Tx_{2n} \to Tu,$$
$$x_{2n} = Sx_{2n-1} \to Su.$$

So we obtain $Tu = Su = u$ and hence u is a common fixed point of T and S. ■

Example 2.2. Let $E = \{1, 2, 3\}$ with the usual metric $d(x, y) = |x - y|$, and let $T, S : E \to E$ the mappings defined by

$$T1 = 1, T2 = T3 = 2$$
$$S1 = S2 = S3 = 1$$

and $\mathcal{R}$ the binary relation on E :

$$\mathcal{R} = \{(1, 1); (1, 2); (1, 3); (2, 1)\}$$

we have:

- (E, d) is a complete metric space (E is closed in the usual complete metric space $\mathbb{R}$).
- $((T, S), \mathcal{R})$ is nonempty $((1, 1) \in E((T, S), \mathcal{R}))$.
- $\mathcal{R}$ is $(T, S)-$closed. Indeed:

$$\begin{aligned}
(1,1) \in \mathcal{R} &\implies (T1, S1) = (1,1) \in \mathcal{R}, (S1, T1) = (1,1) \in \mathcal{R}\\
(1,2) \in \mathcal{R} &\implies (T1, S2) = (1,1) \in \mathcal{R}, (S1, T2) = (1,2) \in \mathcal{R}\\
(1,3) \in \mathcal{R} &\implies (T1, S3) = (1,1) \in \mathcal{R}, (S1, T3) = (1,2) \in \mathcal{R}\\
(2,1) \in \mathcal{R} &\implies (T2, S1) = (2,1) \in \mathcal{R}, (S2, T1) = (1,1) \in \mathcal{R}
\end{aligned}$$

- T and S are continuous on E (Endowed with the induced topology).
- $\mathcal{X} = \{(2, 1)\}$.
- The pair (T, S) is generalized rational $F_{\mathcal{R}}-$contractive in the sense of $(1, 1)$. Indeed:
  - $\mathcal{X} = \{(2, 1)\}$
  - $(x, y) \in \mathcal{X} \iff (x, y) = (2, 1)$
  -

$$\begin{aligned}
\delta + F(d(Tx, Sy)) &\leq F(\max\{d(x, y) + \frac{d(x, Sy).d(y, Tx)}{1 + d(x, y)}; \alpha d(x, Tx); \beta d(y, Sy)\})\\
\delta + F(d(T2, S1)) &\leq F(\max\{d(2, 1) + \frac{d(2, S1).d(1, T2)}{1 + d(2, 1)}; \alpha d(2, T2); \beta d(1, S1)\})\\
\delta + F(1) &\leq F(\max\{1 + \frac{1}{2}; 0; 0\})\\
\delta + F(1) &\leq F(\frac{3}{2}), \quad F \in \mathbb{F},
\end{aligned}$$

we choose: $F(t) = \ln(t)$ we get $\delta + \ln(1) \leq \ln(\frac{3}{2})$.

That is $\exists\delta > 0 : \delta = \frac{1}{2}\ln(\frac{3}{2})$ such that (1.1) is satisfied.

Hence T and S have a common fixed point u (here $u = 1$).

Now we give a result that ensures the uniqueness of the common fixed point found in the previous theorem.

Theorem 2.3. *Let (E, d) be a complete metric space, and $\mathcal{R}$ be a transitive binary relation on E. Assume that $T, S : E \to E$ are two mappings such that:*

(U_1) $\forall(x, y) \in \mathcal{X}, \exists\delta > 0, \exists F \in \mathcal{F}$ *such that*

$$\delta + F(d(Tx, Sy)) \leq F(\lambda \max\{d(x, y) + \frac{d(y, Tx) \cdot d(x, Sy)}{1 + d(x, y)}; \alpha d(x, Tx); \beta d(y, Sy)\}) \tag{2.9}$$

where $\alpha; \beta \in [0, 1]$ and $\lambda \in]0, \frac{1}{2}]$.

(U_2) $E((T, S), \mathcal{R}), \Gamma(x, y, \mathcal{R})$ *are non empty, $\forall x, y \in \mathcal{X}$.*

(U_3) $\mathcal{R}$ *is (T, S)-closed.*

(U_4) T *and S are continuous.*

Then T and S have a unique common fixed point.

Proof. Following the same steps as in the proof of Theorem 2.1, we can easily prove that T and S have a common fixed point, thus we have to show that this common fixed point is unique.
Assume that u and v are two distinct common fixed point of T and S.

$$Tu = Su = u, \quad Tv = Sv = v, \quad d(u, v) > 0.$$

Let $\{w_0, w_1, \cdots, w_p\}$ a path from u and v

$$w_0 = u, w_p = v, \quad (w_j, w_{j+1}) \in \mathcal{R}; \quad j = 0, 1, \cdots, p - 1.$$

Since $\mathcal{R}$ is transitive

$$(u, w_1), (w_1, w_2), \cdots, (w_{p-1}, v) \in \mathcal{R} \Rightarrow (u, v) \in \mathcal{R}.$$

Put in contraction condition (U_1): $x = u, y = v$ we get

$$\delta + F(d(Tu, Sv)) \leq F(\lambda \max\{d(u, v) + \frac{d(v, Tu) \cdot d(u, Sv)}{1 + d(u, v)}; \alpha d(u, Tu); \beta d(v, Sv)\})$$

so

$$\delta + F(d(u, v)) \leq F(\max\{\lambda d(u, v) + \lambda\frac{(d(u, v))^2}{1 + d(u, v)}; 0; 0)\}).$$

Since $\frac{\lambda(d(u, v))^2}{1 + d(u, v)} < \lambda d(u, v)$, we obtain

$$\delta + F(d(u, v)) \leq F(\max\{\lambda d(u, v) + \lambda d(u, v)\})$$

and hence : $\delta + F(d(u,v)) \leq F(2\lambda d(u,v))$.
Since F is strictly increasing, we get

$$\delta + F(d(u,v)) \leq F(d(u,v))$$

which is a contradiction and hence $u = v$, then the common fixed point is unique. ■

Example 2.4. Let $E = [0,1]$ equipped with $d(x,y) = |x-y|$ and $T, S : E \to E$ defined by

$$Tx = \frac{x}{2}, \quad Sx = \frac{x}{4}.$$

And let $\mathcal{R}$ the binary relation defined on E by $\mathcal{R} = \{(0, \frac{1}{n}), \quad n \geq 1)\}$.
Remark that $\mathcal{R}$ is transitive. We claim that (T,S) satisfies Theorem 2.3.
Indeed:

- (E,d) is complete.
- $E((T,S),\mathcal{R})$ is nonempty ($(0,\frac{1}{2}) \in E((T,S),\mathcal{R})$ for example).
- $\mathcal{R}$ is $(T,S)-$closed.

$$\begin{aligned}(0,\frac{1}{n}) \in \mathcal{R} \implies (T0, S\frac{1}{n}) &= (0,\frac{1}{4n}) = (0,\frac{1}{n'}) \in \mathcal{R};\\ (S0, T\frac{1}{n}) &= (0,\frac{1}{2n}) = (0,\frac{1}{n''}) \in \mathcal{R}\end{aligned}$$

- T and S are continuous on E.
- The pair (T,S) is generalized rational $F_{\mathcal{R}}-$contractive in the sense of (1.10)
  - $\mathcal{X} = \mathcal{R} = \{(0,\frac{1}{n}), \quad n \geq 1\}$

$$\begin{aligned}(x,y) \in \mathcal{X} \iff (x,y) &= (0,\frac{1}{n}), \quad n \geq 1\\ d(Tx,Sy) &= d(0,\frac{1}{4n}) = \frac{1}{4n} > 0\\ M_{(x,y)} &= \max\{d(x,y) + \frac{d(x,Sy).d(y,Tx)}{1+d(x,y)}; \alpha d(x,Tx); \beta d(y,Sy)\}\\ &= \max\{\frac{1}{n} + \frac{\frac{1}{4n}.\frac{1}{n}}{1+\frac{1}{n}}; \alpha.0; \beta\frac{3}{4n}\}\\ &= \frac{1}{n} + \frac{1}{4n(n+1)}.\end{aligned}$$

Let $F(t) = \ln(t) \in \mathbb{F}$, we can choose $\delta > 0$ and $\lambda = \frac{1}{2}$ such that

$$\begin{aligned}
\delta + F(\frac{1}{4n}) &\leq F(\frac{1}{2}.(\frac{1}{n} + \frac{1}{4n(n+1)})) \\
&\leq F(\frac{4n+5}{8n(n+1)}) - F(\frac{1}{4n}) = \ln(\frac{4(n+1)+1}{8n(n+1)}) - \ln(\frac{1}{4n}) \\
&\leq \ln(4n+5) - \ln(8n(n+1)) - \ln(\frac{1}{4n}) \\
&\leq \ln(4n+5) - \ln(8n) - \ln(n+1) + \ln(4n) \\
&\leq \ln(4n+5) - \ln(2) - \ln(4n) - \ln(n+1) + \ln(4n) \\
&\leq \ln(4 + \frac{1}{n+1}) - \ln(2)
\end{aligned}$$

since $\ln(4 + \frac{1}{n+1}) - \ln(2) \geq \ln(4) - \ln(2) = \ln(2) > \frac{1}{2}\ln(2)$ then we can choose $\delta = \frac{1}{2}\ln(2)$ such that (1.10) is satisfied.

Hence T and S have a unique common fixed point in E (here $u = 0$ is the fixed point required).

Remark 2.5. Note that the results existing in the literature are not applicable in Examples 2.2 and 2.4.

Remark 2.6. If we take $\alpha = \beta = 0$ in Theorem 2.1, we obtain Theorem 3.3 in [3] and if we consider $\alpha = \beta = 0, \lambda = \frac{1}{2}$ in Theorem 2.3, we get Theorem 3.4 in [3].

Now we get some corollaries:
Put $T = S$ in Theorem 2.1, we get

Corollary 2.7. *Let (E, d) be a complete metric space; $\mathcal{R}$ a binary relation on E and $T : E \to E$ be a mapping. Assume that*

(1) *$\forall(x, y) \in \mathcal{X}, \exists F \in \mathcal{F}$ such that*

$$\delta + F(d(Tx, Ty)) \leq F(\max\{d(x, y) + \frac{d(y, Tx) \cdot d(x, Ty)}{1 + d(x, y)}; \alpha d(x, Tx); \beta d(y, Ty)\})$$

where $\delta > 0$.

(2) *$E(T, \mathcal{R})$ is nonempty.*

(3) *$\mathcal{R}$ is $T-$closed.*

(4) *T is continuous.*

Then T has a fixed point.

And if we take $T = S$ in Theorem 2.3, we obtain

Corollary 2.8. *Let (E, d) be a complete metric space; $\mathcal{R}$ a transitive binary relation on E and $T : E \to E$ be a mapping. Assume that*

(U_1') *$\forall(x, y) \in \mathcal{X}, \exists\delta > 0, \exists F \in \mathcal{F}$ such that*

$$\delta + F(d(Tx, Ty)) \leq F(\lambda \max\{d(x, y); \frac{d(y, Tx) \cdot d(x, Ty)}{1 + d(x, y)}; d(x, Tx); d(y, Ty)\}).$$

(U_2') $E(T,\mathcal{R}),\Gamma(x,y,\mathcal{R})$ *are non empty,* $\forall(x,y)\in\mathcal{X}$.
(U_3') $\mathcal{R}$ *is* $T-$*closed.*
(U_4') T *is continuous.*

Then T *has a unique fixed point in* E.

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