Thai Journal of Mathematics
Volume 21 Number 1 (2023)
Pages 77–100

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

The Multicomplex Numbers and Their Properties on Some Elementary Functions

Md. Nasiruzzaman[1] **and M. Mursaleen**[2,3,*]

[1] *Department of Mathematics, Faculty of Science, University of Tabuk, PO Box-4279, Tabuk-71491, Saudi Arabia*
e-mail : nasir3489@gmail.com (M. Nasiruzzaman)

[2] *Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

[3] *Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
e-mail : mursaleenm@gmail.com (M. Mursaleen)

Abstract In this paper, we introduce the some algebraic properties in idempotent form of bicomplex space and multicomplex space, which is the generalization of the field of complex numbers. We describe how to define elementary functions such as polynomials, exponential functions, trigonometric functions, Taylor series for multicomplex holomorphic functions, algebra of eigenvalues corresponding to an eigenvector on multicomplex space. Finally, our goal are to show that the functions theory on multicomplex space and multicomplex polynomial are in some sense a better generalization than the bicomplex space and bicomplex Polynomial.

MSC: 41A10; 41A25; 41A36

Keywords: bicomplex numbers; multicomplex number system; multicomplex polynomial; Kronecker products and Kronecker sum on multicomplex space

Submission date: 02.10.2014 / Acceptance date: 14.10.2022

1. Introduction and Preliminaries

There exist several ways to generalize complex numbers to higher dimensions. The most well-known extension is given by the quaternions invented by Hamilton [1] which are mainly used to represent rotations in three-dimensional space. However, quaternions are not commutative in multiplication. Another extension was found at the end of the 19^{th} century by Corrado Segre [2], who described special multidimensional algebras. This type of number is now commonly named a multicomplex number. They were studied in details by G.B.Price [3] and N. Fleury [4]. Bicomplex numbers, just like the quaternions, are a generalization of complex numbers to four real dimensions introduced by C. Segre [2]. These two number systems differ because: (i) Quaternions which form a

*Corresponding author.

Published by The Mathematical Association of Thailand.

division algebra, while bicomplex numbers do not, and (ii) bicomplex numbers are commutative, whereas quaternions are not. For such reasons, the bicomplex numbers system has been shown to be more attractive (compared to the quaternions). These properties of bicomplex numbers are preserved when we define multicomplex numbers as the unique higher dimensional analogues to bicomplex numbers. We begin the present paper with an overview of the structure of the multicomplex space $\mathbb{C}^k$ [3]. For more details we refer to see [5–13].

Importantly, we define some form of idempotent elements, convergent of a multicomplex sequence, multicomplex polynomial, multicomplex derivatives and Taylor series representation, characteristic polynomials and characteristic roost of multicomplex matrices, zeros of characteristic polynomial on multicomplex space, Kronecker products, Kronecker sum and some its applications on multicomplex space and a generalization of its characteristic roots, which will be vital for all future advancements. We are then able to prove certain useful properties of functions on $\mathbb{C}^k$. In this paper, we introduce elementary functions, such as polynomials, exponentials, trigonometric functions, Taylor representation for holomorphic function in this algebra as well as their inverses (something that, incidentally, is not possible in the case of quaternions). We will show how these elementary functions enjoy properties that are very similar to those enjoyed by their complex counterparts. To generalize, the observation consists in looking at maps $f = (f_1, f_2)$ in a open set $U \subset \mathbb{C}^k \to \mathbb{C}^k$ and to ask that each component f_1, f_2 be holomorphic in z_1 and in z_2 without assuming any additional relationship between them. Though both generalizations are important, and give rise to large and interesting theories. We believe that there is another even more appropriate generalization, which so far has not received enough attention (see [14–16]). To this purpose, we introduce to multicomplex Cauchy-Riemann system and to apply it to pairs of holomorphic functions (f_1, f_2) in a open set $U \subset \mathbb{C}^k \to \mathbb{C}^k$, so that the pair (f_1, f_2) can be interpreted as a map of $\mathbb{C}^k$ to itself. It is then natural to ask whether it makes any sense to consider pairs (f_1, f_2) for which the following system is satisfied:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}$$

$$\frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}.$$

The bicomplex polynomial was discussed by M.E. Luna-Elizarrara's and M. Shapiro [15], and the eigenvalues for bicomplex matrices was discussed in [17]. We generalized it for multicomplex space $\mathbb{C}^k$ for which ($k = 2$, Bicomplex polynomial, $k = 3$, Tricomplex polynomial). The algebra which one obtains is the bicomplex algebra. In this paper we show how to introduce elementary functions, such as polynomials, characteristic polynomial functions, zeros of characteristic polynomial, on multicomplex space $\mathbb{C}^k$ and the Kronecker products, Kronecker sum and some of its results was discussed in [18].On multicomplex matrices $\mathbb{C}^k_{m\times n}, \mathbb{C}^k_{p\times q}$ (something that, incidentally, is not possible in the case of quaternions). We will show how these elementary functions enjoy properties that are very similar to those enjoyed by their complex counterparts. If $A := \{(a_{lj}) \in \mathbb{C}^k_{m\times n} = A_1 I_x + A_2 I_y\}$ and $Au = \lambda u$ which is equivalent to

$$\begin{cases} A_1 u_1 = \lambda_1 u_1, \\ A_2 u_2 = \lambda_2 u_2. \end{cases}$$

Then λ is eigenvalue of the multicomplex matrix A corresponding to eigenvector u where $\lambda := \lambda_1 I_x + \lambda_2 I_y \in \mathbb{C}^k$ and $u = u_1 I_x + u_2 I_y$. To generalize the above observation consists in looking at

$$A := (a_{lj}) \in \mathbb{C}^k_{m\times n} = B_{i_k} I_x + C_{i_k} I_y := B_{i_{k-1}} I_x + C_{i_{k-1}} I_y$$

where $B_{i_k}, C_{i_k} \in \mathbb{C}^{k-1}_{m\times n}$ and $B_{i_{k-1}}, C_{i_{k-1}} \in \mathbb{C}^{k-1}_{m\times n}$.
For two matrices A and B, the matrix $A \otimes B$ is the Kronecker product and $A \oplus B$ is Kronecker sum of A and B.

$$A \otimes B = \{(a_{lj}B) \in \mathbb{C}^k_{mp\times nq} \mid A = (a_{lj}) \in \mathbb{C}^k_{m\times n}, B = (b_{rs}) \in \mathbb{C}^k_{p\times q}\}$$

$$A \oplus B = \{(I_m \otimes A) + (B \otimes I_n) \mid A \in \mathbb{C}^k_{n\times n}, B \in \mathbb{C}^k_{m\times m}\}.$$

Without assuming any additional relationship between them, both generalizations are important, and give rise to large and interesting theories, we believe that there is another even more appropriate generalization, which so far has not received enough attention.

2. Bicomplex Numbers

Definition 2.1. ([15, 17]) The set of the bicomplex numbers is defined as

$$\mathbb{BC} := \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}^1(i_1)\} \tag{2.1}$$

where i_1, i_2 are the imaginary units and governed by the rules

$$i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j \tag{2.2}$$

and so,

$$j^2 = 1, i_1 j = j i_1 = -i_2, i_2 j = j i_2 = -i_1. \tag{2.3}$$

Note that

$$\mathbb{C}^1(i_k) := \{x + y i_k \mid i_k^2 = -1 \text{ and } x, y \in \mathbb{R} \text{ for } k = 1, 2\} \tag{2.4}$$

where $\mathbb{C}^1$ is the set of all complex numbers with the imaginary units i_k for $k = 1, 2$. Thus the bicomplex numbers are complex numbers with complex coefficients, which explain the name of bicomplex.
With the addition and the multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring (in fact they are the particular case of the so called multicomplex numbers).
Clearly the bicomplex numbers

$$\mathbb{BC} \cong \mathrm{Cl}_{\mathbb{C}}(1, 0) \cong \mathrm{Cl}_{\mathbb{C}}(0, 1) \tag{2.5}$$

are unique among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{BC} := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 i_1 i_2 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}. \tag{2.6}$$

We know the complex conjugation plays an important role for both algebraic and geometric properties of $\mathbb{C}^1$. So for bicomplex numbers there are three possibilities of conjugations. Let $z \in \mathbb{BC}$ and $z_1, z_2 \in \mathbb{C}^1(i_1)$, such that $z := z_1 + z_2 i_2$, then we define the three conjugation as:

$$z^{\dagger_1} = (z_1 + z_2 i_2)^{\dagger_1} = \overline{z}_1 + \overline{z}_2 i_2 \tag{2.7}$$

$$z^{\dagger_2} = (z_1 + z_2 i_2)^{\dagger_2} = z_1 - z_2 i_2 \tag{2.8}$$

$$z^{\dagger_3} = (z_1 + z_2 i_2)^{\dagger_3} = \overline{z}_1 - \overline{z}_2 i_2. \tag{2.9}$$

All the three kinds of conjugations have some of the standard properties of conjugations, such as

$$(z_1 + z_2)^{\dagger_k} = z_1^{\dagger_k} + z_2^{\dagger_k} \tag{2.10}$$

$$(z_1^{\dagger_k})^{\dagger_k} = z_1 \tag{2.11}$$

$$(z_1.z_2)^{\dagger_k} = z_1^{\dagger_k}.z_2^{\dagger_k}. \tag{2.12}$$

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. Thus the analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}^1(i_1)$ and $z := z_1 + z_2 i_2 \in \mathbb{BC}$, then we have:

$$\mid z \mid_{i_1}^2 = z.z^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}^1(i_1) \tag{2.13}$$

$$\mid z \mid_{i_2}^2 = z.z^{\dagger_1} = (\mid z_1 \mid^2 - \mid z_2 \mid^2) + 2Re(z_1 \overline{z}_2) i_2 \in \mathbb{C}^1(i_2) \tag{2.14}$$

$$\mid z \mid_{j}^2 = z.z^{\dagger_3} = (\mid z_1 \mid^2 + \mid z_2 \mid^2) - 2Im(z_1 \overline{z}_2) j \in \mathbb{D}, \tag{2.15}$$

where $\mathbb{D}$ is the subalgebra of hyperbolic numbers, and is defined as

$$\mathbb{D} := \{x + yj \mid j^2 = 1, x, y \in \mathbb{R}, \} \cong \mathrm{Cl}_{\mathbb{R}}(0,1). \tag{2.16}$$

Note that for $z_1, z_2 \in \mathbb{C}^1(i_1)$ *and* $z := z_1 + z_2 i_2 \in \mathbb{BC}$, we can define the usual (Euclidean in $\mathbb{R}^4$) norm of z as $\mid z \mid = \sqrt{\mid z_1 \mid^2 + \mid z_2 \mid^2} = \sqrt{Re(\mid z \mid_j^2)}$. It is easy to verifying that $z.\frac{z^{\dagger_2}}{|z|_{i_1}^2} = 1$. Hence the inverse of z is given by

$$z^{-1} = \frac{z^{\dagger_2}}{\mid z \mid_{i_1}^2}. \tag{2.17}$$

Idempotent basis

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers e_1 and e_2 defined as $e_1 = \frac{1+i_1 i_2}{2}$, $e_2 = \frac{1-i_1 i_2}{2}$. In fact, e_1 and e_2 are hyperbolic numbers ($i_1 i_2 = i_2 i_1 = j$). They make up the so called idempotent basis of the bicomplex numbers, and one easily can check that

$$e_1^2 = e_1, e_2^2 = e_2, e_1 + e_2 = 1, e_1.e_2 = 0, e_k^{\dagger_3} = e_k \text{ (for } k = 1, 2). \tag{2.18}$$

Thus any bicomplex number can be written as

$$z = z_1 + z_2 i_2 = \alpha_1 e_1 + \alpha_2 e_2, \text{where } \alpha_1 = z_1 - z_2 i_1, \alpha_2 = z_1 + z_2 i_1. \tag{2.19}$$

The idempotent representation for a bicomplex number can be expressed in different ways:

$$\begin{aligned} z &:= \eta_1 + \eta_2 i_1 \mid \eta_1, \eta_2 \in \mathbb{BC}(i_2) = (\eta_1 - \eta_2 i_2)(\frac{1 + i_1 i_2}{2}) + (\eta_1 + \eta_2 i_2)(\frac{1 - i_1 i_2}{2}), \\ &:= \beta_1 + \beta_2 i_2 \mid \beta_1, \beta_2 \in \mathbb{BC}(i_1 i_2) = (\beta_1 - \beta_2 i_1 i_2)(\frac{1 + i_1}{2}) + (\beta_1 + \beta_2 i_1 i_2)(\frac{1 - i_1}{2}), \end{aligned}$$

$$:= \gamma_1 + \gamma_2 i_1 \mid \gamma_1, \gamma_2 \in \mathbb{BC}(i_1 i_2) = (\gamma_1 - \gamma_2 i_1 i_2)(\frac{1+i_2}{2}) + (\gamma_1 + \gamma_2 i_1 i_2)(\frac{1-i_2}{2}),$$

$$:= \nu_1 + \nu_2 i_1 i_2 \mid \nu_1, \nu_2 \in \mathbb{BC}(i_1) = (\nu_1 + \nu_2 i_1)(\frac{1+i_2}{2}) + (\nu_1 - \nu_2 i_1)(\frac{1-i_2}{2}),$$

$$:= \mu_1 + \mu_2 i_1 i_2 \mid \mu_1, \mu_2 \in \mathbb{BC}(i_2) = (\mu_1 + \mu_2 i_2)(\frac{1+i_1}{2}) + (\mu_1 - \mu_2 i_2)(\frac{1-i_1}{2}).$$

3. Multicomplex Numbers

Definition 3.1. ([14, 16]) We must firstly define the multicomplex space in which we have to work, that will do so inductively. For the base case $k = 0$, we define $\mathbb{C}^0 := \mathbb{R}$, that is the set of all real numbers with additions, multiplication and norm being defined as usual. The case for $k = 1$ is also familiar to $\mathbb{C}^1$, which is simply the standard complex plane with arithmetic and norm usually defined. The case of $k = 2$ and $k = 3$ are familiar with $\mathbb{C}^2$ and $\mathbb{C}^3$ are the simply bicomplex plane and tricomplex plane. So we define

$$\mathbb{C}^k := \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}, k > 1, i_k^2 = -1 \textit{ and } i_m i_n = i_n i_m \text{ for } m \neq n\}. \tag{3.1}$$

The arithmetic is defined in usual way and if z_1, z_2, z_3 and $z_4 \in \mathbb{C}^{k-1}$ and w_1, w_2 and $w_3 \in \mathbb{C}^k$, then

$$(z_1 + z_2 i_k) + (z_3 + z_4 i_k) = (z_1 + z_3) + (z_2 + z_4) i_k \tag{3.2}$$

$$(z_1 + z_2 i_k)(z_3 + z_4 i_k) = (z_1 z_3 - z_2 z_4) + (z_1 z_4 + z_2 z_3) i_k \tag{3.3}$$

$$w_1(w_2 + w_3) = w_1 w_2 + w_1 w_3. \tag{3.4}$$

With this definition it is simple to show that for all natural numbers k,$\mathbb{C}^k$ is a commutative ring with unity. Further, assuming have defined the norm
$\| \, . \, \|_{k-1}$: $\mathbb{C}^{k-1} \to \mathbb{R}_{\geq 0}$, we define the norm $\| \, . \, \|_k$: $\mathbb{C}^k \to \mathbb{R}_{\geq 0}$ by

$$\| z_1 + z_2 i_k \|_k^2 = \| z_1 \|_{k-1}^2 + \| z_2 \|_{k-1}^2, \tag{3.5}$$

with this definition of the norm, the space $\mathbb{C}^k$ becomes a modified Banach algebra.
Other useful representations of the multicomplex numbers can be found by repetitively applying to the multicomplex coefficients of lower dimension, that is decomposing z_1 and z_2 into lower dimension repetitively. We obtain

$$\mathbb{C}^k := \{z_{11} + z_{12} i_{k-1} + z_{21} i_k + z_{22} i_k i_{k-1} \mid z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{C}^{k-2}\}. \tag{3.6}$$

For any $x_0, \cdots, x_k, \cdots, x_{1\cdots k} \in \mathbb{R}$ we get

$$\mathbb{C}^k := \{x_0 + x_1 i_1 + \cdots + x_k i_k + x_{12} i_1 i_2 + \cdots + x_{k-1k} i_{k-1} i_k + \cdots + x_{1\cdots k} i_1 \cdots i_k \mid\}. \tag{3.7}$$

It is clear that we can represent each element of $\mathbb{C}^k$ with $\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k}$ {where $\binom{k}{r} = \frac{k!}{r!(k-r)!}$}, coefficients in $\mathbb{R}$. One coefficients x_0 for the real part k, and coefficients $x_1, \cdots, x_k$ for the pure imaginary directions and additional coefficients corresponding to 'cross coupled'imaginary directions. We note that the cross directions do not exit in $\mathbb{R}$ or $\mathbb{C}$, but appear only in $\mathbb{C}^k$ for $k \geq 2$.

The multicomplex space for $k \geq 2$ has many idempotents elements, that is elements I with the property that $I^2 = I$

$$I_1 = \frac{1 + i_k i_{k-1}}{2} \text{ and } I_2 = \frac{1 - i_k i_{k-1}}{2} \tag{3.8}$$

$$I_1^2 = (\frac{1 + i_k i_{k-1}}{2})^2 = \frac{1 + i_k i_{k-1}}{2} = I_1 \tag{3.9}$$

$$I_2^2 = (\frac{1 - i_k i_{k-1}}{2})^2 = \frac{1 - i_k i_{k-1}}{2} = I_2 \tag{3.10}$$

$$I_1 + I_2 = (\frac{1 + i_k i_{k-1}}{2}) + (\frac{1 - i_k i_{k-1}}{2}) = 1 \tag{3.11}$$

$$I_1 I_2 = (\frac{1 + i_k i_{k-1}}{2})(\frac{1 - i_k i_{k-1}}{2}) = 0. \tag{3.12}$$

Thus we define a multicomplex number can be written in six different ways:

$$\begin{aligned}
\mathbb{C}^k &= (x_1 + y_1 i_{k-1}) + (x_2 + y_2 i_{k-1}) i_k = z_1 + z_2 i_k = (z_1 - z_2 i_{k-1}) I_1 + (z_1 + z_2 i_{k-1}) I_2 \\
&= (x_1 + x_2 i_k) + (y_1 + y_2 i_k) i_{k-1} = \eta_1 + \eta_2 i_{k-1} = (\eta_1 - \eta_2 i_k) I_1 + (\eta_1 + \eta_2 i_k) I_2 \\
&= (x_1 + y_2 i_k i_{k-1}) + (x_2 - y_1 i_k i_{k-1}) i_k = \beta_1 + \beta_2 i_k \\
&= (\beta_1 - \beta_2 i_k i_{k-1})(\frac{1 + i_{k-1}}{2}) + (\beta_1 + \beta_2 i_k i_{k-1})(\frac{1 - i_{k-1}}{2}) \\
&= (x_1 + y_2 i_k i_{k-1}) + (y_1 - x_2 i_k i_{k-1}) i_{k-1} = \gamma_1 + \gamma_2 i_{k-1} \\
&= (\gamma_1 - \gamma_2 i_k i_{k-1})(\frac{1 + i_k}{2}) + (\gamma_1 + \gamma_2 i_k i_{k-1})(\frac{1 - i_k}{2}) \\
&= (x_1 + y_1 i_{k-1}) + (y_2 - x_2 i_{k-1}) i_k i_{k-1} = \nu_1 + \nu_2 i_k i_{k-1} \\
&= (\nu_1 + \nu_2 i_{k-1})(\frac{1 + i_k}{2}) + (\nu_1 - \nu_2 i_{k-1})(\frac{1 - i_k}{2}) \\
&= (x_1 + x_2 i_k) + (y_2 - y_1 i_k) i_k i_{k-1} = \mu_1 + \mu_2 i_k i_{k-1} \\
&= (\mu_1 + \mu_2 i_k)(\frac{1 + i_{k-1}}{2}) + (\mu_1 - \mu_2 i_k)(\frac{1 - i_{k-1}}{2}).
\end{aligned}$$

We define the definition given below in multicomplex space, in which if we put $k = 2$ definition from [15].

Definition 3.2. Let $w_n = \{\alpha_{1,n} I_1 + \alpha_{2,n} I_2 \mid \alpha_{1,n}, \alpha_{2,n} \in \mathbb{C}^{k-1} \text{ and } w_n \in \mathbb{C}^k \text{ for } n \geq 1\}$ be a sequence of multicomplex numbers then the sequence $\{w_n\}_{n \geq 1}$ is said to be convergent component wise if the sequences $\{\alpha_{1,n}\}$ and $\{\alpha_{2,n}\}$ in $\mathbb{C}^{k-1}$ are convergent in $\mathbb{C}^{k-1}$ to the numbers $\alpha_{1,0}$ and $\alpha_{2,0}$ where $\alpha_{1,0}, \alpha_{2,0} \in \mathbb{C}^{k-1}$, hence we can write $w_n \to w_0 := \alpha_{1,0} I_1 + \alpha_{2,0} I_2$ and we say that w_n has limit w_0.

Theorem 3.3 (Theorem 3, [14])**.** *Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$, then for all $z_1, z_2 \in \mathbb{C}^{k-1}$ and $w \in \mathbb{C}^k$, the following hold :*

$$e^{z_1 + z_2 i_k} = e^{z_1} e^{z_2 i_k} \tag{3.13}$$

$$e^{z_1 i_k} = \cos(z_1) + i_k \sin(z_1) \tag{3.14}$$

$$\cos(-w) := \cos(w) \tag{3.15}$$

$$\sin(-w) := -\sin(w). \tag{3.16}$$

We define the following theorem and corollaries given below in multicomplex space, in which if we put $k = 2$ we get, (cf. [2]).

Theorem 3.4. *Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$be any multicomplex numbers, then the sequence $w_n := (1 + \frac{w}{n})^n$ is convergent to $e^{z_1}(\cos(z_2) + i_k \sin(z_2))$ as $(n \to \infty)$.*

Proof. We have

$$w = z_1 + z_2 i_k = (z_1 - z_2 i_{k-1})I_1 + (z_1 + z_2 i_{k-1})I_2 = \alpha_1 I_1 + \alpha_2 I_2,$$

then,

$$(1 + \frac{w}{n})^n := (1 + \frac{\alpha_1}{n})^n I_1 + (1 + \frac{\alpha_2}{n})^n I_2$$

$$\begin{aligned}\lim_{n\to\infty}(1 + \frac{w}{n})^n &:= \lim_{n\to\infty}(1 + \frac{\alpha_1}{n})^n I_1 + \lim_{n\to\infty}(1 + \frac{\alpha_2}{n})^n I_2 \\ &= \frac{1}{2}(e^{\alpha_1} + e^{\alpha_2}) + \frac{i_k i_{k-1}}{2}(e^{\alpha_1} - e^{\alpha_2}) \\ &= e^{z_1}\{\frac{1}{2}(e^{-i_{k-1}z_2} + e^{i_{k-1}z_2}) + \frac{i_k i_{k-1}}{2}(e^{-i_{k-1}z_2} - e^{i_{k-1}z_2})\} \\ &= e^{z_1}(\cos(z_2) + i_k \sin(z_2)),\end{aligned}$$

thus

$$e^{z_1 + z_2 i_k} = \lim_{n\to\infty}(1 + \frac{w}{n})^n = e^{z_1}(\cos(z_2) + i_k \sin(z_2)). \tag{3.17}$$

■

Corollary 3.5. *If $e^{z_1 i_k} = \cos(z_1) + i_k \sin(z_1)$ and $e^{-z_1 i_k} = \cos(z_1) - i_k \sin(z_1)$. Then,*

$$\cos(z_1) := \frac{e^{z_1 i_k} + e^{-z_1 i_k}}{2} \text{ and } \sin(z_1) := \frac{e^{z_1 i_k} - e^{-z_1 i_k}}{2 i_k}.$$

Proof. Simply on adding and subtracting we get desired result. ■

We define the sine and cosine formulae and transition formula in multicomplex space as.

Definition 3.6. Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$. Then cosine and sine formulae for the multicomplex space are defined as

$$\cos(w) := \frac{e^{w i_k} + e^{-w i_k}}{2} \tag{3.18}$$

$$\sin(w) := \frac{e^{w i_k} - e^{-w i_k}}{2 i_k}. \tag{3.19}$$

Corollary 3.7. *Let I_1 and I_2 be the basis for the multicomplex space $\mathbb{C}^k$, where $I_1 = \frac{1+i_k i_{k-1}}{2}$ and $I_2 = \frac{1-i_k i_{k-1}}{2}$ and $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\} = (z_1 - z_2 i_{k-1})I_1 + (z_1 + z_2 i_{k-1})I_2$. Then the transition formula for multicomplex space as*

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2i_{k-1}} & \frac{1}{2i_{k-1}} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

where $\alpha_1 := (z_1 - z_2 i_{k-1})$ and $\alpha_2 := (z_1 + z_2 i_{k-1})$.

We prove the following theorem given below.

Theorem 3.8. *Let $I_1 = \frac{1+i_k i_{k-1}}{2}$ and $I_2 = \frac{1-i_k i_{k-1}}{2}$. Then*

$$\cos(\frac{i_k}{2}) := \frac{1}{2} e^{-\frac{1}{2}} (e^{I_1} + e^{I_2}) \tag{3.20}$$

$$\sin(\frac{i_k}{2}) := \frac{1}{2i_{k-1}} e^{-\frac{1}{2}} (e^{I_1} - e^{I_2}). \tag{3.21}$$

Proof. $I_1 = 1.I_1 + 0.I_2, I_2 = 0.I_1 + 1.I_2$, then we have

$$e^{I_1} = e.I_1 + 1.I_2 = e^{\frac{1+i_k i_{k-1}}{2}} = e^{\frac{1}{2}} (\cos(\frac{i_k}{2}) + i_{k-1} \sin(\frac{i_k}{2})) \tag{3.22}$$

$$e^{I_2} = 1.I_1 + e.I_2 = e^{\frac{1+i_k i_{k-1}}{2}} = e^{\frac{1}{2}} (\cos(\frac{i_k}{2}) - i_{k-1} \sin(\frac{i_k}{2})) \tag{3.23}$$

■

Theorem 3.9. *Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$ be any multicomplex number. Then*

$$\lim_{w \to \lambda} \frac{w^n - \lambda^n}{w - \lambda} = n\lambda^{n-1}.$$

Proof. Let $w := \alpha_1 I_1 + \alpha_2 I_2$ and $\lambda := \lambda_1 I_1 + \lambda_2 I_2$, where I_1, I_2 as idempotent basis.

$$\text{Then } w^n := \alpha_1^n I_1 + \alpha_2^n I_2 \textit{ and } \lambda^n := \lambda_1^n I_1 + \lambda_2^n I_2$$
$$\text{where } \lambda := \{\psi_1 + \psi_2 i_k \mid \psi_1, \psi_2 \in \mathbb{C}^{k-1}\},$$
$$\text{and } \lambda_1 := \psi_1 - \psi_2 i_{k-1}, \lambda_2 := \psi_1 + \psi_2 i_{k-1},$$

Now

$$\begin{aligned} \lim_{w \to \lambda} \frac{w^n - \lambda^n}{w - \lambda} &:= \lim_{w \to \lambda} \frac{(\alpha_1^n - \lambda_1^n)I_1 + (\alpha_2^n - \lambda_2^n)I_2}{(\alpha_1 - \lambda_1)I_1 + (\alpha_2 - \lambda_2)I_2} \\ &:= \lim_{w \to \lambda} \{(\frac{\alpha_1^n - \lambda_1^n}{\alpha_1 - \lambda_1})I_1 + (\frac{\alpha_2^n - \lambda_2^n}{\alpha_2 - \lambda_2})I_2\} \\ &:= \lim_{\alpha_1 \to \lambda_1} (\frac{\alpha_1^n - \lambda_1^n}{\alpha_1 - \lambda_1})I_1 + \lim_{\alpha_2 \to \lambda_2} (\frac{\alpha_2^n - \lambda_2^n}{\alpha_2 - \lambda_2})I_2 \\ &:= n\lambda_1^{n-1} I_1 + n\lambda_2^{n-1} I_2 = n\lambda^{n-1} \end{aligned}$$

$$\lim_{w \to \lambda} \frac{w^n - \lambda^n}{w - \lambda} = n\lambda^{n-1}. \tag{3.24}$$

■

We define the sine and cosine formulae for hyperbolic functions in multicomplex space and prove the theorem given below.

Definition 3.10. Let $w := z_1 + z_2 i_k \mid z_z, z_2 \in \mathbb{C}^{k-1}$ be a multicomplex number. Then hyperbolic sine and cosine functions for multicomplex variable are defined as

$$\sin hw := \frac{e^w - e^{-w}}{2} \tag{3.25}$$

$$\cos hw := \frac{e^w + e^{-w}}{2} \tag{3.26}$$

We prove the following theorem.

Theorem 3.11. *Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$. Then the following hold:*

$$\sin w := \sin(z_1 + z_2 i_k) := \sin z_1 \cos hz_2 + i_k \cos z_1 \sin hz_2 \tag{3.27}$$

$$\cos w := \cos(z_1 + z_2 i_k) := \cos z_1 \cos hz_2 - i_k \sin z_1 \sin hz_2 \tag{3.28}$$

$$\sin 2w := 2 \sin w \cos w \tag{3.29}$$

$$\cos 2w := \cos^2 w - \sin^2 w. \tag{3.30}$$

Proof. We have

$$\sin w := \frac{1}{2i_k}(e^{wi_k} - e^{-wi_k})$$
$$= \frac{1}{2i_k}(e^{(z_1+z_2 i_k)i_k} - e^{-(z_1+z_2 i_k)i_k})$$
$$= \frac{1}{2i_k}(e^{z_1 i_k - z_2} - e^{-z_1 i_k + z_2})$$
$$= \frac{1}{2i_k}(e^{-z_2}(\cos z_1 + i_k \sin z_1) - e^{z_2}(\cos z_1 - i_k \sin z_1))$$
$$= \sin z_1 (\frac{e^{z_2} + e^{-z_2}}{2}) + i_k \cos z_1 (\frac{e^{z_2} - e^{-z_2}}{2})$$
$$= \sin z_1 \cos hz_2 + i_k \cos z_1 \sin hz_2,$$

and

$$\cos w := \frac{1}{2}(e^{wi_k} + e^{-wi_k})$$
$$= \frac{1}{2}(e^{(z_1+z_2 i_k)i_k} + e^{-(z_1+z_2 i_k)i_k})$$
$$= \frac{1}{2}(e^{z_1 i_k - z_2} + e^{-z_1 i_k + z_2})$$
$$= \frac{1}{2}(e^{-z_2}(\cos z_1 + i_k \sin z_1) + e^{z_2}(\cos z_1 - i_k \sin z_1))$$
$$= \cos z_1 (\frac{e^{z_2} + e^{-z_2}}{2}) - i_k \sin z_1 (\frac{e^{z_2} - e^{-z_2}}{2})$$
$$\cos(z_1 + z_2 i_k) := \cos z_1 \cos hz_2 - i_k \sin z_1 \sin hz_2.$$

Similarly we can obtain the formula for $\sin 2w$ and $\cos 2w$ using the following theorem. ■

Theorem 3.12. *Let* $w_1 = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$ *and* $w_2 = \{z_3 + z_4 i_k \mid z_3, z_4 \in \mathbb{C}^{k-1}\}$ *be any two multicomplex numbers. Then the following hold:*

$$\sin(w_1 + w_2) := \sin w_1 \cos w_2 + \cos w_1 \sin w_2 \tag{3.31}$$

$$\cos(w_1 + w_2) := \cos w_1 \cos w_2 - \sin w_1 \sin w_2 \tag{3.32}$$

Proof.

$$\cos w_1 \cos w_2 := \frac{1}{4}\{e^{i_k(w_1+w_2)} + e^{i_k(w_1-w_2)} + e^{i_k(w_2-w_1)} + e^{-i_k(w_1+w_2)}\} \tag{3.33}$$

$$-\sin w_1 \sin w_2 := \frac{1}{4}\{e^{i_k(w_1+w_2)} - e^{i_k(w_1-w_2)} - e^{i_k(w_2-w_1)} + e^{-i_k(w_1+w_2)}\}, \tag{3.34}$$

on adding these expressions we get:

$$\cos w_1 \cos w_2 - \sin w_1 \sin w_2 = \frac{1}{2}\{e^{i_k(w_1+w_2)} + e^{-i_k(w_1+w_2)}\} = \cos(w_1 + w_2).$$

■

4. Multicomplex Polynomial

We define the definition given below in multicomplex space, in which if we put $k = 2$, bicomplex polynomial (see [15]).

Definition 4.1. Let $w = z_1 + z_2 i_k = \alpha_1 I_1 + \alpha_2 I_2$ be a multicomplex number, where $\alpha_1 = (z_1 - z_2 i_{k-1}), \alpha_2 = (z_1 + z_2 i_{k-1})$ and I_1, I_2 are idempotent basis and let $A_p := \delta_p I_1 + \gamma_p I_2$ be multicomplex coefficients for $p = 0, \cdots, n$. Then $f(w) := \sum_{p=0}^{n} A_p w^p$ is called the multicomplex polynomial and written as

$$f(w) := \sum_{p=0}^{n}(\delta_p \alpha_1^p) I_1 + \sum_{p=0}^{n}(\gamma_p \alpha_2^p) I_2 = f_1(\alpha_1) I_1 + f_2(\alpha_2) I_2.$$

If we denote the set of all r_1 and r_2 distinct roots of $f_1(\alpha_1)$ and $f_2(\alpha_2)$ by ξ_1 and ξ_2, and if we denote by ξ the set of all distinct roots of polynomial $f(w)$, then $f(w)$ has $r_1.r_2$ distinct roots and it is easy to see that $\xi := \xi_1 I_1 + \xi_2 I_2$ and so the structure of the zero set of a multicomplex polynomial $f(w)$ of degree n is fully described by the following lemma.

(i) If both the polynomials $f_1(\alpha_1)$ and $f_2(\alpha_2)$ are of degree at least one, and if $\xi_1 = \{\mu_1, \cdots, \mu_\sigma\}$ has r_1 distint roots and $\xi_2 = \{\nu_1, \cdots, \nu_\tau\}$ has r_2 distinct roots, then the set of the distinct roots of f is given by

$$\xi := w_{s,t} = \mu_s I_1 + \nu_t I_2 \mid s = 1, \cdots, \sigma, \text{ and } t = 1, \cdots, \tau.$$

Example 4.2. Let $f(w) = w^3 - 8$, where $w \in \mathbb{C}^k$. Then we have $f_1(\alpha_1) = \alpha_1^3 - 8$ and $f_2(\alpha_2) = \alpha_2^3 - 8$ the set of zeros of f_1 and f_2 are, respectively

$$\xi_1 := \{\mu_1 = 2, \mu_2 = -1 + i_{k-1}\sqrt{3}, \mu_3 = -1 - i_{k-1}\sqrt{3}\}$$

$$\xi_2 := \{\nu_1 = 2, \nu_2 = -1 + i_{k-1}\sqrt{3}, \nu_3 = -1 - i_{k-1}\sqrt{3}\}$$

then the set of solutions of f is
$\xi := \{w_{s,t} = \mu_s I_1 + \nu_t I_2 \mid s,t = 1,2,3\}$, which has 9 distinct roots

$$\begin{aligned}\xi : \quad = \quad & \Big\{2, (\frac{1+i_{k-1}\sqrt{3}}{2}) + (\frac{3i_{k-1}+\sqrt{3}}{2})i_k, (\frac{1-i_{k-1}\sqrt{3}}{2}) + (\frac{3i_{k-1}-\sqrt{3}}{2})i_k, \\ & (\frac{1-i_{k-1}\sqrt{3}}{2}) + (\frac{-3i_{k-1}-\sqrt{3}}{2})i_k, -1+i_{k-1}\sqrt{3}, -1-i_k\sqrt{3}, \\ & (\frac{1-i_{k-1}\sqrt{3}}{2}) + (\frac{-3i_{k-1}+\sqrt{3}}{2})i_k, -1+i_k\sqrt{3}, -1-i_{k-1}\sqrt{3}\Big\}.\end{aligned}$$

Example 4.3. Let $f(w) := (\frac{1+i_k i_{k-1}}{2})w^5 + \{(-1-4i_{k-1}) + (4-2i_{k-1})i_k\}w^4 + \{(-11+6i_{k-1})-(12+11i_{k-1})i_k\}w^3+\{(\frac{29+26i_{k-1}}{2})+(\frac{-26+47i_{k-1}}{2})i_k\}w^2+\{(\frac{13-34i_{k-1}}{2}) + (\frac{34+13i_{k-1}}{2})i_k\}w + (\frac{-11-2i_{k-1}}{2}) + (\frac{2-11i_{k-1}}{2})$. Then
we have

$$f(w) := f_1(\alpha_1)I_1 + f_2(\alpha_2)I_2$$

where

$$I_1 := (\frac{1+i_k i_{k-1}}{2}),\ I_2 := (\frac{1-i_k i_{k-1}}{2})$$

$$\begin{aligned}& f_1(\alpha_1) := \alpha_1^5 + (-3-8i_{k-1})\alpha_1^4 + (-22+18i_{k-1})\alpha_1^3 + (38+26i_{k-1})\alpha_1^2 \\ & +(13-34i_{k-1})\alpha_1 + (-11-2i_{k-1}) \\ & f_2(\alpha_2) := \alpha_2^4 - 6i_{k-1}\alpha_2^3 - 9\alpha_2^2 \\ & \xi_1 := \{\mu_1 = i_{k-1}, \mu_2 = 1+2i_{k-1}\} \\ & \xi_2 := \{\nu_1 = 0, \nu_2 = 3i_{k-1}\} \\ & \xi := \{W_{s,t} = \mu_s I_1 + \nu_t I_2 \mid s,t = 1,2\},\end{aligned}$$

has 4 distinct roots.

$$\xi := \{\frac{i_{k-1}-i_k}{2}, 2i_{k-1}+i_k, (\frac{1+2i_{k-1}}{2}) + (\frac{-2+i_{k-1}}{2})i_k, (\frac{1+5i_{k-1}}{2}) + (\frac{1+i_{k-1}}{2})i_k\}.$$

(ii) If $f_1(\alpha_1) = 0$, then $\xi_1 = \mathbb{C}^{k-1}$ and $\xi_2 = \{\nu_1, \cdots, \nu_\tau\}$, where $\tau \leq n$; and $\xi := w_t = \omega I_1 + \nu_t I_2 \mid \omega \in \mathbb{C}^{k-1}, t = 1, \cdots, \tau$. If $f_2(\alpha_2) = 0$, then $\xi_2 = \mathbb{C}^{k-1}$ and $\xi_1 = \{\mu_1, \cdots, \mu_\sigma\}$, where $\sigma \leq n$; and

$$\xi := w_s = \mu_s I_1 + \omega I_2 \mid \omega \in \mathbb{C}^{k-1}, s = 1, \cdots, \sigma.$$

Example 4.4. Let $f(w) := (1 - i_k i_{k-1})w^2 + i_k - i_{k-1}$. Then
we have

$$f(w) := f_1(\alpha_1)I_1 + f_2(\alpha_2)I_2$$

where

$$I_1 := (\frac{1+i_k i_{k-1}}{2}),\ I_2 := (\frac{1-i_k i_{k-1}}{2})$$

$$f_1(\alpha_1) := 2(\alpha_1^2 - i_{k-1})I_1$$

$$f_1(\alpha_1) := 0$$

$$\xi := w_s = \mu_s I_1 + \omega I_2 = \{\pm(\frac{1+i_{k-1}}{\sqrt{2}})I_1 + \omega I_2 \mid \omega \in \mathbb{C}^{k-1}(\omega = \sqrt{i_{k-1}})\}.$$

(iii) If all the coefficients A_p with the exception of $A_0 = \delta_0 I_1 + \gamma_0 I_2$ are not multicomplex multiples of I_1 (respectively I_2), but A_0 has $\gamma_0 \neq 0$ (respectively $\delta_0 \neq 0$), then polynomial f has no root.

Example 4.5. Let $f(w) := (1 - i_k i_{k-1})w^2 + 1 + i_k - i_{k-1} - i_k i_{k-1}$. Then we have

$$f(w) := f_1(\alpha_1)I_1 + f_2(\alpha_2)I_2$$

where

$$f_1(\alpha_1) = 2(\alpha_1^2 - i_{k-1}) \text{ and } f_2(\alpha_2) = 2,$$

clearly polynomial has no root.

(iv) (Analogue of Fundamental Theorem of Algebra for Multicomplex Polynomials)
Let $f(w) := \sum_{p=0}^{n} A_p w^p$ be multicomplex Polynomial, where $A_p := \delta_p I_1 + \gamma_p I_2$, and $w^p = \alpha_1^p I_1 + \alpha_2^p I_2$, with $\alpha_1 = (z_1 - z_2 i_{k-1}), \alpha_2 = (z_1 + z_2 i_{k-1})$. If all the coefficients A_p with the exception of $A_0 = \delta_0 I_1 + \gamma_0 I_2$ are not multicomplex multiples of I_1 (respectively I_2), but A_0 has $\gamma_0 \neq 0$ (respectively $\delta_0 \neq 0$), then polynomial f has no root. In all other cases f has at least one root.

Remark 4.6. Let $w = z_1 + z_2 i_k = \alpha_1 I_1 + \alpha_2 I_2$ be a multicomplex number and $f(w) := \sum_{p=0}^{n} A_p w^p$ be any multicomplex polynomial. If we put $k = 2$, then it becomes Bicomplex polynomial, and if put $k = 3$, then becomes tricomplex polynomial.

Remark 4.7. A multicomplex polynomial may not have a unique factorization into linear polynomials.

Example 4.8. Let $f(w) := w^3 + 1$. Then we have

$$f_1(\alpha_1) = \alpha_1^3 + 1, f_2(\alpha_2) = \alpha_2^3 + 1$$

$$\xi_1 = \{\mu_1 = -1, \mu_2 = \frac{1 + \sqrt{3} i_k}{2}, \mu_3 = \frac{1 - \sqrt{3} i_k}{2}\}$$

$$\xi_2 = \{\nu_1 = -1, \nu_2 = \frac{1 + \sqrt{3} i_k}{2}, \nu_3 = \frac{1 - \sqrt{3} i_k}{2}\}$$

$$\xi := w_{s,t} = \mu_s I_1 + \nu_t I_2 \mid s, t = 1, 2, 3$$

$$w^3 + 1 := (w + 1)(w - \frac{1}{2} - \frac{\sqrt{3}}{2} i_k)(w - \frac{1}{2} + \frac{\sqrt{3}}{2} i_k)$$

$$w^3 + 1 := (w + 1)(w - \frac{1}{2} - \frac{\sqrt{3}}{2} i_{k-1})(w - \frac{1}{2} + \frac{\sqrt{3}}{2} i_{k-1}).$$

Note: It is clear from what we have indicated that the multicomplex polynomials do not satisfy the Fundamental theorem of algebra in its original form.

Theorem 4.9 (Theorem 2, [16]). *Let $w = z_1 + z_2 i_k$ be any multicomplex number. Then the functions* sin, cos *and exponential in the form of the power series is defined as*

$$\sin w := \sin(z_1 + z_2 i_k) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z_1 + z_2 i_k)^{2n-1}}{(2n-1)!} \tag{4.1}$$

$$\cos w := \cos(z_1 + z_2 i_k) := \sum_{n=0}^{\infty} \frac{(-1)^{n}(z_1 + z_2 i_k)^{2n}}{(2n)!} \tag{4.2}$$

$$\exp(w) := e^{z_1+z_2 i_k} = \sum_{n=0}^{\infty} \frac{(z_1+z_2 i_k)^n}{(n)!}. \tag{4.3}$$

Definition 4.10 (Definition 1, [14])**.** A function $f : \mathbb{C}^k \to \mathbb{C}^k$ is said to be multicomplex differentiable at $w_0 \in \mathbb{C}^k$ if the limit

$$\lim_{w\to w_0} \frac{f(w)-f(w_0)}{w-w_0} \tag{4.4}$$

exists. This limit is called first derivative of f at w_0 and will be denoted by $f'(w_0)$.

Definition 4.11 (Definition 2, [14])**.** A function f is said to be holomorphic in an open set $U \subset \mathbb{C}^k$ if $f'(w)$ exits for all $w \in U$.
This definition is not very restrictive, most usual functions are holomorphic in $\mathbb{C}^k$. Examples of the non holomorphic functions are the modulus and absolute value functions at zero.

Theorem 4.12 (Theorem 2, [14])**.** *Let $w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$ be a multicomplex number and f be a function such that $f : U \subset \mathbb{C}^k \to \mathbb{C}^k$ defined by*

$$f(z_1 + z_2 i_k) = f_1(z_1, z_2) + f_2(z_1, z_2) i_k. \tag{4.5}$$

Then the following are equivalent,

(*i*) *f is holomorphic in U,*
(*ii*) *f_1 and f_2 are holomorphic in z_1 and z_2 and satisfy the multicomplex Cauchy-Riemann equations:*

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}, \tag{4.6}$$

(*iii*) *f can be represented, near every point $w_0 \in U$, by Taylor series.*

We prove the theorem given below.

Theorem 4.13. *Let $w = z_1 + z_2 i_k$ a multicomplex number, and $\exp(w), \sin w$ and $\cos w$ are holomorphic in an open set $U \subset \mathbb{C}^k$. Then the following hold:*

$$\frac{\mathrm{d}}{\mathrm{d}w} w^n := n w^{n-1} \text{ where } n \in \mathbb{N} \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}w} \exp(w) := \exp(w) \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}w} \sin w := \cos w \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}w} \cos w := -\sin w. \tag{4.10}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}w} \exp(w) := \frac{\mathrm{d}}{\mathrm{d}w} \sum_{n=0}^{\infty} \frac{1}{n!} w^n = \sum_{n=1}^{\infty} \frac{n}{n!} w^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} w^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} w^{n} = \exp(w)$$

$$\frac{\mathrm{d}}{\mathrm{d}w} \sin w := \frac{\mathrm{d}}{\mathrm{d}w} (\sum_{n=0}^{\infty} \frac{(-1)^n (z_1 + z_2 i_k)^{2n+1}}{(2n+1)!})$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(z_1 + z_2 i_k)^{2n}}{(2n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (z_1 + z_2 i_k)^{2n}}{(2n)!} = \cos w.$$

Similarly we can prove,

$$\frac{\mathrm{d}}{\mathrm{d}w} \cos w := -\sin w$$

$$\frac{\mathrm{d}}{\mathrm{d}w} w^n := n w^{n-1}.$$

■

Definition 4.14 (Definition 3, [14]). Let $\mathbb{C}^k := w = \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}\}$ be a multicomplex number, and let $f : U \subset \mathbb{C}^k \to \mathbb{C}^k$ be a multicomplex holomorphic function in U. Then f can be expanded in a Taylor series about a real point a as follows:

$$f(a + hi_1 + \cdots + hi_k) := f(a) + h(i_1 + \cdots + i_k) f'(a) + h^2 (i_1 + \cdots + i_k)^2 \frac{f''(a)}{2} + \cdots$$

$$+ h^n (i_1 + \cdots + i_k)^n \frac{f^{(n)}(a)}{n!} + h^{n+1} (i_1 + \cdots + i_k)^{n+1} \frac{f^{(n+1)}(a)}{(n+1)!} + O(h^{(n+2)}) \quad (4.11)$$

where f^n denotes the n^{th} order derivative, and

$$(i_1 + \cdots + i_k)^n := \sum_{\substack{x_1, x_2, \cdots, x_k \\ x_1 + x_2 + \cdots + x_k = n}} \frac{n!}{x_1! \cdots x_k!} i_1^{x_1} \cdots i_k^{x_k}. \quad (4.12)$$

Using the above definition we can prove the theorem given below.

Theorem 4.15. *Let $f, g : U \subset \mathbb{C}^k \to \mathbb{C}^k$ be multicomplex holomorphic functions in U and if $f(a) = 0$ and $g(a) = 0$, but $g'(a) \neq 0$. Then*

$$\lim_{w \to a} \frac{f(w)}{g(w)} = \frac{f'(a)}{g'(a)} \quad (4.13)$$

and hence, in general, if $f^n(a) = 0 = g^n(a)$, but $g^{(n+1)}(a) \neq 0$. Then

$$\lim_{w \to a} \frac{f(w)}{g(w)} = \frac{f^{n+1}(a)}{g^{n+1}(a)}. \quad (4.14)$$

Proof. From Taylor series we have,

$$f(a+h(i_1+\cdots+i_k)) := \sum_{r=0}^{n+1} h^r(i_1+\cdots+i_k)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)}) \tag{4.15}$$

$$g(a+h(i_1+\cdots+i_k)) := \sum_{r=0}^{n+1} h^r(i_1+\cdots+i_k)^r \frac{g^{(r)}(a)}{r!} + O(h^{(n+2)}).$$

Put $a+h(i_1+\cdots+i_k)=w$.
Then $h(i_1+\cdots+i_k)=w-a$.

$$f(w) := \sum_{r=0}^{n+1}(w-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})$$

$$g(w) := \sum_{r=0}^{n+1}(w-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})$$

$$\frac{f(w)}{g(w)} := \frac{\sum_{r=0}^{n+1}(w-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})}{\sum_{r=0}^{n+1}(w-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})}.$$

If $f(a)=0=g(a)$, but $g'(a)\neq 0$. Then

$$\lim_{w\to a}\frac{f(w)}{g(w)} = \frac{f'(a)}{g'(a)}.$$

If $f'(a)=0=g'(a)$, but $g''(a)\neq 0$, then

$$\lim_{w\to a}\frac{f(w)}{g(w)} = \frac{f''(a)}{g''(a)}.$$

Hence in general, if $f^n(a)=0=g^n(a)$, but $g^{(n+1)}(a)\neq 0$. Then

$$\lim_{w\to a}\frac{f(w)}{g(w)} = \frac{f^{n+1}(a)}{g^{n+1}(a)}.$$

■

5. Multicomplex Matrices

We define the definition for the multicomplex matrices given below, in which if we put $k=2$ definition for bicomplex matrices (see [17]).

Definition 5.1 (Multicomplex Matrices). The set of $m\times n$ matrices $\mathbb{C}^k_{m\times n}$ with multicomplex entries, is denoted as $A := \{(a_{lj}) \in \mathbb{C}^k_{m\times n}, 1\leq l\leq m, 1\leq j\leq n\} = B_{i_k}I_x + C_{i_k}I_y := B_{i_{k-1}}I_x + C_{i_{k-1}}I_y$, where $B_{i_k}, C_{i_k} \in \mathbb{C}^{k-1}_{m\times n}$ and $B_{i_{k-1}}, C_{i_{k-1}} \in \mathbb{C}^{k-1}_{m\times n}$ and $I_x = \frac{1+i_k i_{k-1}}{2}, I_y = \frac{1-i_k i_{k-1}}{2}$.

Corollary 5.2. *We can do easily the following results in the field of multicomplex space:*
(i) Let A be an $n\times n$ multicomplex matrix

$$A = B_{i_k}I_1 + C_{i_k}I_2 := B_{i_{k-1}}I_x + C_{i_{k-1}}I_y,$$

then its determinant is given by

$$detA = detB_{i_k}I_x + detC_{i_k}I_y := detB_{i_{k-1}}I_x + detC_{i_{k-1}}I_y, \tag{5.1}$$

(ii) Let A and B be any two square multicomplex matrices then

$$det(AB) = detA.detB, \tag{5.2}$$

(iii) Let $A = B_{i_k}I_x + C_{i_k}I_y := B_{i_{k-1}}I_x + C_{i_{k-1}}I_y \in \mathbb{C}^k_{n\times n}, B_{i_k}, C_{i_k} \in \mathbb{C}^k_{n\times n}$ and $B_{i_{k-1}}, C_{i_{k-1}} \in \mathbb{C}^{k-1}_{n\times n}$, be a multicomplex matrix. Then A is invertible if and only if B_{i_k}, C_{i_k} are invertible in $\mathbb{C}^k_{n\times n}$ and $B_{i_{k-1}}, C_{i_{k-1}}$ are invertible in $\mathbb{C}^{k-1}_{n\times n}$.

We define the definition for the eigenvalues of a matrix in multicomplex space in which if we put $k = 2$ then the definition of eigenvalues for bicomplex matrices (see [17]).

Definition 5.3 (Eigenvalues for Multicomplex Matrices). Let $A := \{(a_{lj}) \in \mathbb{C}^k_{m\times n} = A_1I_x + A_2I_y\}$ and $Au = \lambda u$ which is equivalent to

$$\begin{cases} A_1u_1 = \lambda_1 u_1, \\ A_2u_2 = \lambda_2 u_2. \end{cases}$$

Then λ is called the eigenvalue of the multicomplex matrix A corresponding to eigenvector u where $\lambda := \lambda_1 I_x + \lambda_2 I_y \in \mathbb{C}^k$ and $u = u_1I_x + u_2I_y$. If λ is not a zero divisor and $u_1 \neq 0, u_2 \neq 0$ then λ is an eigenvalue of A if and only if λ_1 and λ_2 be an eigenvalue of A_1 and A_2 corresponding to eigenvector of u_1 and u_2.

We define and prove the following theorem given bellow.

Theorem 5.4. *Let $A := \{(a_{lj}) \in \mathbb{C}^k_{m\times n} = A_1I_x + A_2I_y\}$ and $Au = \lambda u$ which is equivalent to*

$$\begin{cases} A_1u_1 = \lambda_1 u_1, \\ A_2u_2 = \lambda_2 u_2. \end{cases}$$

Where $\lambda = \lambda_1I_x + \lambda_2I_y \in \mathbb{C}^k$ and $u = u_1I_x + u_2I_y$. Then multicomplex matrix A has $\{\lambda = p_1.q_1\}\}$ distinct eigenvalues if and only if A_1 has $\{\lambda_1 = p_1\}$ distinct eigenvalues and A_2 has $\{\lambda_2 = q_1\}$ distinct eigenvalues.

Proof. We have $\lambda = \{\alpha_sI_x + \beta_tI_y \mid 1 \leq s \leq p_1, 1 \leq t \leq q_1\}$

$$= \{\alpha_1, \alpha_2, \cdots, \alpha_{p_1}\}I_x + \{\beta_1, \beta_2, \cdots, \beta_{q_1}\}I_y = \lambda_1I_x + \lambda_2I_y$$

$$Au = \lambda u \Rightarrow (A_1I_x + A_2I_y)u = (\lambda_1I_x + \lambda_2I_y)u$$

$$\begin{cases} A_1u_1 = \lambda_1 u_1, \\ A_2u_2 = \lambda_2 u_2. \end{cases}$$

Conversely: If $\lambda_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_{p_1}\}, \lambda_2 = \{\beta_1, \beta_2, \cdots, \beta_{q_1}\}$

$$Au = (A_1I_x + A_2I_y)u = (\lambda_1I_x + \lambda_2I_y)u$$

$$Au = \lambda u.$$

Implies that

$$\lambda = \{\lambda_1I_x + \lambda_2I_y = \alpha_sI_x + \beta_tI_y \mid 1 \leq s \leq p_1, 1 \leq t \leq q_1\}.$$

■

Example 5.5. Take

$$A_{2,2} = \begin{pmatrix} 1 - i_{k-1} + i_k + i_{k-1}i_k & 1 + i_{k-1} + i_k - i_{k-1}i_k \\ 1 - i_{k-1} - i_k + i_{k-1}i_k & -1 + i_{k-1} + i_k - i_{k-1}i_k \end{pmatrix}$$

$$A = A_1 I_x + A_2 I_y$$

$$A = \begin{pmatrix} 2(1 - i_{k-1}) & 0 \\ 2 & -2 \end{pmatrix} I_x + \begin{pmatrix} 0 & 2(1 + i_{k-1}) \\ -2i_{k-1} & 2i_{k-1} \end{pmatrix} I_y$$

$$det(A - \lambda I) := \lambda^2 - 2i_k\lambda + 4(i_{k-1} - 1)$$

$$det(A_1 - \lambda_1 I) := \lambda_1^2 + 2i_{k-1}\lambda_1 + 4(i_{k-1} - 1)$$

$$det(A_2 - \lambda_2 I) := \lambda_2^2 - 2i_k\lambda_2 + 4(i_{k-1} - 1)$$

$\lambda = \{(-i_{k-1} + \sqrt{3 - 4i_{k-1}})I_x + (i_{k-1} + \sqrt{3 - 4i_{k-1}})I_y, (-i_{k-1} + \sqrt{3 - 4i_{k-1}})I_x + (i_{k-1} - \sqrt{3 - 4i_{k-1}})I_y, (-i_{k-1} - \sqrt{3 - 4i_{k-1}})I_x + (i_{k-1} + \sqrt{3 - 4i_{k-1}})I_y, (-i_{k-1} - \sqrt{3 - 4i_{k-1}})I_x + (i_{k-1} - \sqrt{3 - 4i_{k-1}})I_y\}$.

Note: If we put $k = 2$, we get bicomplex eigenvalues for bicomplex matrix.

Theorem 5.6. *Let $A := \{(a_{lj}) \in \mathbb{C}^k_{n\times n} = (\alpha_{lj})I_x + (\beta_{lj})I_y, 1 \le l, j \le n\}$ be any Multicomplex matrix and $det(\lambda I_n - A)$ be the characteristic polynomial, then the matrix A is zero of $det(\lambda I_n - A)$.*

Proof. We have

$$det(\lambda I_n - A) := det(\lambda_1 I_n - \alpha_{lj})I_x + det(\lambda_2 I_n - \beta_{lj})I_y,$$

where

$$det(\lambda I_n - A) = \sum_{p=0}^{n} a_p \lambda^p = (\sum_{p=0}^{n} \delta_p \lambda_1^p)I_x + (\sum_{p=0}^{n} \gamma_p \lambda_2^p)I_y$$

$$det(\lambda I_n - A) = (\lambda I_n - A).Adj(\lambda I_n - A) = Adj(\lambda I_n - A).(\lambda I_n - A).$$

And

$$Adj(\lambda I_n - A) = \sum_{p=0}^{n-1} \omega_p \lambda^p = (\sum_{p=0}^{n-1} \phi_p \lambda_1^p)I_x + (\sum_{p=0}^{n-1} \psi_p \lambda_2^p)I_y.$$

Take

$$A = A_1 I_x + A_2 I_y, \lambda = \lambda_1 I_x + \lambda_2 I_y.$$

Then we have

$$\begin{gathered} \phi_{n-1} = \delta_n I \\ \phi_{n-2} - A_1\phi_{n-1} = \delta_{n-1} I \\ \phi_{n-3} - A_1\phi_{n-2} = \delta_{n-2} I \\ \vdots \ddots \vdots \\ \phi_0 - A_1\phi_1 = \delta_1 I \\ -A_1\phi_0 = \delta_0 I. \end{gathered}$$

And

$$\begin{gathered} \psi_{n-1} = \gamma_n I \\ \psi_{n-2} - A_1\psi_{n-1} = \gamma_{n-1} I \\ \psi_{n-3} - A_1\psi_{n-2} = \gamma_{n-2} I \\ \vdots \ddots \vdots \\ \psi_0 - A_1\psi_1 = \gamma_1 I \end{gathered}$$

$$-A_1\psi_0 = \gamma_0 I.$$

Multiplying by $A_1^n, A_1^{n-1}, \cdots, A_1, I$

$$\begin{gathered}
A_1^n\phi_{n-1} = A_1^n\delta_n I \\
A_1^{n-1}\phi_{n-2} - A_1^n\phi_{n-1} = A_1^{n-1}\delta_{n-1} I \\
A_1^{n-2}\phi_{n-3} - A_1^{n-1}\phi_{n-2} = A_1^{n-2}\delta_{n-2} I \\
\vdots \ \ddots \ \vdots \\
A_1\phi_0 - A_1^2\phi_1 = A_1\delta_1 I \\
-A_1\phi_0 = \delta_0 I
\end{gathered}$$

$$\delta_n A_1^n + \delta_{n-1}A_1^{n-1} + \cdots + \delta_1 A_1 + \delta_0 I = 0. \tag{5.3}$$

Similarly multiplying by $A_2^n, A_2^{n-1}, \cdots, A_2, I$.
We have

$$\gamma_n A_2^n + \gamma_{n-1}A_2^{n-1} + \cdots + \gamma_1 A_2 + \psi_0 I = 0. \tag{5.4}$$

From above equation we have

$$a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0. \tag{5.5}$$

■

Theorem 5.7. *Let $A := \{(a_{lj}) \in \mathbb{C}^k_{n\times n} = A_1 I_x + A_2 I_y\}$ be any Multicomplex matrix, then A is zero of $det(\lambda I_n - A)$ if and only if A_1 is zero of $det(\lambda_1 I_n - A_1)$ and A_2 is zero of $det(\lambda_2 I_n - A_2)$ where $\lambda = \lambda_1 I_x + \lambda_2 I_y$.*

Proof. Very simple, can be easily proved. ■

Example 5.8. From above example clearly

$$\begin{gathered}
f(A) := A^2 - 2i_k A + 4(i_{k-1} - 1) = 0 \\
f_1(A_1) := A_1^2 + 2i_{k-1}A_1 + 4(i_{k-1} - 1) = 0 \\
f_2(A_2) := A_2^2 - 2i_k A_2 + 4(i_{k-1} - 1) = 0.
\end{gathered}$$

6. Kronecker Product on Multicomplex Space

The following information is interpreted from the paper On the history of Kronecker product by Henderson, Pukelsheim and Searle (see [7]). Apparently, the first documented work on Kronecker products was written by Johann Georg Zehfuss between 1858 and 1868.

Now we do it for the multicomplex space.

Definition 6.1 (Kronecker Product). Let A and B are Multicomplex valued matrices, and if $A = (a_{lj}) \in \mathbb{C}^k_{m\times n}, B = (b_{rs}) \in \mathbb{C}^k_{p\times q}$ then their Kronecker product is denoted by $A \otimes B$ and is defined as

$$A \otimes B = (a_{lj}B) \in \mathbb{C}^k_{mp\times nq} = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{pmatrix}.$$

Which imply

$$A\otimes B=(\alpha_{lj}B_1)I_x+(\beta_{lj}B_2)I_y=\begin{pmatrix}\alpha_{1,1}B_1I_x+\beta_{1,1}B_2I_y & \cdots & \alpha_{1,n}B_1I_x+\beta_{1,n}B_2I_y\\ \alpha_{2,1}B_1I_x+\beta_{2,1}B_2I_y & \cdots & \alpha_{2,n}B_1I_x+\beta_{2,n}B_2I_y\\ \vdots & \ddots & \vdots\\ \alpha_{m,1}B_1I_x+\beta_{m,1}B_2I_y & \cdots & \alpha_{m,n}B_1I_x+\beta_{m,n}B_2I_y\end{pmatrix},$$

where $A=\{\alpha_{lj}I_x+\beta_{lj}I_y \mid 1\le l\le m, 1\le j\le n\}$ and $B=B_1I_x+B_2I_y$.
Clearly $A\otimes B\neq B\otimes A$.

We do the theorem for multicomplex space.

Theorem 6.2. *Let $A\in\mathbb{C}^k_{m\times n}, B\in\mathbb{C}^k_{r\times s}, C\in\mathbb{C}^k_{n\times p}, D\in\mathbb{C}^k_{s\times t}$. Then $(A\otimes B)(C\otimes D)=AC\otimes BD$.*

Proof. We have

$$A\otimes B=(\alpha_{lj}B_1)I_x+(\beta_{lj}B_2)I_y=\begin{pmatrix}\alpha_{1,1}B_1I_x+\beta_{1,1}B_2I_y & \cdots & \alpha_{1,n}B_1I_x+\beta_{1,n}B_2I_y\\ \alpha_{2,1}B_1I_x+\beta_{2,1}B_2I_y & \cdots & \alpha_{2,n}B_1I_x+\beta_{2,n}B_2I_y\\ \vdots & \ddots & \vdots\\ \alpha_{m,1}B_1I_x+\beta_{m,1}B_2I_y & \cdots & \alpha_{m,n}B_1I_x+\beta_{m,n}B_2I_y\end{pmatrix}$$

$$C\otimes D=(\gamma_{uv}D_1)I_x+(\delta_{uv}D_2)I_y=\begin{pmatrix}\gamma_{1,1}D_1I_x+\delta_{1,1}D_2I_y & \cdots & \gamma_{1,p}D_1I_x+\delta_{1,p}D_2I_y\\ \gamma_{2,1}D_1I_x+\delta_{2,1}D_2I_y & \cdots & \gamma_{2,p}D_1I_x+\delta_{2,p}D_2I_y\\ \vdots & \ddots & \vdots\\ \gamma_{n,1}D_1I_x+\delta_{n,1}D_2I_y & \cdots & \gamma_{n,p}D_1I_x+\delta_{n,p}D_2I_y\end{pmatrix}$$

$$\begin{aligned}(A\otimes B)(C\otimes D)&=\begin{pmatrix}P & \cdots & Q\\ \vdots & \ddots & \vdots\\ R & \cdots & S\end{pmatrix}\\ &=AC\otimes BD,\end{aligned}$$

where,

$$P=\sum_{k=1}^{n}(\alpha_{1,k}\gamma_{k,1}B_1D_1I_x+\beta_{1,k}\delta_{k,1}B_2D_2I_y)$$

$$Q=\sum_{k=1}^{n}(\alpha_{1,k}\gamma_{k,p}B_1D_1I_x+\beta_{1,k}\delta_{k,p}B_2D_2I_y)$$

$$R=\sum_{k=1}^{n}(\alpha_{m,k}\gamma_{k,1}B_1D_1I_x+\beta_{m,k}\delta_{k,1}B_2D_2I_y)$$

$$S=\sum_{k=1}^{n}(\alpha_{m,k}\gamma_{k,p}B_1D_1I_x+\beta_{m,k}\delta_{k,p}B_2D_2I_y).$$

■

The following results can be do easily for the multicomplx space.

(i) Let A and B be non singular Multicomplex valued matrices. Then $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$.

(ii) Let $A\in\mathbb{C}^k_{m\times n}, B\in\mathbb{C}^k_{r\times s}$. Then $(A\otimes B)^T=A^T\otimes B^T$.

(iii) If $A\in\mathbb{C}^k_{n\times n}, B\in\mathbb{C}^k_{m\times m}$ are normal. Then $A\otimes B$ is normal.

(iv) If $A\in\mathbb{C}^k_{n\times n}, B\in\mathbb{C}^k_{m\times m}$ are symmetric. Then $A\otimes B$ is symmetric.

Now we prove the theorem in multicomplex space on the eigenvalues.

Theorem 6.3. *Let $A \in \mathbb{C}^k_{n\times n}, B \in \mathbb{C}^k_{m\times m}$ for which $Au = \lambda u$ and $Bv = \mu v$. If λ be $(p_1.q_1)$ distinct eigenvalues of A and μ be $(p_2.q_2)$ distinct eigenvalues for B then $A \otimes B$ has $(p_1q_1.p_2q_2)$ distinct eigenvalues and written as in the form of $\alpha_k\nu_r I_x + \beta_s\sigma_t I_y$ where $1 \le k \le p_1, 1 \le r \le p_2, 1 \le s \le q_1, 1 \le t \le q_2$.*

Proof. Let $A = A_1I_x + A_2I_y, B = B_1I_x + B_2I_y, \lambda = \lambda_1I_x + \lambda_2I_y, \mu = \mu_1I_x + \mu_2I_y$
$Au = \lambda u, Bv = \mu v$ with

$$\begin{cases} A_1u_1 = \lambda_1u_1, \\ A_2u_2 = \lambda_2u_2. \end{cases}$$

And

$$\begin{cases} B_1v_1 = \mu_1v_1, \\ B_2v_2 = \mu_2v_2. \end{cases}$$

If $\lambda_1 = \alpha_1, \alpha_2, \cdots, \alpha_{p_1}$ having p_1 distinct eigenvalues
$\lambda_2 = \beta_1, \beta_2, \cdots, \beta_{q_1}$ having q_1 distinct eigenvalues
then λ has mn distinct eigenvalues in the form of

$$\{\lambda = \alpha_kI_x + \beta_sI_y \mid 1 \le k \le p_1, 1 \le s \le q_1\}. \tag{6.1}$$

Similarly, if $\mu_1 = \nu_1, \nu_2, \cdots, \nu_{p_2}$ having p_2 distinct eigenvalues
$\mu_2 = \sigma_1, \sigma_2, \cdots, \sigma_{q_2}$ having q_2 distinct eigenvalues
then μ has p_2q_2 distinct eigenvalues in the form of

$$\{\mu = \nu_rI_x + \sigma_tI_y \mid 1 \le r \le p_2, 1 \le t \le q_2\} \tag{6.2}$$

$$\begin{aligned}
(A \otimes B)(u \otimes v) &= ((A_1 \otimes B_1)I_x + (A_2 \otimes B_2)I_y)(u \otimes v) \\
&= (A_1 \otimes B_1)I_x(u \otimes v) + (A_2 \otimes B_2)I_y(u \otimes v) \\
&= (A_1u \otimes B_1v)I_x + (A_2u \otimes B_2v)I_y \\
&= (\lambda_1u \otimes \mu_1v)I_x + (\lambda_2u \otimes \mu_2v)I_y \\
&= \lambda_1\mu_1(u \otimes v)I_x + \lambda_2\mu_2(u \otimes v)I_y \\
(A \otimes B)(u \otimes v) &= (\lambda_1\mu_1I_x + \lambda_2\mu_2I_y)(u \otimes v) \\
(A \otimes B)(u \otimes v) &= (\lambda.\mu)(u \otimes v)
\end{aligned}$$

$$\lambda\mu = \{\alpha_k\nu_rI_x + \beta_s\sigma_tI_y \mid 1 \le k \le p_1, 1 \le r \le p_2, 1 \le s \le q_1, 1 \le t \le q_2\}.$$

Implies that $A \otimes B$ has $(p_1q_1.p_2q_2)$ distinct eigenvalues. ∎

Example 6.4. Take

$$A = \begin{pmatrix} 1 - i_{k-1} + i_k + i_{k-1}i_k & 1 + i_{k-1} + i_k - i_{k-1}i_k \\ 1 - i_{k-1} - i_k + i_{k-1}i_k & -1 + i_{k-1} + i_k - i_{k-1}i_k \end{pmatrix}$$

$$B = \begin{pmatrix} 1 + i_{k-1} + i_k + i_{k-1}i_k & 1 - i_{k-1} - i_k - i_{k-1}i_k \\ 1 + i_{k-1} + i_k - i_{k-1}i_k & 1 - i_{k-1} + i_k + i_{k-1}i_k \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 4(1 - i_{k-1}) & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 4 & 0 & -4 & 0 \\ 0 & 4(1 - i_{k-1}) & 0 & 4(-1 + i_{k-1}) \end{pmatrix} I_x$$

$$+\begin{pmatrix} 0 & 0 & 4(-1+i_{k-1}) & 8 \\ 0 & 0 & 8i_{k-1} & 0 \\ 4 & 4(-1-i_{k-1}) & -4 & 4(1+i_{k-1}) \\ 4(1-i_{k-1}) & 0 & 4(-1+i_{k-1}) & 0 \end{pmatrix} I_y$$

where

$$A = A_1 I_x + A_2 I_y$$

$$A = \begin{pmatrix} 2(1-i_{k-1}) & 0 \\ 2 & -2 \end{pmatrix} I_x + \begin{pmatrix} 0 & 2(1+i_{k-1}) \\ -2i_{k-1} & 2i_{k-1} \end{pmatrix} I_y$$

$$det(A-\lambda I) := \lambda^2 - 2i_k\lambda + 4(i_{k-1}-1)$$

$$det(A_1-\lambda_1 I) := \lambda_1^2 + 2i_{k-1}\lambda_1 + 4(i_{k-1}-1)$$

$$det(A_2-\lambda_2 I) := \lambda_2^2 - 2i_k\lambda_2 + 4(i_{k-1}-1)$$

$$\lambda_1 = \{\alpha_1, \alpha_2 \mid \alpha_1 = -i_{k-1} + \sqrt{3-4i_{k-1}}, \alpha_2 = -i_{k-1} - \sqrt{3-4i_{k-1}}\}$$

$$\lambda_2 = \{\beta_1, \beta_2 \mid \beta_1 = i_{k-1} + \sqrt{3-4i_{k-1}}, \beta_2 = i_{k-1} - \sqrt{3-4i_{k-1}}\},$$

where λ_1 are set of eigenvalues for A_1 and λ_2 be the set of eigenvalues of A_2.

Similarly, if $B = B_1 I_x + B_2 I_y$ having eigenvalues μ for which $\mu = \mu_1 I_x + \mu_2 I_y$ where μ_1 are set of eigenvalues for B_1 and μ_2 be the set of eigenvalues of B_2 then we have

$$\mu_1 = \{\nu_1, \nu_2 \mid \nu_1 = 2 - i_{k-1} + i_k, \nu_2 = 2 - i_{k-1} - i_k\}$$

$$\mu_2 = \{\sigma_1, \sigma_2 \mid \sigma_1 = i_{k-1} + \sqrt{7}, \sigma_2 = i_{k-1} - \sqrt{7}\}$$

$$\lambda.\mu = \lambda_1.\mu_1 I_x + \lambda_2.\mu_2 I_y,$$

where

$$\lambda_1\mu_1 = \{\alpha_1\nu_1, \alpha_1\nu_2, \alpha_2\nu_1, \alpha_2\nu_2\}$$

$$\lambda_2\mu_2 = \{\beta_1\sigma_1, \beta_1\sigma_2, \beta_2\sigma_1, \beta_2\sigma_2\}$$

hence set of all eigenvalues of $A \otimes B$ is in the form of

$$\{\alpha_k\nu_r I_x + \beta_s\sigma_t I_y \mid k, r, s, t = 1, 2\}.$$

7. Kronecker Sum on Multicomplex Space

As for the Kronecker product, we can define the Kronecker sum and some of its results on the eigenvalues in the multicomplex space.

Definition 7.1 (Kronecker Sum). Let $A \in \mathbb{C}^k_{n\times n}, B \in \mathbb{C}^k_{m\times m}$ then the Kronecker sum of the Multicomplex valued matrices A and B is denoted as $A \oplus B$ and is the $mn \times mn$ matrix and is defined as $A \oplus B = (I_m \otimes A) + (B \otimes I_n)$ and in general $A \oplus B \neq B \oplus A$.

We prove the following theorem on eigenvalues for Kronecker sum in the multicomplex space.

Theorem 7.2. *Let $A \in \mathbb{C}^k_{n\times n}, B \in \mathbb{C}^k_{m\times m}$ for which $Au = \lambda u$ and $Bv = \mu v$. Let λ be $(p_1.q_1)$ distinct eigenvalues of A and μ be $(p_2.q_2)$ distinct eigenvalues for B then $A \oplus B$ has $(p_1p_2.q_1q_2)$ distinct eigenvalues and written as in the form of $(\alpha_k+\nu_r)I_x+(\beta_s+\sigma_t)I_y$ where $1 \le k \le p_1, 1 \le r \le p_2, 1 \le s \le q_1, 1 \le t \le q_2$.*

Proof.

$$\begin{aligned}
(A \oplus B)(u \otimes v) &= ((I_m \otimes A) + (B \otimes I_n))(u \otimes v)\\
&= ((I_m \otimes A_1 + B_1 \otimes I_n)I_x + (I_m \otimes A_2 + B_2 \otimes I_n)I_y)(u \otimes v)\\
&= u \otimes (A_1 + B_1)vI_x + (A_2 + B_2)u \otimes vI_y\\
&= u \otimes (A_1v + B_1v)I_x + (A_2u + B_2u) \otimes vI_y\\
&= u \otimes (\lambda_1v + \mu_1v)I_x + (\lambda_2u + \mu_2u) \otimes vI_y\\
&= (\lambda_1 + \mu_1)I_x(u \otimes v) + (\lambda_2 + \mu_2)I_y(u \otimes v)\\
&= ((\lambda_1 + \mu_1)I_x + (\lambda_2 + \mu_2)I_y)(u \otimes v)\\
&= (\lambda + \mu)(u \otimes v).
\end{aligned}$$

Since $\lambda_1 = \alpha_1, \alpha_2, \cdots, \alpha_{p_1}$ having p_1 distinct eigenvalues,
$\lambda_2 = \beta_1, \beta_2, \cdots, \beta_{q_1}$ having q_1 distinct eigenvalues,
$\mu_1 = \nu_1, \nu_2, \cdots, \nu_{p_2}$ having p_2 distinct eigenvalues,
$\mu_2 = \sigma_1, \sigma_2, \cdots, \sigma_{q_2}$ having q_2 distinct eigenvalues.
Therefore $\lambda_1+\mu_1 = \{\alpha_k+\nu_r \mid 1 \le k \le p_1, 1 \le r \le p_2\}$ has p_1p_2 distinct values. $\lambda_2+\mu_2 = \{\beta_s+\sigma_t \mid 1 \le s \le q_1, 1 \le t \le q_2\}$ has q_1q_2 distinct values. $\lambda+\mu = (\alpha_k+\nu_r)I_x+(\beta_s+\sigma_t)I_y$ has $(p_1p_2.q_1q_2)$ distinct eigenvalues. ■

Example 7.3. Take A and B Multicomplex matrices from the above example

$$A \oplus B = (I_2 \otimes A) + (B \otimes I_2)$$

where

$$I_2 \otimes A = \begin{pmatrix} 2(1-i_{k-1}) & 0 & 0 & 0\\ 2 & -2 & 0 & 0\\ 0 & 0 & 2(1-i_{k-1}) & 0\\ 0 & 0 & 2 & -2 \end{pmatrix} I_x$$

$$+ \begin{pmatrix} 0 & 2(1+i_{k-1}) & 0 & 0\\ -2i_{k-1} & 2i_{k-1} & 0 & 0\\ 0 & 0 & 0 & 2(1+i_{k-1})\\ 0 & 0 & -2i_{k-1} & 2i_{k-1} \end{pmatrix} I_y$$

$$B \otimes I_2 = \begin{pmatrix} 2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2(1-i_{k-1}) & 0\\ 0 & 0 & 2 & 2(1-i_{k-1}) \end{pmatrix} I_x$$

$$+ \begin{pmatrix} 2i_{k-1} & 0 & 2(1-i_{k-1}) & 0\\ 0 & 2i_{k-1} & 0 & 2(1-i_{k-1})\\ 2(1+i_{k-1}) & 0 & 0 & 0\\ 0 & 2(1+i_{k-1}) & 0 & 0 \end{pmatrix} I_y$$

$$A \otimes B = \begin{pmatrix} 2(2-i_{k-1}) & 0 & 0 & 0\\ 2 & 0 & 0 & 0\\ 0 & 0 & 4(1-i_{k-1}) & 0\\ 0 & 0 & 2 & -2i_{k-1} \end{pmatrix} I_x$$

$$+\begin{pmatrix} 2i_{k-1} & 2(1+i_{k-1}) & 2(1-i_{k-1}) & 0 \\ -2i_{k-1} & 4i_{k-1} & 0 & 2(1-i_{k-1}) \\ 2(1+i_{k-1}) & 0 & 0 & 2(1+i_{k-1}) \\ 0 & 2(1+i_{k-1}) & -2i_{k-1} & 2i_{k-1} \end{pmatrix} I_y,$$

where

$$\lambda + \mu = \lambda_1.\mu_1 I_x + \lambda_2.\mu_2 I_y,$$

and

$$\lambda_1 + \mu_1 = \{\alpha_1 + \nu_1, \alpha_1 + \nu_2, \alpha_2 + \nu_1, \alpha_2 + \nu_2\}$$

$$\lambda_2 + \mu_2 = \{\beta_1 + \sigma_1, \beta_1 + \sigma_2, \beta_2 + \sigma_1, \beta_2 + \sigma_2\}$$

hence set of all eigenvalues of $A \oplus B$ is in the form of

$$\{(\alpha_k + \nu_r)I_x + (\beta_s + \sigma_t)I_y \mid k, r, s, t = 1, 2\}.$$

References

[1] W.R. Hamilton, On Quaternions; or on a new System of Imaginaries in Algebra, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science 15 (1844) 489–495.

[2] C. Segre, The real representation of complex elements and hyperalgebraic entities (Italian), Mathematische Annalen 40 (3) (1892) 413–467.

[3] G.B. Price, An Introduction to Multicomplex Spaces and Functions, Marcel Dekker, 1991.

[4] N. Fleury, M.R. Detraubenberg, R.M. Yamaleev, Commutative Extended Complex Numbers and Connected Trigonometry, Journal of Mathematical Analysis and Applications 180 (1993) 431–457.

[5] K.S. Charak, D. Rochon, On Factorization of Bicomplex Meromorphic Functions Quaternionic and Clifford Analysis Trends in Mathematics, 55–68, 2008 Birkhauser Verlag Basel/Switzerland.

[6] K.S. Charak, D. Rochon and N. Sharma, Normal families of bicomplex holomorphic functions, World Scientific Publishing Company 17 (3) (2009) 257–268.

[7] H.V. Henderson, F. Pukelsheim, S.R. Searle, On the History of the Kronecker Product, Linear and Multilinear Algebra 14 (1983) 113–120.

[8] A. Kilicman, Z.A.A. Zhour, On the Connection between Kronecker and Hadamard Convolution Products of Matrices and Some Applications, Journal of Inequalities and Applications 2009, 2009:736243 doi:10.1155/2009/736243

[9] M. Mursaleen, M. Nasiruzzaman, H.M. Srivastava, Approximation by bicomplex beta operators in compact BC-disks, Math. Meth. Appl. Sci. 39 (2016) 2916–2929.

[10] J. Ryan, Complexified Clifford analysis, Complex Variables and Elliptic Equations 1 (1982) 119–149.

[11] D. Rochon, M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, Analele Universitatii Oradea, Fasc. Matematica 11 (2004) 71–110.

[12] Z.A.A. Zhour, A. Kilicman, M.A. Hasan, A new Algebraic Method for Solving the Balanced Transportation Problem (BTP) Based on the Kronecker Product, IRSIAM Journal on Applied Mathematics (IJAM) 1 (2005) 64–80.

[13] Z.A.A. Zhour, A. Kilicman, Some Topics on Weighted Generalized Inverse and Kronecker Product of Matrices, Malaysian Journal of Mathematical Sciences 1 (1) (2007) 109–122.

[14] G. Lantoine, R.P. Russell, T. Dargent, Using Multicomplex Variables for Automatic Computation of High-order Derivatives, ACM Transactions on Mathematical Software 38 (3) Article id: 16, April 2012, 21 pages.

[15] M.E. Luna-Elizarrara's, M.Shapiro, Bicomplex Numbers and their Elementary Functions CUBO A Mathematical Journal 14 (02) (2012) 61–80.

[16] F.L. Reid, A. Robert V. Gorder, A Multicomplex Riemann Zeta Function, Adv. Appl. Cliff ord Algebras 23 (2013) 237-251.

[17] D. Alpay, Maria Elena Luna-Elizarrar's Michael Shapiro and Daniele C. Struppa Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis, Springer (2014).

[18] A.J. Laub, Matrix Analysis for Scientists and Engineers, SIAM, 2004.