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On the Diophantine Equation $p^x + (p+5)^y = z^2$, where p is Odd Prime

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Abstract This paper studies the Diophantine equation $p^x + (p+5)^y = z^2$, where p is an odd prime number. Results on Legendre and Jacobi symbols are used, and the transformation to an elliptic curve of rank zero is done to show that this equation has a unique positive integer solution $(p, x, y, z) = (5, 3, 2, 15)$ whenever $x \not\equiv 1 \pmod{12}$.

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1. Introduction

In Number Theory, mathematicians are usually interested in the study of integers or integer-valued functions. In this field of pure mathematics, the concept of Diophantine equation is one of the interesting topics. A *Diophantine equation* is normally a polynomial equation of degree $n \geq 1$ that seeks for integer solutions. By integer solution, we mean the values of the unknowns must be integers. Similarly, by nonnegative or positive integer solutions, we mean that all unknowns in the given equation must take nonnegative or positive integer values, respectively.

The most famous type is the *linear Diophantine equation* in two unknowns. This is of the form $ax + by = c$, where the given coefficients a and b (not both zero), and the constant c are all integers. The solution to such type is already well-established, and can be found in any elementary number theory books.

Theorem 1.1. *Let a, b and c be integers such that a and b cannot be both equal to 0. Let $d = \gcd(a, b)$.*

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(1) If d does not divide c, then the Diophantine equation $ax + by = c$ has no solution.

(2) If d divides c, then the Diophantine equation $ax + by = c$ has infinitely many solutions. If (x_0, y_0) is a particular solution, then the other solutions (x, y) can be obtained from the following formulas:

$$x_0 = x_0 + \frac{b}{d}t, \quad y_0 = y_0 - \frac{a}{d}t, \quad t \in \mathbb{Z}.$$

Another type of Diophantine equation that number theorists continue to explore is the so-called *exponential Diophantine equation*. This can be easily identified because some unknowns are the exponents. One of the famous exponential Diophantine equations is the equation in the Catalan problem — a seemingly very easy problem, but it was not until Preda Mihailescu provided a complete proof after more than 150 years of existence as conjecture [1].

Theorem 1.2 (Mihailescu's Theorem/Catalan's Conjecture)**.** *Let $a, b, x,$ and y be positive integers with $a, b > 1$ satisfying the Diophantine equation $a^x - b^y = 1$. Then the 4-tuple $(a, b, x, y) = (3, 2, 2, 3)$ is the unique solution of this equation.*

For the past decade, a handful of mathematicians have been studying an exponential Diophantine equation of the form

$$p^x + q^y = z^2, \tag{1.1}$$

where p and/or q are usually fixed primes, and the solutions (x, y, z) are sought in the set of nonnegative or positive integers. Unlike the linear types, up to this time, there are no definite strategies that could be used in solving all exponential Diophantine equations. However, one strategy is by taking the equations in a particular modulus. This strategy was used in solving Diophantine equations of the form (1.1) which include the following: $2^x + 5^y = z^2$ by Acu [2]; $3^x + 5^y = z^2$ and $8^x + 19^y = z^2$ by Sroysang [3, 4], and $3^x + 19^y = z^2, 3^x + 91^y = z^2, 17^x + 19^y = z^2$, and $71^x + 73^y = z^2$ by Rabago [5, 6]. Rabago studied other Diophantine equations of the form (1.1) (cf. [7], [8].) In 2015, Bacani and Rabago [9] studied a more general variant of (1.1), that is,

$$p^x + (p + 2)^y = z^2, \tag{1.2}$$

where p and $p + 2$ are primes (known as twin primes). They showed that (1.2) has infinitely many nonnegative integer solutions. In 2018, Burshtein [10] made a study on $p^x + (p + 4)^y = z^2$, where p and $p + 4$ are primes (known as cousin primes), and for the cases where $x + y = 2, 3, 4$. He also studied the Diophantine equation $p^x + (p + 5)^y = z^2$, where $p+5 = 2^{2u}$ [11]. On the other hand, Mina and Bacani [12] presented some theorems that guarantee non-existence of solutions of (1.1) over the set of positive integers using values of Legendre and Jacobi symbols. Other related studies on (1.1) can be seen in [13] and [14].

Motivated by the papers that were mentioned above, we consider another variant of (1.1). This time, we will study the Diophantine equation of the form

$$p^x + (p + 5)^y = z^2, \tag{1.3}$$

where p is an odd prime number. Similar to what was presented in [12], our strategy to determine the solutions is by using the values of Legendre and Jacobi symbols. Furthermore, to find the complete set of solutions of (1.3), the method of transforming the Diophantine equation (1.3) to elliptic curves will also be utilized.

Throughout the paper, we will denote the set of integers and the set of positive integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively.

2. Preliminaries

Before we present the main result of this paper, we first provide some basic results that are essential in the proof of the main theorem.

Definition 2.1. Let n be a natural number and a be an integer such that $\gcd(a, n) = 1$. If the congruence $x^2 \equiv a \pmod{n}$ has a solution $x \in \mathbb{Z}$, then a is said to be a *quadratic residue of* n. Otherwise a is a *quadratic nonresidue of* n.

We now define the *Legendre symbol*:

Definition 2.2 (Legendre Symbol). Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. The *Legendre symbol of a modulo p* (or simply $\left(\frac{a}{p}\right)$) is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p \end{cases}.$$

Here are some standard results on the Legendre symbol that are needed for our results.

Theorem 2.3. *If p is an odd prime, then*

$$\left(\frac{-2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 3 \pmod{8} \\ -1 & \text{if } p \equiv 5, 7 \pmod{8} \end{cases}.$$

Theorem 2.4. *If $p \neq 5$ is an odd prime, then*

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 9, 11, 19 \pmod{20} \\ -1 & \text{if } p \equiv 3, 7, 13, 17 \pmod{20} \end{cases}.$$

We now define a generalization of the Legendre symbol.

Definition 2.5 (Jacobi Symbol). Let $\gcd(a, b) = 1$, where a is an integer and $b > 1$ is odd. If $b = p_1 p_2 \cdots p_k$ is the prime factorization of b, where the p_i's are not necessarily distinct, then the Jacobi symbol $\left(\frac{a}{b}\right)$ is defined to be

$$\left(\frac{a}{b}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right),$$

where the expressions $\left(\frac{a}{p_i}\right)$ are Legendre symbols.

Below is a result about Jacobi symbol that is also needed in our result.

Theorem 2.6. *Let $q \neq \pm 5$ be an odd integer. Then, the Jacobi symbol $\left(\frac{-5}{q}\right)$ is equal to 1 if and only if $q \equiv 1, 3, 7, 9 \pmod{20}$.*

3. Main Results

We now present the main result of the study.

Theorem 3.1. *The exponential Diophantine equation* (1.3) *has a unique solution* $(p, x, y, z) = (5, 3, 2, 15)$ *in* $\mathbb{N}$ *whenever* $x \not\equiv 1 \pmod{12}$.

Proof. Suppose we have a solution (p, x, y, z) over $\mathbb{N}$ where y is odd. Assume first that $p \neq 5$. Taking (1.3) modulo p, we get $z^2 \equiv 5^y \pmod{p}$. This gives us the Legendre symbol $\left(\frac{5^y}{p}\right) = 1$. Using the properties of the Legendre symbol, we have

$$1 = \left(\frac{5^y}{p}\right) = \left(\frac{5}{p}\right)^y = \left(\frac{5}{p}\right),$$

where the last equality is true because y is odd. From Theorem 2.4, we obtain

$$p \equiv 1, 9, 11, 19 \pmod{20}. \tag{3.1}$$

Now, take Equation (1.3) modulo 5. Then, we obtain

$$p^x + p^y \equiv 1 \text{ or } 4 \pmod{5}$$

since $z^2 \equiv 1$ or $4 \pmod{5}$. From (3.1), we get

$$p \equiv 1 \pmod{5} \quad \text{or} \quad p \equiv 4 \pmod{5}.$$

We have the following cases:

a. $p \equiv 1 \pmod{5}$. For this case, we have $1^x + 1^y \equiv 2 \not\equiv 1, 4 \pmod{5}$. Hence, we get no solutions.

b. $p \equiv 4 \pmod{5}$. We have

$$4^x + 4^y \equiv \begin{cases} 3 \pmod{5} & \text{if } x \text{ and } y \text{ are odd} \\ 0 \pmod{5} & \text{if } x \text{ and } y \text{ are of different parity} \\ 2 \pmod{5} & \text{if } x \text{ and } y \text{ are even} \end{cases}.$$

Thus, $p^x + (p+5)^y \not\equiv z^2 \pmod{5}$ in any of the cases. This tells us that y cannot be odd. So, consider y to be even. Let $y = 2y_1$ for some $y_1 \in \mathbb{N}$. In this case, we have

$$p^x = z^2 - (p+5)^{2y_1} = (z + (p+5)^{y_1})(z - (p+5)^{y_1}).$$

Since p is prime, there exist integers $\alpha, \beta \geq 0$ with $\alpha < \beta$ such that $p^\alpha = z - (p+5)^{y_1}$ and $p^\beta = z + (p+5)^{y_1}$ with $\alpha + \beta = x$. Subtracting these equations will give us $p^\alpha(p^{\beta-\alpha} - 1) = p^\beta - p^\alpha = (z + (p+5)^{y_1}) - (z - (p+5)^{y_1}) = 2(p+5)^{y_1}$. Note that $p \nmid (p+5)^{y_1}$ assuming $p \neq 5$. Hence, $\alpha = 0$ and we get

$$p^x - 1 = 2(p+5)^{y_1}. \tag{3.2}$$

Factoring the left hand side of (3.2) yields

$$(p-1)(p^{x-1} + p^{x-2} + \cdots + 1) = 2(p+5)^{y_1}.$$

This implies $p - 1$ divides $2(p+5)^{y_1}$; or $\frac{p-1}{2}$ divides $(p+5)^{y_1}$. This yields

$$(p+5)^{y_1} = ((p-1) + 6)^{y_1} \equiv 6^{y_1} \equiv 0 \left(\operatorname{mod} \frac{p-1}{2}\right).$$

Furthermore, this means $\frac{p-1}{2} = 2^k 3^l$ or $p = 2^{k+1}3^l + 1$ for some $0 \leq k, l \leq y_1$.

Assume that x is odd.

a. Suppose $k = 0$.
By taking (3.2) modulo 4, we obtain the following congruences:

$$p^x - 1 = (2 \cdot 3^l + 1)^x - 1 \equiv 3^x - 1 \pmod 4 \quad \text{and} \quad 2(p+5)^{y_1} \equiv 2(2 \cdot 3^l + 6)^{y_1} \equiv 0 \pmod 4.$$

Combining these two congruences will give us $3^x - 1 \equiv 0 \pmod 4$, which implies that x is even. This is a contradiction to the assumption that x is odd.

b. Suppose $k > 0$.
Substituting $p = 2^{k+1}3^l + 1$ in (3.2), we obtain

$$(2^{k+1}3^l + 1)^x - 1 = 2(2^{k+1}3^l + 6)^{y_1}.$$

By factoring, we get

$$((2^{k+1}3^l + 1) - 1)((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = 2^{y_1+1}(2^k3^l + 3)^{y_1}.$$

By further simplification, we obtain

$$2^{k+1}3^l((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = 2^{y_1+1}(2^k3^l + 3)^{y_1}.$$

For the case where $k < y_1$, we have

$$3^l((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = 2^{y_1-k}(2^k3^l + 3)^{y_1}.$$

Taking this equation modulo 2 will also give us

$$1^{x-1} + 1^{x-2} + \cdots + 1 \equiv 0 \pmod 2.$$

This tells us that $x \equiv 0 \pmod 2$. Hence, we obtain another contradiction. Therefore, y_1 must be equal to k.

We have

$$3^l((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = (2^k3^l + 3)^{y_1}.$$

Substituting $k = y_1$, we get

$$3^l((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = (2^k3^l + 3)^k.$$

Now, if $l = 0$, then $p = 2^{k+1} + 1$. It is well-known that a prime of this form implies that $k+1$ is a power of 2. Since $k > 0$, then $k + 1 \geq 2$. Then (3.2) becomes

$$(2^{k+1} + 1)^x - 1 = 2(2^{k+1} + 6)^{y_1},$$

where $k+1 \geq 2$ is even. Taking this equation modulo 3, we get $2^x - 1 \equiv 2 \pmod 3$ which is not possible. Hence, $l > 0$. By factoring, we get

$$3^l((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = 3^{y_1}(2^k3^{l-1} + 1)^k. \tag{3.3}$$

Assume that $l < y_1$. In this case, we have from Equation (3.3) that

$$((2^{k+1}3^l + 1)^{x-1} + (2^{k+1}3^l + 1)^{x-2} + \cdots + 1) = 3^{y_1-l}(2^k3^{l-1} + 1)^k.$$

Taking this equation modulo 3, we get

$$1^{x-1} + 1^{x-2} + \cdots + 1 \equiv 0 \pmod 3$$

which implies that $x \equiv 0 \pmod 3$. Let $x = 3x_1$ for some $x_1 \in \mathbb{N}$. From Equation (3.2), we have

$$(p^{x_1})^3 - 1 = 2(p+5)^{y_1}.$$

Suppose y_1 is even, i.e., $y_1 = 2y_2$ for some $y_2 \in \mathbb{N}$. We have

$$(p^{x_1})^3 - 1 = 2((p+5)^{y_2})^2.$$

Multiplying both sides of the equation by 8 will give us

$$(2p^{x_1})^3 - 8 = (4(p+5)^{y_2})^2.$$

Let $r = 4(p+5)^{y_2}$ and $s = 2p^{x_1}$. Our equation becomes

$$r^2 = s^3 - 8,$$

a nonsingular elliptic curve, which we denote it by E. Now, the set of all $\mathbb{Q}$-rational points of E forms an abelian group $E(\mathbb{Q})$ called the *Mordell-Weil group*. This is isomorphic to $E_{tors}(\mathbb{Q}) \oplus \mathbb{Z}^r$, where $E_{tors}(\mathbb{Q})$ is the set of rational points of finite order and $r \geq 0$ is called the *rank of* E. We use *SAGE* [15] to calculate the rank r and the torsion subgroup $E_{tors}(\mathbb{Q})$ of E. We get $r = 0$ and $E_{tors}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, respectively. This implies that the Mordell-Weil group of E contains two rational points only, namely, $(r,s) = (0,2)$ and the point at infinity $\mathcal{O}$. The point $(r,s) = (0,2)$ corresponds to the equations $(p+5)^{y_2} = 0$ and $2p^{x_1} = 2$, which clearly gives no solution in $\mathbb{N}$. Hence, we get no solution when y_1 is even.

Now, assume that y_1 is odd. In this case, take Equation (3.2) modulo p to get

$$-1 \equiv 2 \cdot 5^{y_1} \pmod p.$$

Multiplying both sides by 10 gives

$$2^2 \cdot 5^{y_1+1} = (2 \cdot 5^{y_2})^2 \equiv -10 \pmod p.$$

This implies that the Legendre symbol

$$\left(\frac{-10}{p}\right) = \left(\frac{-2}{p}\right)\left(\frac{5}{p}\right) = 1.$$

Using Theorems 2.3 and 2.4, the above congruence is only possible if

$$p \equiv 1, 7, 9, 11, 13, 19, 23, 37 \pmod{40}. \tag{3.4}$$

Now, taking Equation (1.3) modulo 5 with x being odd and y being even, will give us the following cases:

a. For $p \equiv 1 \pmod 5$, we get $1^x + 6^y \equiv 2 \not\equiv z^2 \pmod 5$. Hence we get no solutions.

b. For $p \equiv 4 \pmod 5$, we get $4^x + 4^y \equiv 4 + 1 \not\equiv z^2 \pmod 5$. Hence we get no solutions either. (We exclude the case where $p = 5$).

Thus, we can only have $p \equiv 2 \pmod 5$ or $p \equiv 3 \pmod 5$. We can immediately conclude from (3.4) that

$$p \equiv 7, 13, 23, 37 \pmod{40}. \tag{3.5}$$

Now, since $p = 2^{k+1}3^l + 1$, we get

$$p + 5 = 2^{k+1}3^l + 6 = 6(2^k 3^{l-1} + 1),$$

where $2^k3^{l-1} + 1$ is odd. Taking Equation (3.2) modulo $\frac{p+5}{6}$ (odd), we obtain

$$p^x - 1 \equiv 0 \left(\operatorname{mod} \frac{p+5}{6}\right).$$

This implies that

$$p^{x+1} \equiv p \left(\operatorname{mod} \frac{p+5}{6}\right).$$

This gives us the Jacobi symbol

$$\left(\frac{p}{\frac{p+5}{6}}\right) = \left(\frac{-5}{\frac{p+5}{6}}\right) = 1.$$

Using Theorem 2.6, we get

$$\frac{p+5}{6} \equiv 1, 3, 7, 9 \pmod{20};$$

that is,

$$p \equiv 1, 13, 17, 9 \pmod{20}.$$

This tells us, from Equation (3.5), that $p \equiv 13, 37 \pmod{40}$. Since $p = 2^{k+1}3^l + 1$ and $p \equiv 5 \pmod 8$, we conclude that $k = 1$ and thus $y_1 = 1$. This further implies that $l < y_1 = 1$, which is a contradiction.

We now consider the case where $l = y_1 = k$. From Equation (3.3), we have

$$(2^{k+1}3^k + 1)^{x-1} + (2^{k+1}3^k + 1)^{x-2} + \cdots + 1 = (2^k 3^{k-1} + 1)^k.$$

By doing some manipulation, we can get

$$\begin{aligned}(2^{k+1}3^k + 1)^{x-1} + (2^{k+1}3^k + 1)^{x-2} \; + \cdots + (2^{k+1}3^k + 1)& \\ = (2^k 3^{k-1} + 1)^k - 1& \\ = (2^k 3^{k-1})((2^k 3^{k-1} + 1)^{k-1} + \cdots + 1)&.\end{aligned}$$

If $k > 1$, taking the above equation modulo 12 will give us

$$1^{x-1} + 1^{x-2} + \cdots + 1 \equiv 0 \pmod{12}.$$

This implies that $x - 1 \equiv 0 \pmod{12}$, which is a contradiction to the fact that $x \not\equiv 1 \pmod{12}$. Hence, $k = 1$ and we get

$$13^{x-1} + 13^{x-2} + \cdots + 1 = 3.$$

Clearly, this has no solutions.

We are left with the case where x is even. With y also being even, from (3.2), we have the equation

$$p^{2x_1} - 1 = 2(p+5)^{y_1}. \tag{3.6}$$

Factoring the left hand side of (3.6) will give us

$$(p^2 - 1)(p^{2(x_1-1)} + p^{2(x_1-2)} + \cdots + 1) = 2(p+5)^{y_1}.$$

This implies that $p + 1$ divides $2(p+5)^{y_1}$ or

$$2^{2y_1+1} \equiv 2(p + 1 + 4)^{y_1} \equiv 2(p+5)^{y_1} \equiv 0 \pmod{p+1}.$$

It follows that $p = 2^m - 1$, where $0 < m \leq 2y_1 + 1$. On the other hand, we know that $p = 2^{k+1}3^l + 1$. Combining these two equations will give us the following equation:

$$2^{m-1} = 2^k 3^l + 1.$$

If $k > 0$, taking this equation modulo 2 gives us a contradiction. This leads to $k = 0$. In this case, we have

$$2^{m-1} - 3^l = 1.$$

This has no solution if $\min\{m-1, l\} > 1$ according to Mihailescu's Theorem (Catalan's conjecture). We only need to inspect the case where $l = 1$. In this case, we get $2^{m-1} = 4$.

This implies that $m = 3$ and $p = 2^m - 1 = 2^3 - 1 = 7$. Consequently, $p + 5 = 12$. Substituting these values to (3.6), we obtain

$$7^{2x_1} - 1 = 2 \cdot 12^{y_1}. \tag{3.7}$$

By factoring, we get

$$(7^2 - 1)(7^{2(x_1-1)} + 7^{2(x_1-2)} + \cdots + 1) = 2^{2y_1+1}3^{y_1}.$$

Simplifying the above equation yields

$$7^{2(x_1-1)} + 7^{2(x_1-2)} + \cdots + 1 = 2^{2y_1-3}3^{y_1-1}.$$

Taking this equation modulo 6 (assuming that $2y_1 - 3 > 0$ and $y_1 - 1 > 0$), we obtain

$$1^{x_1-1} + 1^{x_1-2} + \cdots + 1 \equiv 0 \pmod 6.$$

This implies that $x_1 \equiv 0 \pmod 6$. Let $x_1 = 6x_2$ for some $x_2 \in \mathbb{N}$. Substituting this to (3.7), we get

$$7^{12x_2} - 1 = 2 \cdot 12^{y_1}.$$

By factoring, we get

$$(7^{12} - 1)(7^{12(x_2-1)} + 7^{12(x_2-2)} + \cdots + 1) = 2^{2y_1+1}3^{y_1}.$$

Note that $7^{12} - 1 = 2^5 \cdot 3^2 \cdot 48060025$, and 48060025 is not divisible by 2 or 3. This is a contradiction since the right-hand side of the previous equation has only factors that are multiples of 2 and 3. Thus, $2y_1 - 3 = 0$ or $y_1 - 1 = 0$. This implies that $y_1 = 1$. Substituting this to (3.7) gives us a contradiction. Thus, we get no solution if x and y are both even.

We are left with the case where $p = 5$. In this case we have $5^x + 10^y = z^2$. Assume that $x \leq y$. We have

$$5^x(1 + 2^y5^{y-x}) = z^2.$$

This means that x is even and $1 + 2^y5^{y-x} = u^2$ for some $u \in \mathbb{N}$. By factoring, we get

$$2^y5^{y-x} = (u + 1)(u - 1).$$

Let $d = \gcd(u + 1, u - 1)$. Note that d is even. Also, we have $d \mid ((u + 1) - (u - 1))$ or that is $d \mid 2$. Thus $d = 2$ which further implies that $y > 1$. We have the following:

$$\begin{cases} u + 1 = 2^{y-1} \\ u - 1 = 2 \cdot 5^{y-x} \end{cases}, \quad \begin{cases} u + 1 = 2^{y-1}5^{y-x} \\ u - 1 = 2 \end{cases}.$$

For the first system, we have $2 \cdot 5^{y-x} + 2 = 2^{y-1}$. This can be simplified to $2^{y-2} - 5^{y-x} = 1$. By Mihailescu's Theorem, this has no solution except when $y - 2$ is equal to 1 or 0 or when $y - x$ is equal to 1 or 0. Any of the cases yields to a contradiction.

For the second system, we have $4 = 2^{y-1}5^{y-x}$. This implies that $x = y$ and $y = 3$. This is a contradiction because x is even.

We now assume $x > y$. In this case, we have

$$5^y(5^{x-y} + 2^y) = z^2.$$

This means y is even and $5^{x-y} + 2^y = u^2$ for some $u \in \mathbb{N}$. This was studied by Acu in [2] and proven to have a positive solution $(y, x - y, u) = (2, 1, 3)$. This means that $y = 2$ and $x = 3$. Therefore, the only solution is $(p, x, y, z) = (5, 3, 2, 15)$. This completes the proof of the main theorem. ∎

4. Conclusion

In this paper, we have proven rigorously that the exponential Diophantine equation (1.3) has a unique solution $(p, x, y, z) = (5, 3, 2, 15)$ in the set of positive integers whenever $x \not\equiv 1 \pmod{12}$.

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