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Existence and Uniqueness Results for Nonlocal Problem with Fractional Integro-Differential Equation in Banach Space

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Abstract In the present paper we investigate the existence and uniqueness of solutions for a Cauchy problem governed by a Caputo fractional integro-differential equation with nonlocal initial condition in Banach Space. We shall prove existence and uniqueness results by using Banach and Krasnoselskii fixed point theorems. Some examples are presented for illustrate our main results.

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1. Introduction

Fractional calculus generalizes the derivative and the integral of some function to the non-integer order. The study of fractional differential equations has become a very important area of mathematics due to its numeros applications in various fields of physics, biophysics, mechanics, chemistry and engineering [1–3]. For more details, interested authors can consult for example Kilbas et al. [4], Miller and Ross [5], Oldham and Spanier [6], Podlubny [7] and Samko et al. [8]. Existence and uniqueness results of solution for fractional differential equations drew the attention of many researchers (see [9–18]).In addition, many properties of solutions of this type of problem such as stability, positivity, etc., have been studied and establish these properties to various abstract boundary value problems. Such a importance led to the publication of many research papers in this field, which revealed the flexibility of fractional calculus theory in designing various mathematical models. The main methods conducted in these papers are by terms of fixed point techniques [19–21].

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Integro-differential equations play an important role in various specialties of engineering sciences. Several authors have worked on this type of equations (see [22–32]).

In [33], Momani et al. studied the local and global existence for the following Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t,u(t)) + \int_{t_0}^{t} K(t,s,u(s))ds, \\ \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $f \in C\big([0,1]\times\mathbb{R}^n,\mathbb{R}^n\big)$, $K \in C\big([0,1]\times[0,1]\times\mathbb{R}^n,\mathbb{R}^n\big)$ and ${}^{C}D^{\alpha}$ is the Caputo fractional operator.

Ahmed and Sivasundaram in [34], considered the fractional integro-differential equation in (1.1) with nonlocal condition $u(0) = u_0 - g(u)$, where $0 < \alpha < 1$, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative, $f : [0,T]\times X \longrightarrow X$, $K : [0,T]\times[0,T]\times X \longrightarrow X$ are continuous functions and $g \in C([0,T],X) \longrightarrow X$ where X is a Banach space.

In this paper we study existence and uniqueness results for the following fractional integro-differential problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = h(u(t)) + f(t,u(t)) + \int_{0}^{t} K(t,s,u(s))ds, \quad t \in [0,1], \\ \\ u(0) = \sigma \int\limits_{0}^{\xi} u(s)ds, \quad 0 < \xi < 1. \end{cases} \tag{1.2}$$

where σ is a real constant, $0 < \alpha < 1$, ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$, $K : [0,1]\times[0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$, $h \in C([0,1],\mathbb{R})$ are appropriate functions satisfying some conditions which will be stated later.

2. Notations and Notion Preliminaries

In the present section, we present some notations, definitions and auxiliary lemmas concerning fractional calculus and fixed point throrems. Let $J = [0,1]$ and $C(J,\mathbb{R})$, $C^n(J,\mathbb{R})$ denotes respectively the Banach spaces of all continuous bounded functions and n times continuously differentiable functions on J. In addition, we define the norm $\|g\| = \max\{|g(t)| : t \in J\}$ for any continuously function $g : J \longrightarrow \mathbb{R}$.

Definition 2.1. [4, 7] Let $\alpha > 0$ and $g : J \longrightarrow \mathbb{R}$. The left sided Riemann-Liouville fractional integral of order α of a function g is defined by

$$I^{\alpha}_{0^+}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t \in J. \tag{2.1}$$

Definition 2.2. [4, 8] Let $n-1 < \alpha < n$, $(n \in \mathbb{N}^{\star})$ and $g \in C^n(J,\mathbb{R})$. The left sided Caputo fractional derivative of order α of a function g is given by

$$\begin{aligned} {}^{C}D^{\alpha}_{0^+}g(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}g^{(n)}(s)ds \\ &= I^{n-\alpha}_{0^+}\frac{d^n}{dt^n}g(t), \quad t \in J. \end{aligned} \tag{2.2}$$

Lemma 2.3. [4, 8] *For real numbers* $\alpha, \beta > 0$ *and appropriate function* g, *we have for all* $t \in J$:
1) $I^{\alpha}_{0+} I^{\beta}_{0+} g(t) = I^{\beta}_{0+} I^{\alpha}_{0+} g(t) = I^{\alpha+\beta}_{0+} g(t)$.

2) $I^{\alpha}_{0+} {}^{C}D^{\alpha}_{0+} g(t) = g(t) - g(0), \quad 0 < \alpha < 1$.

3) ${}^{C}D^{\alpha}_{0+} I^{\alpha}_{0+} g(t) = g(t)$.

Lemma 2.4. [35](Banach fixed point theorem) *Let* U *be a non-empty complete metric space and* $T : U \longrightarrow U$ *is contraction mapping. Then, there exists a unique point* $u \in U$ *such that* $T(u) = u$.

Lemma 2.5. [35](Krasnoselskii fixed point theorem) *Let* E *be bounded, closed and convex subset in a Banach space* X. *If* $T_1, T_2 : E \longrightarrow E$ *are two operators that satisfy the following conditions*
1) $T_1 x + T_2 y \in E, \quad$ *for every* $x, y \in E$
2) T_1 *is a contraction*
3) T_2 *is compact and continuous.*
then, there exists $z \in E$ *such that* $T_1 z + T_2 z = z$.

3. Existence and Uniqueness Result

Before presenting our main results, we need the following auxiliary lemma.

Lemma 3.1. *Let* $0 < \alpha < 1$ *and* $\sigma \neq \frac{1}{\xi}$. *Assume that* h, f *and* K *are three continuous functions. If* $u \in C(J, \mathbb{R})$ *then* u *is solution of* (1.2) *if and only if* u *satisfies the integral equation*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[h(u(s)) + f(s, u(s)) + \int_0^s K(s, \tau, u(\tau)) d\tau \right] ds \\ &+ \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^{\xi} (\xi - \tau)^{\alpha} \left[h(u(\tau)) + f(\tau, u(\tau)) + \int_0^{\tau} K(\tau, \lambda, u(\lambda)) d\lambda \right] d\tau. \end{aligned}$$

Proof. Let $u \in C(J, \mathbb{R})$ be a solution of (1.2). Firstly, we show that u is solution of integral equation (3.1). By Lemma 2.3, we obtain

$$I^{\alpha}_{0+} {}^{C}D^{\alpha}_{0+} u(t) = u(t) - u(0). \tag{3.1}$$

In addition, from equation in (1.2) and Definition 2.1, we have

$$\begin{aligned} I^{\alpha}_{0+} {}^{C}D^{\alpha}_{0+} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[h(u(s)) + f(s, u(s)) \\ &\quad + \int_0^s K(s, \tau, u(\tau)) d\tau \Big] ds. \end{aligned} \tag{3.2}$$

By substituting (3.2) in (3.1) with nonlocal condition in problem (1.2), we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[h(u(s)) + f(s, u(s)) \\ &\quad + \int_0^s K(s, \tau, u(\tau)) d\tau \Big] ds + u(0), \end{aligned} \tag{3.3}$$

but, we have

$$\begin{aligned} u(0) &= \sigma\int_0^\xi u(s)ds \\ &= \frac{\sigma}{\Gamma(\alpha)}\int_0^\xi \left[\int_0^s (s-\tau)^{\alpha-1}\Big(h(u(\tau))+f(\tau,u(\tau))+\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big)d\tau\right]ds \\ &\quad +\sigma\xi u(0) \\ &= \frac{\sigma}{\Gamma(\alpha)}\Big[\int_0^\xi\int_0^s (s-\tau)^{\alpha-1}h(u(\tau))d\tau ds+\int_0^\xi\int_0^s(s-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau ds \\ &\quad +\int_0^\xi\int_0^s(s-\tau)^{\alpha-1}\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda d\tau ds\Big]+\sigma\xi u(0). \end{aligned}$$

Consequently,

$$\begin{aligned} u(0) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha)}\Big[\int_0^\xi\int_0^s (s-\tau)^{\alpha-1}h(u(\tau))d\tau ds+\int_0^\xi\int_0^s(s-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau ds \\ &\quad +\int_0^\xi\int_0^s(s-\tau)^{\alpha-1}\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda d\tau ds\Big]. \end{aligned}$$

Using Fubini's theorem and after some manipulations we obtain:

$$u(0)=\frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^\xi(\xi-\tau)^\alpha\left[h(u(\tau))+f(\tau,u(\tau))+\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\right]d\tau.$$

Now, by substituting the last value of $u(0)$ in (3.3) we find (3.1).
Conversely, in view of Lemma 2.3 and by applying the operator ${}^CD^\alpha_{0+}$ on both sides of (3.1), we get

$$\begin{aligned} {}^CD^\alpha_{0+}u(t) &= {}^CD^\alpha_{0+}I^\alpha_{0+}h(u(t))+{}^CD^\alpha_{0+}I^\alpha_{0+}f(t,u(t))+{}^CD^\alpha_{0+}I^\alpha_{0+}\left(\int_0^t K(t,s,u(s))ds\right) \\ &= h(u(t))+f(t,u(t))+\int_0^t K(t,s,u(s))ds, \end{aligned} \tag{3.4}$$

this means that u satisfies the equation in problem (1.2). Furthermore, by substituting t by 0 in integral equation (3.1), we have clearly that the nonlocal condition in (1.2) holds. Therefore, u is solution of problem (1.2), which completes the proof. ∎

We will prove an existence and uniqueness result of the problem (1.2) in $C(J,\mathbb{R})$ by using Banach's fixed point theorem. For this fact, we will need some assumptions about the functions h, f and K previously defined.
(H_1) : $|h(u(t))-h(v(t))|\leq k_1\|u-v\|, \quad t\in J, \quad u,v\in\mathbb{R}$.
(H_2) : $|f(t,u(t))-f(t,v(t))|\leq k_2\|u-v\|, \quad t\in J,\ u,v\in\mathbb{R}$.
(H_3) : $|K(t,s,u(s))-K(t,s,v(s))|\leq k_3\|u-v\|, \quad (t,s)\in D,\ u,v\in\mathbb{R}$.
where k_1,k_2,k_3 are three positive real constants and $D=\{(t,s):\ \ 0\leq s\leq t\leq 1\}$.

Theorem 3.2. *Assume that the assumptions $(H_1),(H_2)$ and (H_3) hold. If*

$$\begin{aligned} &\frac{k_1+k_2}{\Gamma(\alpha+1)}+\frac{k_3}{\Gamma(\alpha+2)}+\frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} \\ &+\frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}<1, \end{aligned} \tag{3.5}$$

then the fractional integro-differential problem (1.2) *has a unique solution on* $C(J,\mathbb{R})$.

Proof. Firstly, we define an operator $P: C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by

$$\begin{aligned} Pu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[h(u(s)) + f(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau\Big]ds \\ &+ \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau \end{aligned}$$

and we consider the subset B_r of $C(J,\mathbb{R})$ defined by

$$B_r = \{u \in C(J,\mathbb{R}): \quad \|u\| \le r\} \tag{3.6}$$

where r is a strictly positive real number chosen so that

$$r \ge \frac{\frac{M_1+M_2}{\Gamma(\alpha+1)} + \frac{M_3}{\Gamma(\alpha+2)} + \frac{|\sigma|M_1+|\sigma|M_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|M_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}}{1 - \frac{k_1+k_2}{\Gamma(\alpha+1)} - \frac{k_3}{\Gamma(\alpha+2)} - \frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} - \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}}, \tag{3.7}$$

with $M_1 = \big|h(0)\big|$, $\quad M_2 = \sup_{s\in J}\big|f(s,0)\big|$, $\quad$ and $M_3 = \sup_{(s,\tau)\in D}\big|K(s,\tau,0)\big|$.
Now, we show that the operator P has a unique fixed point on B_r which represents the unique solution of the problem (1.2). Our proof is down in two steps.

First step: We have to show that $PB_r \subset B_r$. For each $t \in J$ and for any $u \in B_r$, we have

$$\begin{aligned} &\big|(Pu)(t)\big| \\ \le\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|h(u(s))\big| + \big|f(s,u(s))\big| + \int_0^s \big|K(s,\tau,u(\tau))\big|d\tau\Big]ds \\ &+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[|h(u(\tau))| + |f(\tau,u(\tau))| + \int_0^\tau |K(\tau,\lambda,u(\lambda))|d\lambda\Big]d\tau \\ \le\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|h(u(s))-h(0)\big| + \big|h(0)\big|\Big]ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|f(s,u(s))-f(s,0)\big| + \big|f(s,0)\big|\Big]ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \Big[\big|K(s,\tau,u(\tau))-K(s,\tau,0)\big| + \big|K(s,\tau,0)\big|\Big]d\tau ds \\ &+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[\big|h(u(\tau))-h(0)\big| + \big|h(0)\big|\Big]d\tau \\ &+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[\big|f(\tau,u(\tau))-f(\tau,0)\big| + \big|f(\tau,0)\big|\Big]d\tau \\ &+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\int_0^\tau \Big[\big|K(\tau,\lambda,u(\lambda))-K(\tau,\lambda,0)\big| + \big|K(\tau,\lambda,0)\big|\Big]d\lambda d\tau \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\Big[k_1\|u\|+M_1\Big]t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Big[k_2\|u\|+M_2\Big]t^{\alpha}}{\Gamma(\alpha+1)}+\frac{k_3\|u\|t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{M_3t^{\alpha+1}}{\Gamma(\alpha+2)}\\
& +\frac{|\sigma|\Big[k_1\|u\|+M_1\Big]\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)}+\frac{|\sigma|\Big[k_2\|u\|+M_2\Big]\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)}+\frac{|\sigma|k_3\|u\|\xi^{\alpha+2}}{|1-\sigma\xi|\Gamma(\alpha+3)}\\
& +\frac{|\sigma|M_3\xi^{\alpha+2}}{|1-\sigma\xi|\Gamma(\alpha+3)}\\
\leq\ & \left[\frac{k_1+k_2}{\Gamma(\alpha+1)}+\frac{k_3}{\Gamma(\alpha+2)}+\frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}+\frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right]r\\
& +\frac{M_1+M_2}{\Gamma(\alpha+1)}+\frac{M_3}{\Gamma(\alpha+2)}+\frac{|\sigma|M_1+|\sigma|M_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}+\frac{|\sigma|M_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\\
\leq\ & r.
\end{aligned}
$$

Therefore $\|Pu\| \leq r$, which means that $PB_r \subset B_r$.

Second step: We shall show that $P : B_r \longrightarrow B_r$ is a contraction.
In view of the assumptions $(H_1), (H_2)$ and (H_3), we have for any $u, v \in B_r$ and for each $t \in J$

$$
\begin{aligned}
& \big|(Pu)(t)-(Pv)(t)\big|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|h(u(s))-h(v(s))\big|ds\\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s,u(s))-f(s,v(s))\big|ds\\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \big|K(s,\tau,u(\tau))-K(s,\tau,v(\tau))\big|d\tau ds\\
& +\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^{\xi}(\xi-\tau)^{\alpha}\big|h(u(\tau))-h(v(\tau))\big|d\tau\\
& +\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^{\xi}(\xi-\tau)^{\alpha}\big|f(\tau,u(\tau))-f(\tau,v(\tau))\big|d\tau\\
& +\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^{\xi}(\xi-\tau)^{\alpha}\int_0^{\tau}\big|K(\tau,\lambda,u(\lambda))-K(\tau,\lambda,v(\lambda))\big|d\lambda d\tau\\
\leq\ & \left[\frac{k_1t^{\alpha}}{\Gamma(\alpha+1)}+\frac{k_2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{k_3t^{\alpha+1}}{\Gamma(\alpha+2)}\right]\|u-v\|\\
& +\left[\frac{|\sigma|k_1}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}+\frac{|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}+\frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right]\|u-v\|\\
\leq\ & \left[\frac{k_1+k_2}{\Gamma(\alpha+1)}+\frac{k_3}{\Gamma(\alpha+2)}+\frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}+\frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right]\|u-v\|.
\end{aligned}
$$

By exploiting estimation (3.5), it follows that P is a contraction. All assumptions of Lemma 2.4 are satisfied, then there exists $u \in C(J, \mathbb{R})$ such that $Pu = u$ which is the unique solution of problem (1.2) in $C(J, \mathbb{R})$. This completes the proof of Theorem 3.2. ■

Example 3.3. Consider the following nonlocal fractional integro-differential problem

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}} u(t) = \dfrac{1}{48}\sin(u(t)) + \dfrac{u(t)}{90+e^{-t}} + \displaystyle\int_0^t \dfrac{e^{s-t}}{64} u(s)ds, \quad t \in [0,1], \\ u(0) = \dfrac{1}{10} \displaystyle\int_0^{\frac{1}{4}} u(s)ds. \end{cases} \tag{3.8}$$

where $\alpha = \dfrac{1}{2}$, $\sigma = \dfrac{1}{10}$, $\xi = \dfrac{1}{4}$, $h(u) = \dfrac{1}{48}\sin(u)$, $f(t,u) = \dfrac{u}{90+e^{-t}}$, and $K(t,s,u) = \dfrac{e^{s-t}}{64}u$. For $u, v \in \mathbb{R}^+$ and $t \in [0,1]$, we have:

$$\big|h(u(t)) - h(v(t))\big| \leq \frac{1}{48}\|u-v\|,$$

$$\big|f(t,u) - f(t,v)\big| \leq \frac{1}{90}\|u-v\|,$$

and

$$\big|K(t,s,u(s)) - K(t,s,v(s))\big| \leq \frac{1}{64}\|u-v\|.$$

Now, the assumptions (H_1), (H_2) and (H_3) are satisfied with $k_1 = \dfrac{1}{48}$, $k_2 = \dfrac{1}{90}$ and $k_3 = \dfrac{1}{64}$, then after some computations, we find that:

$$\frac{k_1+k_2}{\Gamma(\alpha+1)} + \frac{k_3}{\Gamma(\alpha+2)} + \frac{|\sigma|k_1 + |\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2} \approx 0.0479 < 1.$$

Therefore, by applying Theorem 3.2 the problem (3.8) has a unique solution on $[0,1]$.

4. Existence Result

In the present section, we will demonstrate an existence result of the fractional integro-differential problem (1.2). For this fact, we need the following assumptions.
(H_4) $h : J \longrightarrow \mathbb{R}$ is continuous and there exists $0 < M < 1$ such that

$$|h(u(t)) - h(v(t))| \leq M\|u-v\|, \ t \in J, \ u, v \in \mathbb{R}. \tag{4.1}$$

(H_5) $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists $\phi \in L^\infty(J, \mathbb{R}^+)$ such that

$$|f(t,u(t)) - f(t,v(t))| \leq \phi(t)\|u-v\|, \ t \in J, \ u, v \in \mathbb{R}. \tag{4.2}$$

(H_6) $K : D \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on D and there exists $\rho \in L^1(J, \mathbb{R}^+)$ such that

$$|K(t,s,u(s)) - K(t,s,v(s))| \leq \rho(t)\|u-v\|, (t,s) \in D\ , \ u, v \in \mathbb{R}, \tag{4.3}$$

where $D = \{(t,s) : 0 \leq s \leq t \leq 1\}$.

Theorem 4.1. *Suppose that the assumptions (H_4), (H_5) and (H_6) hold. If*

$$\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1}\Big]\xi^{\alpha+1} < 1. \tag{4.4}$$

Then, the fractional integro-differential problem (1.2) has at least one solution in $C(J,\mathbb{R})$ on J.

Proof. First, we transform the problem (1.2) into a fixed point problem. For this fact we define the operator $P : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by

$$\begin{aligned} Pu(t) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^{\xi} (\xi-\tau)^{\alpha} \left[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^{\tau} K(\tau,\lambda,u(\lambda))d\lambda \right] d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t} (t-s)^{\alpha-1} \left[h(u(s)) + f(s,u(s)) + \int_0^{s} K(s,\tau,u(\tau))d\tau \right] ds \end{aligned}$$

Before starting the proof of our theorem, we decompose the operator P into a sum of two operators F and G, where

$$Fu(t) = \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^{\xi} (\xi-\tau)^{\alpha} \left[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^{\tau} K(\tau,\lambda,u(\lambda))d\lambda \right] d\tau$$

and

$$Gu(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t} (t-s)^{\alpha-1} \left[h(u(s)) + f(s,u(s)) + \int_0^{s} K(s,\tau,u(\tau))d\tau \right] ds$$

For any function $u \in C(J,\mathbb{R})$, we define the norm

$$\|u\| = \max\big\{|u(t)| : t \in J\big\}.$$

Now, our existence result will be discussed in several steps:

Step (1):

Let $\mu = \sup_{(s,u)\in J\times S_r} |f(s,u)|$, $\mu^{\star} = \sup_{(s,\tau,u)\in D\times S_r} \int_0^{s} |K(s,\tau,u(\tau))|d\tau$ and $\eta = \sup_{u\in S_r} |h(u)|$, define the set $S_r = \big\{u \in C(J,\mathbb{R}) : \|u\| \le r\big\}$, where r is a real constant positive number such that

$$r \ge \left[\frac{|\sigma|\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)} \right] (\eta+\mu+\mu^{\star}) \tag{4.5}$$

and prove that $Fu + Gv \in S_r \subset C(J,\mathbb{R})$, for every $u, v \in S_r$.
For $u \in S_r$ and $t \in J$, we have

$$\begin{aligned} |Fu(t)| &\le \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^{\xi} (\xi-\tau)^{\alpha} \left[|h(u(\tau))| + |f(\tau,u(\tau))| + \int_0^{\tau} |K(\tau,\lambda,u(\lambda))|d\lambda \right] d\tau \\ &\le \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^{\xi} (\xi-\tau)^{\alpha} \\ &\quad \times \left[\sup_{u\in S_r} |h(u)| + \sup_{(\tau,u)\in J\times S_r} |f(\tau,u)| + \sup_{(\tau,\lambda,u)\in D\times S_r} \int_0^{\tau} |K(\tau,\lambda,u)|d\lambda \right] d\tau \\ &= \frac{|\sigma|\left[\eta+\mu+\mu^{\star}\right]}{|1-\sigma\xi|\Gamma(\alpha+2)} \xi^{\alpha+1}. \end{aligned}$$

Thus,

$$\|Fu\| \le \frac{|\sigma|\left[\eta+\mu+\mu^{\star}\right]}{|1-\sigma\xi|\Gamma(\alpha+2)} \xi^{\alpha+1}. \tag{4.6}$$

In a similar way, for $v \in S_r$ and $t \in J$, we find

$$\begin{aligned} |Gv(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[|h(v(s))| + |f(s,v(s))| + \int_0^s |K(s,\tau,v(\tau))| d\tau \Big] ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \Big[\sup_{v \in S_r} |h(v)| + \sup_{(s,v) \in J \times S_r} |f(s,v)| + \sup_{(s,\tau,v) \in D \times S_r} \int_0^s |K(s,\tau,v)| d\tau \Big] ds \\ &\leq \frac{\eta + \mu + \mu^\star}{\Gamma(\alpha+1)}. \end{aligned}$$

Therefore,

$$\|Gv\| \leq \frac{\eta + \mu + \mu^\star}{\Gamma(\alpha+1)}. \tag{4.7}$$

Consequently, in view of inequalities (4.5)-(4.7), we get

$$\begin{aligned} \|Fu + Gv\| &\leq \|Fu\| + \|Gv\| \\ &\leq \frac{|\sigma| \big[\eta + \mu + \mu^\star \big]}{|1 - \sigma\xi| \Gamma(\alpha+2)} \xi^{\alpha+1} + \frac{\eta + \mu + \mu^\star}{\Gamma(\alpha+1)} \\ &\leq r. \end{aligned}$$

This means that $Fu + Gv \in S_r$.

Step (2):
We show that F is contraction map on S_r. From the definition of the operator F and by using Fubini's theorem, we can write

$$\begin{aligned} Fu(t) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^\tau K(\tau,\lambda,u(\lambda)) d\lambda \Big] d\tau \\ &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[h(u(\tau)) + f(\tau,u(\tau)) \Big] d\tau \\ &\quad + \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\tau)^\alpha K(\tau,\lambda,u(\lambda)) d\tau d\lambda. \end{aligned}$$

Therefore, for $u, v \in S_r$ and $t \in J$ we find

$$\begin{aligned} &|Fu(t) - Fv(t)| \\ &\leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[|h(u(\tau)) - h(v(\tau))| + |f(\tau,u(\tau)) - f(\tau,v(\tau))| \Big] d\tau \\ &\quad + \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\tau)^\alpha |K(\tau,\lambda,u(\lambda)) - K(\tau,\lambda,v(\lambda))| d\tau d\lambda \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi(\xi-\tau)^\alpha\Big[M|u(\tau)-v(\tau)|+\phi(\tau)\|u(\tau)-v(\tau)|\Big]d\tau\\
&\quad+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi\int_\lambda^\xi(\xi-\tau)^\alpha\rho(\tau)|u(\lambda)-v(\lambda)|d\tau d\lambda.\\
&\leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi(\xi-\tau)^\alpha\Big[M|u(\tau)-v(\tau)|+\phi(\tau)|u(\tau)-v(\tau)|\Big]d\tau\\
&\quad+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi\int_\lambda^\xi(\xi-\lambda)^\alpha\rho(\tau)|u(\lambda)-v(\lambda)|d\tau d\lambda.\\
&\leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M\|u-v\|+\|\phi\|_{L^\infty}\|u-v\|+\|\rho\|_{L^1}\|u-v\|\Big]\xi^{\alpha+1}\\
&\leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M+\|\phi\|_{L^\infty}+\|\rho\|_{L^1}\Big]\xi^{\alpha+1}\|u-v\|.
\end{aligned}
$$

Thus,

$$
\|Fu-Fv\|\leq\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M+\|\phi\|_{L^\infty}+\|\rho\|_{L^1}\Big]\xi^{\alpha+1}\|u-v\|
$$

Therefore, by using (4.4) we conclude that F is a contraction map on S_r.

Step (3):

To show that G is a compact operator, we claim that $\overline{G(S_r)}$ is a compact subset of $C(J,\mathbb{R})$. To show this, we need only to prove that $G(S_r)$ is uniformly bounded and equicontinuous subset of $C(J,\mathbb{R})$.
Firstly, it is clear by inequality (4.7), that $G(S_r)$ is uniformly bounded.
Next, we will prove that $G(S_r)$ is equicontinuous subset of $C(J,\mathbb{R})$..
For this we have for any $u\in S_r$ and for each $t_1,t_2\in J$ where $t_1\leq t_2$:

$$
\begin{aligned}
&\Big|Gu(t_2)-Gu(t_1)\Big|\\
\leq\;&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big[|h(u(s))|+|f(s,u(s))|+\int_0^s|K(s,\tau,u(\tau))|d\tau\Big]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big|\Big[|h(u(s))|+|f(s,u(s))|+\int_0^s|K(s,\tau,u(\tau))|d\tau\Big]ds\\
\leq\;&\frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+\int_0^{t_1}\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big|ds\Big]\\
&\times\Big[\sup_{v\in S_r}|h(u)|+\sup_{(s,u)\in J\times S_r}|f(s,u)|+\sup_{(s,\tau,u)\in D\times S_r}\int_0^s|K(s,\tau,u)|d\tau\Big]\\
\leq\;&\frac{\eta+\mu+\mu^\star}{\Gamma(\alpha+1)}\Big[2(t_2-t_1)^\alpha+t_2^\alpha-t_1^\alpha\Big]
\end{aligned}
$$

where η, μ, and $\mu^\star$ are the constants defined in step (1). The right hand side of the above inequality is independent of u and tends to zero when $t_2\longrightarrow t_1$, then $\|Gu(t_2)-Gu(t_1)\|\longrightarrow 0$, which means that $G(S_r)$ is equicontinuous.
Finally, from the continuity of h, f and K, it follows that the operator $G:S_r\longrightarrow S_r$ is continuous. So the operator G is compact on S_r. Now, all assumptions of lemma 2.5 are satisfied. Therefore, the operator $P=F+G$ has a fixed point on S_r. Then the fractional

integro-differential problem (1.2) has a solution $u \in C(J, \mathbb{R})$. This completes the proof of the theorem 4.1. ■

Example 4.2. Consider the following nonlocal fractional integro-differential problem

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2}} u(t) = \frac{1}{10}\sin^2(u(t)) + \frac{2u(t)}{13+e^t} + \int_0^t \frac{e^{2t}}{5+e^s} u(s)ds, \ t \in [0,1], \\ u(0) = \frac{1}{3}\int_0^{\frac{1}{4}} u(s)ds. \end{cases} \tag{4.8}$$

In this example, we have: $\alpha = \frac{1}{2}$, $\sigma = \frac{1}{3}$, $\xi = \frac{1}{4}$, $h(u) = \frac{1}{10}\sin^2(u)$, $f(t,u) = \frac{2u}{13+e^t}$, and $K(t,s,u) = \frac{e^{2t}}{5+e^s}u$. Then for $u, v \in \mathbb{R}^+$ and $t \in J$, we have:

$$\left|h(u(t)) - h(v(t))\right| \leq \frac{1}{5}\left\|u - v\right\|,$$

$$\left|f(t,u) - f(t,v)\right| \leq \frac{2}{13+e^t}\left\|u - v\right\|$$

and

$$\left|K(t,s,u(s)) - K(t,s,v(s))\right| \leq \frac{1}{6}e^{2t}\left\|u - v\right\|.$$

So, The assumptions (H_4), (H_5) and (H_6) are satisfied with $M = \frac{1}{5}$, $\phi(t) = \frac{2}{13+e^t}$ and $\rho(t) = \frac{1}{6}e^{2t}$ where $\|\phi\|_{L^\infty} = \frac{1}{7}$ and $\|\rho\|_{L^1} = \frac{1}{12}(e^2 - 1)$. Now, some elementary computations give us

$$\frac{|\sigma|}{|1 - \sigma\xi|\Gamma(\alpha+2)}\Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1}\Big]\xi^{\alpha+1} \approx 0.0299 < 1$$

which means that the condition (4.4) holds. Therefore, by applying theorem 4.1, we deduce that the nonlocal fractional integro-differential problem (4.8) has a solution on $[0, 1]$.

Conclusion

In this paper, we considered a new fractional class of integro-differential equation in the context of the standard Caputo fractional derivative. The main goal in the present paper is to derive several criteria of the existence and uniqueness of solutions for mentioned initial value problem. To acheive our aim, we first transformed our main problem into equivalent fixed point problem. After that with the help of the fixed point theorems of Banach and Krasnoselskii we proved our results of existence and uniqueness of solutions to our problem in a well-defined Banach space. Finally, we have illustrated our theoretical results with examples.

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