Thai Journal of Mathematics
Volume 7 (2009) Number 1 : 35–47

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

General Cesàro mean approximation methods for nonexpansive mappings in Hilbert spaces*

P. Markshoe and R. Wangkeeree

Abstract : Let C be a nonempty closed convex subset of a real Hilbert space H, f a contraction on C and A a strongly bounded linear operator on H with coefficient $\bar{\gamma} > 0$. Consider a general Cesáro mean iterative method

$$x_0 \in C, \;\; x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I + \alpha_n A)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \;\; n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ satisfying certain conditions, and T is a nonexpansive mapping of C into itself. It is proved that the sequence $\{x_n\}$ generated by above method, converges strongly to a point $\tilde{x} \in F(T)$ which solves the variational inequality

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - x \rangle \leq 0, \;\; x \in F(T). \tag{0.1}$$

The results presented in this paper generalize, extend and improve the corresponding results of Shimizu and Takahashi [14], Matsushita and Kuroiwa [8] and many others.

Keywords : Fixed point; Variational inequality; Viscosity approximation; Nonexpansive mapping; Hilbert space.
2000 Mathematics Subject Classification : 47H09, 47H10, 47H17.

1 Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and let x be an element of C. Let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. For each t with $0 < t < 1$, let x_t be a unique point of C which satisfies

$$x_t = tx + (1-t)Tx_t.$$

*Supported by Faculty of Science, Naresuan University, THAILAND.

Browder [1] showed that $\{x_t\}$ converges strongly as $t \longrightarrow 0$ to the element of $F(T)$ which is nearest to x in $F(T)$. This result was extended to a Banach space by Reich [12] and Takahashi and Ueda [18]. On the other hand, Wittmann [20] showed that each sequence $\{x_n\}$ defined by

$$x_0 \in C, x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n \text{ for } n = 0, 1, 2, ...,$$

converges strongly to the element of $F(T)$ which is nearest to x if $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1, \alpha_n \longrightarrow 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Using an idea of Browder [1], Shimizu and Takahashi [15] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = \alpha_n x + (1-\alpha_n)\frac{1}{n}\sum_{j=1}^{n} T^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 < \alpha_n < 1$ and $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. Shimizu and Takahashi [14] also studied the convergence of another iteration process for a family of nonexpansive mappings in the framework of a Hilbert space. The iteration process is a mixed iteration process of Wittmann's [20] and Shimizu and Takahash's [15]. For simplicity, we state their result for a nonexpansive mapping T with $F(T)$ is nonempty. They show that each sequence $\{x_n\}$ defined by

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.2}$$

converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ if $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1, \alpha_n \longrightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. But this approximation method is not suitable for some nonexpansive nonself-mappings. In the framework of a real Hilbert space, Matsushita and Kuroiwa [8] studied the strong convergence of the iterative method below. For $x \in C$,

$$x_0 \in C, \;\; x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} (P_C T)^j x_n \text{ for } n = 0, 1, 2, \ldots, \tag{1.3}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, P_C is the metric projection of H onto C, and T is a nonexpansive nonself-mapping of C into H. Using the nowhere normal outward condition on T and the appropriate assumptions imposed upon the parameters sequences $\{\alpha_n\}$, they proved that $\{x_n\}$ generated by (1.3) converges strongly as $n \longrightarrow \infty$ to an element of fixed point of T when $F(T)$ is nonempty.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [3, 23, 22] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2 \;\; \text{for all } x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu ([22]) proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \; n \geq 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\alpha_n\}$ satisfies certain conditions that will be made precise in Section 3.

Using the viscosity approximation method, Moudafi [10] introduced the following iterative iterative process for nonexpansive mappings (see [21] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \; n \geq 0, \tag{1.7}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [10, 21] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, x \in C. \tag{1.8}$$

Recently, Marino and Xu [6] was combine the iterative method (1.6) with the viscosity approximation method (1.7) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), n \geq 0. \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, x \in C, \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for $\gamma f(i.e., h'(x) = \gamma f(x)$ for $x \in H)$

Inspired and motivated by the above research, we introduce the general Cesáro mean iterative method for a nonexpansive mapping in a real Hilbert space as follows:

$$x_0 \in C, \;\; x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I + \alpha_n A)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \; n \geq 0.$$

where f is a contraction on C, A a strongly bounded linear operator on H with coefficient $\bar{\gamma} > 0$, and investigate the problem of approximating a fixed point of a nonexpansive mapping which solves some variational inequality. The results of this paper extend and improve the results of Shimizu and Takahashi [14], Matsushita and Kuroiwa [8] and many others.

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H.

A space X is said to satisfy Opials condition [11] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|, \ \ \forall y \in X, y \neq x.$$

Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $x = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Definition 2.1. *A mapping $T : C \longrightarrow H$ is said to satisfy nowhere normal outward condition ((NNO) for short) if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C.*

Lemma 2.2. ([7, Proposition 2, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty, then T satisfies (NNO) condition.*

Lemma 2.3. ([7, Proposition 1, P. 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C and $T : C \longrightarrow H$ a nonself-mapping. Suppose that T satisfies (NNO) condition. Then $F(PT) = F(T)$.*

Lemma 2.4. ([8]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \longrightarrow C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\{x_{n+1} - \frac{1}{n+1}\sum_{i=1}^{n+1} T^i x_n\}$ converges strongly to 0 as $n \longrightarrow \infty$ and let $\{x_{n_j}\}$ be a subsequnec of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x. Then x is a fixed point of T.*

Lemma 2.5. ([21]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.
(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.6. ([17]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$.*

Lemma 2.7. ([6]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

3 Main Results

In this section, we prove the strong convergence theorem for a nonexpansive mappings in a real Hilbert space. Before proving it, we need the following lemma.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \longrightarrow C$ be a contraction with coefficient $\alpha \in (0,1)$, and A be a strongly bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \ \ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \ n \geq 1. \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ satisfying
(C1) $\alpha_n + \beta_n < 1, \lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and either*
(C2) $\lim_{n\longrightarrow\infty} \beta_n = 0$ *or*
(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.
Then
(i) $\{x_n\}$ *is bounded and*
(ii) $\lim_{n\longrightarrow\infty} \|x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n\| = 0$.

Proof. Note that from the condition $\lim_{n\longrightarrow\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.7, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I-\alpha_n A)x,x\rangle &= 1-\beta_n-\alpha_n\langle Ax,x\rangle\\ &\geq 1-\beta_n-\alpha_n\|A\|\\ &\geq 0,\end{aligned}$$

this show that $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{|\langle((1-\beta_n)I-\alpha_n A)x,x\rangle| : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned}$$

For any $p\in F(T)$, we can calculate

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n-p)\|\\ &\leq (1-\beta_n-\alpha_n\bar{\gamma})\|\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\|\\ &\leq (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\|\\ &\leq (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|\\ &\leq (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|+(\bar{\gamma}-\gamma\alpha)\alpha_n\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}.\end{aligned}$$

It follows from induction that

$$\|x_n-p\|\leq\max\left\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\beta}\right\}, n\geq 1. \tag{3.2}$$

Hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}, \{\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n\}$. Further, we note that

$$\begin{aligned}\|x_{n+1}-\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n\| &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n-\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n\|\\ &\leq \alpha_n\|\gamma f(x_n)-A[\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n]\|+\beta_n\|x_n-\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n\|. \end{aligned}\tag{3.3}$$

Assume that (C2) holds. It follows from (3.3) and the conditions (C1) and (C2) that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-\frac{1}{n+1}\sum_{j=0}^{n}T^j x_n\|=0.$$

Assume that (C3) holds. For all $n\geq 0$, we define $T_n := \frac{1}{n+1}\sum_{j=0}^{n}T^j$. Setting

$$y_n=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)W_n x_n}{1-\beta_n},$$

we have $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, n \geq 1$. It follows that

$$\begin{aligned} y_{n+1} - y_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)T_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)T_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}\gamma f(x_n) + T_{n+1}x_{n+1} - T_n x_n \\ &\quad + \frac{\alpha_n}{1-\beta_n}AT_n x_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}}AT_{n+1}x_{n+1} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - AT_{n+1}x_{n+1}) + \frac{\alpha_n}{1-\beta_n}(AT_n x_n - \gamma f(x_n)) \\ &\quad + T_{n+1}x_{n+1} - T_n x_n. \qquad (3.4) \end{aligned}$$

Consider

$$\begin{aligned} \|T_{n+1}x_{n+1} - T_n x_n\| &= \|\frac{1}{n+2}\sum_{j=0}^{n+1} T^j x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n\| \\ &= \|\frac{1}{n+2}\sum_{n=0}^{n} T^j x_{n+1} + \frac{1}{n+2}T^{n+1}x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n + \frac{1}{n+1}\sum_{j=0}^{n} T^j x_{n+1} \\ &\quad - \frac{1}{n+1}\sum_{j=0}^{n} T^j x_{n+1}\| \\ &= \|\frac{1}{(n+1)}\sum_{j=0}^{n}\left[T^j x_{n+1} - T^j x_n\right] + \frac{1}{(n+2)}T^{n+1}x_{n+1} \\ &\quad - \frac{1}{(n+1)(n+2)}\sum_{j=0}^{n} T^j x_{n+1}\| \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{(n+1)}\sum_{j=0}^{n}\|T^j x_{n+1} - T^j x_n\| + \frac{1}{(n+2)}\|T^{n+1}x_{n+1} - p + p\| \\ &\quad - \frac{1}{(n+2)}\|\frac{1}{(n+1)}\sum_{j=0}^{n} T^j x_{n+1} - p + p\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{n+2}\|x_{n+1} - p\| + \frac{1}{n+2}\|x_{n+1} - p\| + \frac{2}{n+2}\|p\| \\ &= \|x_{n+1} - x_n\| + \frac{2}{n+2}\|x_{n+1} - p\| + \frac{2}{n+2}\|p\|. \qquad (3.5) \end{aligned}$$

Then

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma(f(x_{n+1})\| + \|AT_{n+1}x_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|AT_n x_n\| + \|\gamma f(x_n)\|) + \\ &\quad + \|x_{n+1} - x_n\| + \frac{2}{n+2}\|x_{n+1} - p\| + \frac{2}{n+2}\|p\|, \qquad (3.6) \end{aligned}$$

which implies that

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma(f(x_{n+1})\|+\|AT_{n+1}x_{n+1}\|)\\ &+\frac{\alpha_n}{1-\beta_n}(\|AT_nx_n\|+\|\gamma f(x_n)\|)+\frac{2}{n+2}\|x_{n+1}-p\|\\ &+\frac{2}{n+2}\|p\|. \qquad (3.7)\end{aligned}$$

Thus, we get

$$\lim_{n\to\infty}\sup(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, by Lemma 2.6, we obtain $\|y_n-x_n\|\to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|y_n-x_n\|=0. \qquad (3.8)$$

Now, we will prove $\|T_nx_n-x_n\|\to 0$ as $n\to\infty$.
Observe that

$$\begin{aligned}\|x_{n+1}-T_nx_n\| &\leq \alpha_n\|\gamma f(x_n)-T_nx_n\|+\beta_n\|x_n-T_nx_n\|\\ &\leq \alpha_n\|\gamma f(x_n)-T_nx_n\|+\beta_n\|x_n-x_{n+1}\|+\beta_n\|x_{n+1}-T_nx_n\|\end{aligned}$$

it follows that

$$(1-\beta_n)\|x_{n+1}-T_nx_n\| \leq \alpha_n\|\gamma f(x_n)-T_nx_n\|+\beta_n\|x_n-x_{n+1}\|. \qquad (3.9)$$

It follows from (C1), (C3), (3.8) and (3.9) that

$$\|x_{n+1}-T_nx_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty \qquad (3.10)$$

that is

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-\frac{1}{n+1}\sum_{j=0}^{n}T^jx_n\|=0.$$

This completes the proof.

□

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f: C\longrightarrow C$ be a contraction with coefficient $\alpha\in(0,1)$, and A be a strongly bounded linear operator on H with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let the sequence $\{x_n\}$ be generated by*

$$x_1\in C,\ \ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)\frac{1}{n+1}\sum_{j=0}^{n}T^jx_n,\ \ n\geq 1. \qquad (3.11)$$

where $\{\alpha_n\}, \{\beta_n\}$ *are the sequences in* $[0, 1]$ *satisfying*
(C1) $\alpha_n + \beta_n < 1, \lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and either*
(C2) $\lim_{n\longrightarrow\infty} \beta_n = 0$ *or*
(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.
Then the sequence $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in F(T)$ *which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, z \in F(T). \tag{3.12}$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$.

Proof. Observe that $P_F(I - A + \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}\|P_F(I - A + \gamma f)(x) - P_F(I - A + \gamma f)(y)\| &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ &\leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ &\leq (1 - \bar{\gamma})\|x - y\| + \gamma\beta\|x - y\| \\ &= (1 - (\bar{\gamma} - \gamma\beta))\|x - y\|.\end{aligned}$$

Since H is complete, there exists a unique element $\tilde{x} \in C$ such that $\tilde{x} = P_F(I - A + \gamma f)(\tilde{x})$. Next, we show that

$$\limsup_{n\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n\rangle \leq 0. \tag{3.13}$$

Since $\{x_n\} \subseteq C$ is bounded, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n\rangle = \lim_{j\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - x_{n_j}\rangle. \tag{3.14}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to $\tilde{q} \in C$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup \tilde{q}$. Applying Lemma 3.1 (ii) and Lemma 2.4, we obtain $\tilde{q} \in F(T)$. It follows from the variational inequality (3.12) and (3.14) that

$$\limsup_{n\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n\rangle = \langle (A - \gamma f)\tilde{x}, \tilde{x} - \tilde{q}\rangle \leq 0.$$

Using Lemma 3.1 (ii) and (3.13), we obtain

$$\limsup_{n\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - T_n x_n\rangle \leq \limsup_{n\longrightarrow\infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - x_{n+1}\rangle \leq 0.$$

Finally, we show that $x_n \longrightarrow \tilde{x}$.

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)T_n x_n-\tilde{x}\|^2 \\
&= \|((1-\beta_n)I-\alpha_n A)(T_n x_n-\tilde{x})+\beta_n(x_n-\tilde{x})+\alpha_n(\gamma f(x_n)-A\tilde{x})\|^2 \\
&= \|((1-\beta_n)I-\alpha_n A)(T_n x_n-\tilde{x})+\beta_n(x_n-\tilde{x})\|^2+\alpha_n^2\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(x_n)-A\tilde{x}\rangle \\
&\quad +2\alpha_n\langle((1-\beta_n)I-\alpha_n A)(T_n x_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\rangle \\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})\|T_n x_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|)^2+\alpha_n^2\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\beta_n\alpha_n\gamma\langle x_n-\tilde{x},f(x_n)-f(\tilde{x})\rangle+2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad +2(1-\beta_n)\gamma\alpha_n\langle T_n x_n-\tilde{x},f(x_n)-f(\tilde{x})\rangle+2(1-\beta_n)\alpha_n\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad -2\alpha_n^2\langle A(T_n x_n-\tilde{x}),\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|)^2+\alpha_n^2\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\beta_n\alpha_n\gamma\alpha\|x_n-\tilde{x}\|^2+2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad +2(1-\beta_n)\gamma\alpha_n\alpha\|x_n-\tilde{x}\|^2+2(1-\beta_n)\alpha_n\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad -2\alpha_n^2\langle A(T_n x_n-\tilde{x}),\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&= [(1-\alpha_n\bar{\gamma})^2+2\beta_n\alpha_n\gamma\alpha+2(1-\beta_n)\gamma\alpha_n\alpha]\|x_n-\tilde{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle+2(1-\beta_n)\alpha_n\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad -2\alpha_n^2\langle A(T_n x_n-\tilde{x}),\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\leq [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-\tilde{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle+2(1-\beta_n)\alpha_n\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad +2\alpha_n^2\|A(T_n x_n-\tilde{x})\|\|\gamma f(\tilde{x})-A\tilde{x}\| \\
&= [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\alpha_n\{\alpha_n(\bar{\gamma}^2\|x_n-\tilde{x}\|^2+\|\gamma f(x_n)-A\tilde{x}\|^2 \\
&\quad +2\|A(T_n x_n-\tilde{x})\|\|\gamma f(\tilde{x})-A\tilde{x}\|)+2\beta_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle \\
&\quad +2(1-\beta_n)\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle\}. \qquad (3.15)
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{T_n x_n\}$ are bounded, we can take a constant $M>0$ such that

$$\bar{\gamma}^2\|x_n-\tilde{x}\|^2+\|\gamma f(x_n)-A\tilde{x}\|^2+2\|A(T_n x_n-\tilde{x})\|\gamma f(\tilde{x})-A\tilde{x}\|\leq M,$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-\tilde{x}\|^2\leq[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\alpha_n\sigma_n, \qquad (3.16)$$

where

$$\sigma_n=2\beta_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle+2(1-\beta_n)\langle T_n x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\rangle+\alpha_n M.$$

We get $\limsup_{n\longrightarrow\infty}\sigma_n\leq 0$. Applying Lemma 2.5 to (3.16), we conclude that $x_n\longrightarrow\tilde{x}$. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \longrightarrow C$ be a contraction with coefficient $\alpha \in (0,1)$. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \;\; x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \;\; n \geq 1. \quad (3.17)$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ satisfying
(C1) $\alpha_n + \beta_n < 1, \lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and either*
(C2) $\lim_{n\longrightarrow\infty} \beta_n = 0$ *or*
(C3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.
Then the sequence $\{x_n\}$ converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (I-f)\tilde{x}, \tilde{x} - z \rangle \leq 0, z \in F(T).$$

Corollary 3.4. [8, Theorem 1] *Let H be a real Hilbert space, C a closed convex subset of H, P the metric projection of H onto C, T a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ defined by (1.3) converges strongly to Qx, where Q is the metric projection from C onto $F(T)$.*

Proof. Setting $\beta_n \equiv 0$ and $f := x$ for some $x \in C$ and applying the Theorem 3.2 with the nonexpansive self-mapping $P_C T$, we obtain that $\{x_n\}$ converges strongly as $n \longrightarrow \infty$ to a fixed point of $P_C T$. Since $F(T) \neq \emptyset$, using Lemma 2.2 and 2.3, we obtain $F(T) = F(P_C T)$. The proof is complete. □

Acknowledgements
The authors would like to thank the Faculty of Science, Naresuan University, THAILAND for the financial support.

References

[1] F. K. Browder, Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces, Arch. Rational Mech. Anal. 24, 82-90 (1967).

[2] R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, Israel J. Math. 32 (1979) 107-116.

[3] F. Deutsch and I. Yamada, Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998) 33–56.

[4] J. P. Gossez, E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, Pacific J. Math. 40 (1972) 565-573.

[5] B. R. Halpern and G. M. Bergman, A fixed-point theorem for inward and outward maps, Trans. Amer. Math. Soc. 130 (1968) 353-358.

[6] G. Marino and H. K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, J. Math. Anal. Appl. 318(2006) 43-52.

[7] S. Matsushita and D. Kuroiwa, Approximation of fixed points of nonexpansive nonself-mappings, Sci. Math. Jpn. 57 (2003) 171-176.

[8] S. Matsushita and D. Kuroiwa, Strong convergence of averaging iterations of nonexpansive nonself-mappings, J. Math. Anal. Appl. 294 (2004) 206-214.

[9] S. Matsushita and W. Takahashi, Strong convergence theorem for nonexpansive nonself-mappings without boundary conditions, Nonlinear Analysis (2006), doi:10.1016/j.na.2006.11.007.

[10] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. **241** (2000) 46–55.

[11] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[12] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. 75 (1980), 278-292

[13] N. Shioji and W. Takahashi, A Strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces, Arch. Math. 72 (1999) 354-359.

[14] T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1997) 71-83.

[15] T. Shimizu and W. Takahashi, Strong convergence theorem for asymptotically nonexpansive mappings, Nonlinear Analysis 26(1996),265-272.

[16] Y. Song and R. Chen, Viscosity approximation methods to Cesaro means for nonexpansive mappings, Applied Mathematics and Computation 186 (2007) 1120-1128.

[17] T. Suzuki, Strong convergence of krasnoselskii and manns type sequences for one-parameter nonexpansive semigroups without bochner integrals. J. Math. Anal. Appl. 305:227-239 (2005).

[18] W. Takahashi and Y. Ueda, On Reich's strong convergence theorems for resolvents of accretive operators, J. Math. Anal. Appl., 104 (1984) 546-553.

[19] W. Takahashi, Nonlinear Functional Analysis Fixed Point Theory and its Applications, Yokohama Publishers Inc., Yokohama, 2000.

[20] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58(1992), 486-491.

[21] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. **298** (2004) 279–291.

[22] H. K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116(2003)659-678.

[23] H. K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66(2002) 240-256.

(Received 30 May 2007)

Pramote Markshoe
Department of Mathematics,
Faculty of Science, Naresuan University,
Phitsanulok 65000, THAILAND.,
e-mail : `pramotem@nu.ac.th`

Rabian Wangkeeree
Department of Mathematics,
Faculty of Science, Naresuan University,
Phitsanulok 65000, THAILAND.,
e-mail : `pramotem@nu.ac.th`