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Image Quality Assessment: From Difference to Relative Entropy

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Abstract Assessing image quality via objective methods basically attempts to quantify or measure the differences in visibility between a reference image and a distorted image. In this paper, a more general objective image quality index is given, which is simple to implement and has vast application in image processing. The relative entropy index is based on the assumption that the image is an information entropy. Experimental comparisons demonstrate the effectiveness of the proposed method.

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1. Introduction

Objective assessment of image quality is used to construct metrics that can forecast the quality of the perceived image . It has several roles to play in image processing applications. Firstly, it can be used to check and adjust image quality. Secondly, it can be used to optimize parameter settings and algorithms of image and signal processing. Thirdly, it can be used to standardize image processing systems, signal processing systems and algorithms. Moreover, quality measurement has an important role in applications such as image restoration, signal recovery, display and analysis. The widely used reference image and distortion image estimation algorithms are signal-to-noise ratio index (SNR) [1], peak signal-to-noise ratio index (PSNR) [1], and structural similarity index (SSIM) [2].

Among simplest quality metrics are the mean squared error (MSE) and peak signal to noise (PSNR). They are easily calculated and they have clear physical meanings. Besides, in the context of optimization, they are mathematically convinient. Since the peak signal to noise (PSNR) use ratio, it is carefully. However, rarely matches the perceived visual quality [3–5]. There are a lot of efforts to develop methods for assessing quality that make use of knowledge of existing features of the human visual system (HVS). Most of the proposed assessment models that are of perceptual quality go through a process of modifying MSE metric such that errors are corrected based on their visibility. There are no precise rule for choosing signal to noise ratio (SNR), peak signal to noise (PSNR), mean squared error (MSE) and structural similarity index (SSIM) when the assessment of the quality of image is needed. Therefore, the way of belief is a way to interpret the given numbers during the evaluation process [6]. Really, some studies have revealed that as opposed to the structural similarity index (SSIM), the mean squared error (MSE) and also the peak signal to noise (PSNR) have a poor performance when separating structural content in images because different types of degradation's working on the same image output same value of MSE.

Mathematically determined measures are interesting. First because they are not difficult to calculate and often have computational complexity. Secondly, they are free from particular observers and viewing conditions. However, it is believed that viewing the conditions at which viewing is done play a significant role in how image quality is perceived by human. In general, it is not available for image analysis systems. Now suppose there are different viewing conditions, then a method of veiwing which is condition-dependent will beget different measurement results that are not easy to use. Besides, it will become the responsibility of the user to quantify the conditions under viewing , calculate and provide the parameters for the quantifying systems. In contrast, an self-reliant measurement of viewing conditions gives only one quality value that gives an idea of how good is the image.

In this article, we present a universal mathematical definition of image quality index. By universal, we mean that the quality measurement method does not depend on the images being tested, particular observers or viewing conditions. More importantly, it must apply to several image processing applications and provide reasonable comparisons in distorting various types of images. Presently, the signal to noise ratio (SNR), peak signal-to-noise ratio (PSNR), mean squared error (MSE) and structural similarity (SSIM) are still widely used without considering their shortcomings. This work focus on providing an alternative index to substitute their functions. This index is called the Relative entropy index (RE).

2. Image Quality Assessment

The image signal with quality evaluation is the sum of unsorted references and error signals. The most widely used hypothesis is that loss of quality, perception is strongly related to the vision of error signal. Implementing this concept is easy by the use of mean square error (MSE) which quantifies the strength of error signal of an object. Distorted images with same MSE may have different types of errors , some of which are more visible than others. Guidelines for most assessment of perceived image quality available in text try to weigh, various looks of the signal according to the error visibility as determined by psychological metrics in human or physiological metrics in animals. Reviews about the algorithm for evaluating the quality of images, videos and signal processing can be found in [7–10].

The basic principle of the error-sensitivity approach is approximately the best vision error. This is achieved by simulating the performance characteristics of the first phase of the human visual system (HVS). The human visual system (HVS) is a complicated and very highly dynamical. However, the basic model of vision in the initial stage is based on linear or quasi-linear operators that have been described by the use of restricted and simplistic stimuli. Therefore, the methods of error sensitivity has to be based on strong assumptions and generalizations.

2.1. Quality Definition

It is unclear whether the error display is equal to some loss of quality, because some distortions may be clearly visible but not so obnoxious. This is a shortcoming with the usual definition of quality of image. A trivial instance is increase in the number of intensities via an acceptable scale factor. In [11] it was pointed out that there is a moderate correlation between image quality and image fidelity.

2.2. Suprathreshold

Experiments that form the basis of several error sensitivity models are in particular developed to assess almost invisible criteria. These measurable criteria will be used to determine the sensitivity of the visual error measure. However, very little mental study states whether the model can be generalized to classify the characteristics of perceived distortions that are significantly bigger than the threshold level as in most cases of images. Efforts have been made recently to combine suprathreshold psycophysics for anlysing image distortions [12, 13].

2.3. Natural Image Complexity

The deception phenomenon often overlays the use of at least two distinct forms. The format is easier than the image in the real world, which is considered a very simple format. A model for the interaction between a few simple forms to assess the response. This is an experiment with a limited amount of stimulation to create a model that can predict the visual quality of natural images with complex structures. The festival of recently created models are both easy and complicated and should trigger further reserach [14].

2.4. De-correlation

The moment people choose using indicators for spatial consolidation, then they are indirectly suppose in-dependency of errors at distinct locations. This is correct if processing before pooling remove dependencies in the input signals. Unfortunately, the reverse is the case for wavelet transform which is a linear channel decomposition method. In addition, direct dependency exists between inter- and intra-channel wavelet coefficients of natural images [6], [15]. From statistics point of view, there is a huge decline of dependencies of the transform coefficients can be achieved by a well constructed non-linear gain control model where the optimal parameters are used to decrease dependencies instead of fitting data from masking experiments [16]. Furthermore, in [17, 18] optimal design of transformationand masking models were shown to have reduced both perceptual and statistical dependencies.

2.5. Cognitive Interaction

A popular fact is that quality of images are affected cognitive understanding and interactive visual processing. For example, given different instructions, a human observer give different quality scores to the same image [7]. Other things that may affect evaluation are, already known idea of what the image entails or observation and addiction [7]. However, a very good number of metrics for rating quality of image do not take into account such consequences because quantifying them is not easy.

3. Philosophy

Metrics for quality of image are based on the assumption that HVS adapt extremely when extracting structural information [19]. A very good approximation to perceived image distortion is expected when structural information change metric is considered. A better understanding of the ideology by comparing with the error sensitivity ideology via three contents.

Firstly, the error sensitivity procedure estimates perceived errors when quantifying image degradation. However, latest ideology used degradation of images as perceived variation in structural information. An instance is given in Figure 1 in which the original "Cmeraman" image is adjusted using disparate distortions, each of which is altered to produce closely uniform MSE comparable to the exact. Moreover, the images considered may have gravely different perceived quality. Taking into account the visual difference of the exact image is easily distinguished when error sensitivity ideology is used, it will not be easy to expatiate why the quality of the contrast-stretched image is very high. However with latest ideology, it can be easily understood because almost all structured insight of the exact image is maintained. This means that the main input can be almost fully recovered by a effortless point-wise inverse linear transform. In different circumstances, a few structural information from the exact image will be indefinitely lost in the blurry images, hence a lower quality score than motion blur, Gaussian blur, and salt & pepper blur image should be obtained.

Secondly, the criterion for error-sensitivity is a grassroot process, replicating relevant duty of the first-phase component in the HVS. However, the latest criterion is a top-bottom process copying the supposed functionality of the whole HVS. From another angle, it avoids the suprathreshold mentioned previously since it is independent of threshold psycho-physics to measure the distortion. Moreover, the cognitive interaction reduces to a definite level because checking the structures of the observed entities is believed to be the reason for the whole activity of visual observations including effective and collective process.

Thirdly, complexity of natural images and interiors can be steer clear of to a certain level because the latest ideology try not to predict the quality of images by compiling errors related to the patterns that are easy to understand. rather, compute directly the structural changes between two complicated signals.

FIGURE 1. Main image (top left), degraded image (top right) with MSE= 15.0742, PSNR= 36.3485, SNR= 30.7657, SSIM= 0.9501, the motion blurred image (bottom left) with MSE= 770.5704, PSNR= 19.2627, SNR= 13.6800, SSIM= 0.6138 and the salt & pepper blurred image (bottom left) with MSE= 434.2821, PSNR= 21.7531, SNR= 16.1704, SSIM= 0.5690.

3.1. Mathematics & Entropy

In this subsection, we give some useful definitions and theorems.

Definition 3.1. A *random variable* $\bar{X}$ is a function from the sample space to the set of real numbers. The range of a random variable $\bar{X}$, denoted by $R_{\bar{X}}$, is the set of possible values of $\bar{X}$.

Definition 3.2. Denote $\bar{X}$ as a discrete random variable (drv) with range $R_{\bar{X}} = \{x_1, ..., x_n\}$. The map $P_{\bar{X}}(x_k) = P(X = x_k)$, for $k = 1, \ldots, n$ is called probability density function (pdf) of $\bar{X}$.

Definition 3.3. The *entropy* of $\bar{X}$ with$p_{\bar{X}}(x)$ as pdf, is

$$H(\bar{X}) = -\sum_{x} p_{\bar{X}}(x) \log_2 p_{\bar{X}}(x). \tag{3.1}$$

As $p_{\bar{X}}(x) \in [0, 1]$, Then, $log_2 p_{\bar{X}}(x) \leq 0$ and $H(\bar{X}) \geq 0$.

Definition 3.4. Kullback-Leibler distance between distributions, $p_{\bar{X}}(x)$ and $q_{\bar{X}}(x)$ is defined as

$$H(p_{\bar{X}}(x)||q_{\bar{X}}(x)) = \sum_x p_{\bar{X}}(x) \log_2 \frac{p_{\bar{X}}(x)}{q_{\bar{X}}(x)}. \tag{3.2}$$

It is sometimes called relative entropy. It is not a metric mathematically because it is not symmetric, that is

$$H(p_{\bar{X}}(x)||q_{\bar{X}}(x)) \neq H(q_{\bar{X}}(x)||p_{\bar{X}}(x)). \tag{3.3}$$

Theorem 3.5. [20] *The Kullback-Leibler distance between two distributions, $p_{\bar{X}}(x)$ and $q_{\bar{X}}(x)$ of the drv $\bar{X}$, is positive and zero only when $p_{\bar{X}}(x) = q_{\bar{X}}(x)$.*

Definition 3.6. A map $g : M \to N$ is *continuous* at a point $b \in M$ if $\lim_{m \to b} g(m) = g(b)$. Also, it is continuous on $E \subseteq M$ if g for all $e \in E$, g is continuous at e.

Theorem 3.7. [21] **Sequential Continuity** *Suppose $g : M \to N$ and $m \in M$. g is continuous at m if and only if $g(m_n) \to g(m)$ for all sequences m_n in M with $m_n \to m$.*

3.2. Relative Entropy (RE) Index

In this subsection, we defined probability density function as a vector. Thus we can compute entropy of a vector and the relative entropy between any two vectors. It is called the Relative Entropy (RE) Index.

Consider the following functions defined as $f : \mathbb{R}^n \to (-\frac{\pi}{2}, \frac{\pi}{2})^n$ by

$$f(x_1, \ldots, x_n) = (\arctan(x_1), \ldots, \arctan(x_n)), \tag{3.4}$$

$g : (-\frac{\pi}{2}, \frac{\pi}{2})^n \to (0, \frac{\pi}{2})^n$ by

$$g(x_1, \ldots, x_n) = (\frac{x_1 + 0.5\pi}{2}, \ldots, \frac{x_n + 0.5\pi}{2}), \tag{3.5}$$

and $h : (0, \frac{\pi}{2})^n \to \mathbb{A}$ by

$$h(x_1, \ldots, x_n) = (\frac{\cos^2(x_1)}{n}, \frac{\sin^2(x_1)}{n}, \ldots, \frac{\cos^2(x_n)}{n}, \frac{\sin^2(x_n)}{n}), \tag{3.6}$$

where $\mathbb{A} = \{(a_1, b_1, \ldots, a_n, b_n) \in (0,1)^{2n} : \forall i \in \{1, \ldots, n\}, a_i + b_i = \frac{1}{n}\}$. Let $u, v \in \mathbb{R}^n$ such that $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Let X be a discrete random variable with range $R_x = \{1, \ldots, 2n\}$. Define p and q probability density function of X as

$$p_X(i) = \Pi_i(h(g(f(u)))) \text{ and } q_X(i) = \Pi_i(h(g(f(v)))), \tag{3.7}$$

where Π_i is a projection at i-th entry. So we can defined the entropy of a vector u by

$$E(u) = H(X). \tag{3.8}$$

Moreover, we can define the relative entropy between vector u and vector v by

$$RE(u, v) = H(p_X(x)||q_X(x)). \tag{3.9}$$

It is called *Relative Entropy Index* where u is a reference vector and v is an approximate vector. See Fig 2.

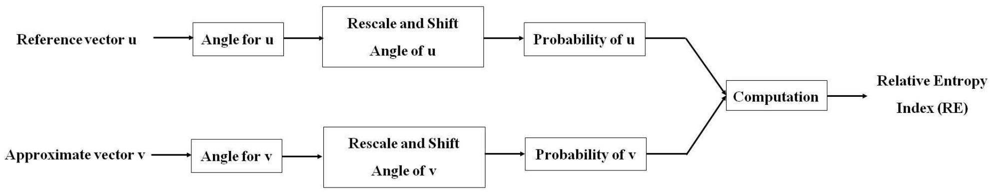

FIGURE 2. Diagram of the Relative Entropy Index (RE)

Theorem 3.8. *Let $\bar{u}, \bar{v} \in \mathbb{R}^n$ such that $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n)$. Then the Relative Entropy Index (RE) satisfy these properties:*

(1) $RE(\bar{u}, \bar{v}) \geq 0$,
(2) $RE(\bar{u}, \bar{v}) = 0$ if and only if $\bar{u} = \bar{v}$,
(3) RE is a continuous on $\mathbb{R}^n \times \mathbb{R}^n$.
(4) $RE(\bar{u}_j, \bar{v}) \to 0$ if and only $\bar{u}_j \to v$.

Proof. Let $\bar{X}$ be a discrete random variable with range $R_x = \{1, \ldots, 2n\}$. Define p and q pdf of $\bar{X}$ by

$$p_{\bar{X}}(i) = \Pi_i(h(g(f(\bar{u})))) \text{ and } q_{\bar{X}}(i) = \Pi_i(h(g(f(\bar{v})))),$$

where Π_i is a projection at i-th entry. Thus $RE(\bar{u}, \bar{v}) = H(p_{\bar{X}}(x)||q_{\bar{X}}(x))$.

(1) Since Relative Entropy Index defined by relative entropy, $RE(\bar{u}, \bar{v}) \geq 0$ by Theorem 3.5.
(2) By Theorem 3.5, $RE(\bar{u}, \bar{v}) = 0$ if and only if $p_{\bar{X}}(x) = q_{\bar{X}}(x)$. Clearly $h \circ g \circ f$ is a bijective function. Therefore, $\bar{u} = \bar{v}$.
(3) Let $(\bar{u}_j, \bar{v}_j)$ be a sequence in $\mathbb{R}^n \times \mathbb{R}^n$ such that $(\bar{u}_j, \bar{u}_j) \to (\bar{u}, \bar{v})$. Define p_j and q_j probability density function of X by

$$p_{j_X}(i) = \Pi_i(h(g(f(\bar{u})))) \text{ and } q_{j_X}(i) = \Pi_i(h(g(f(\bar{v})))),$$

where Π_i is a projection at i-th entry. Thus $RE(\bar{u}_j, \bar{v}_j) = H(p_{j_X}(x)||q_{j_X}(x))$. It easy to see that $\Pi_i \circ h \circ g \circ f$ is continuous at any vector in $\mathbb{R}^n$. Therefore, $RE(\bar{u}_j, \bar{v}_j) \to RE(\bar{u}, \bar{v})$. By Theorem 3.7, RE is a continuous on $\mathbb{R}^n \times \mathbb{R}^n$.
(4) Since RE is continuous on $\mathbb{R}^n$, $RE(\bar{u}_j, \bar{v}) \to 0$ if and only if $(\bar{u}_j, \bar{v}) \to (\bar{v}, \bar{v})$ if and only if $\bar{v}_j \to \bar{v}$.

∎

We applied Relative Entropy Index (RE) to measuring image quality. Let I be a reference image and let R be an approximate image of I. Assume that I and R image have size $K \times L \times D$. Set up $I' = h(g(f(I)))$ and $R' = h(g(f(R)))$ where $n = NMD$. The RE index is defined by

$$RE(I, R) = H(R' \| I').$$

4. Experiment

All the program were coded in MatLabR2018 and compute on PC Intel(R), Core(TM) i5-72000U, CPU @2.50 GHz (4CPUs), ~2.7GHz and Ram 4 GB. We test image quality index by using result's in Padcharoen et al. in [22]. The test images and restored images are given in Figure 3. Table 1, 2, 3 4 gives image quality index. See Code 5.1 and 5.2.

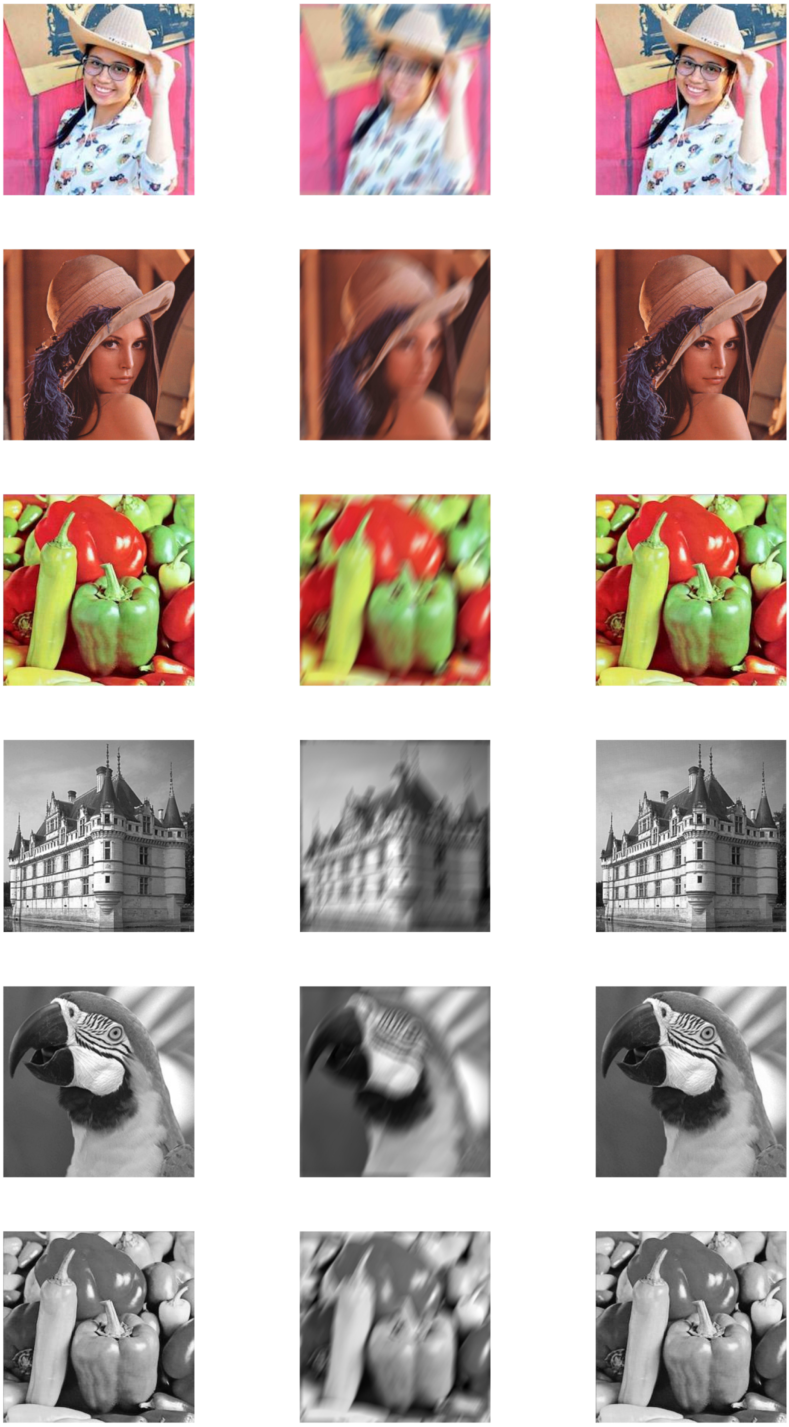

FIGURE 3. Restoration results. First column is original images, second column is blur and noisy images, and third column is restored results.

Table 1. SSIM

Figure	SSIM	
	original with blur and noise	original with restored
1	0.9226	0.9978
2	0.9628	0.9982
3	0.9592	0.9995
4	0.8493	0.9872
5	0.9242	0.9938
6	0.9088	0.9935

Table 2. SNR

Figure	SNR	
	original with blur and noise	original with restored
1	21.3928	44.2628
2	23.5586	43.5520
3	21.1065	44.6516
4	18.7265	35.2650
5	21.3746	42.8516
6	21.3211	45.0002

Table 3. PSNR

Figure	PSNR	
	original with blur and noise	original with restored
1	22.6380	45.5081
2	25.9929	45.9863
3	22.9896	46.5347
4	20.5175	37.0560
5	23.3222	44.7992
6	23.1583	46.8374

Table 4. RE

Figure	RE	
	original with blur and noise	original with restored
1	8.6331×10^{-3}	4.3238×10^{-4}
2	5.4414×10^{-3}	7.1372×10^{-7}
3	7.6×10^{-2}	2.2×10^{-3}
4	1.2674×10^{-2}	3.9216×10^{-5}
5	7.9019×10^{-3}	5.3147×10^{-6}
6	1.0072×10^{-2}	7.4178×10^{-5}

5. Conclusion

This article initiates a relative entropy index (RE). However, a theoretical study was carried out to compare the MSE, SNR, PSNR, SSIM with the RE index. There are many iterative algorithms to solve such problems thatb constructs sequence which converge to the solution.As the relative entropy index is continuous, the sequence converges to the solution if and only if the relative entropy index converges to zero. As a conclusion, it look like the values of the MSE, SNR, PSNR, SSIM can be forcast from the RE index and conversely. The mean square error (MSE), signal to noise ratio (SNR), peak signal to noise ratio (PSNR), structural similarity (SSIM), and relative entropy index (RE). The above metrics vary on the level of their sensitivity to image degradation.

Appendix

Structural Similarity

Structural similarity (SSIM) [2, 23] is a way of forecasting quality of perceived images. SSIM measures the similarity between two images. The SSIM index is a metric for assessing image quality using a reference image. The index is defined as

$$SSIM(I,R) = \frac{(2\mu_I\mu_R+C_1)(2\sigma_{IR}+C_2)}{(\mu_I^2+\mu_R^2+C_1)(\sigma_I^2+\sigma_R^2+C_2)}$$

where I is a reference image, R is an approximate image of I, μ_I is the mean of I, μ_R is the mean of R, σ_I is the variance of I, σ_R is the variance of R, σ_{IR} is the covariance of I and R, C_1 and C_2 are small constant by $C_1 = (K_1L)^2$ and $C_2 = (K_2L)^2$ where L is the range of the pixel and $K_1, K_2 \ll 1$. It satisfy the following properties:

(1) symmetry: $SSIM(I,R) = SSIM(R,I)$,
(2) boundedness: $0 \leq SSIM(I,R) \leq 1$,
(3) unique maximum: $SSIM(I,R) = 1$ if and only if $I = R$.
(4) continuous: $SSIM(I,R)$ at point (I,R).

Signal to Noise Ratio

Signal to noise ratio (SNR) [1] is a metric used in science and engineering to differentiate the degree of a desired signal to the level of background noise. SNR is the ratio of signal power to that of the noise with decibels as its unit. The index is defined as

$$SNR(I,R) = 10\log_{10}\frac{\|I\|^2}{\|I-R\|^2}$$

where I is a reference image, and R is the approximate image of I. If $I = R$, then we defined $SNR(I,R) = \infty$. It satisfy the following properties:

(1) unboundedness: There is a $SNR(I,R) = \infty$ when $I = R$,
(2) unique maximum: $SNR(I,R) = \infty$ if and only if $I = R$.
(3) continuous: $SNR(I,R)$ at point (I,R) except $I \neq R$ or $I = 0$.

Peak Signal to Noise Ratio

Peak signal to noise ratio (PSNR) [1] is the ratio between the maximum possible power of a signal to that of the noise that affects the loyalty of its representation with decibels as its unit. Due to very wide range of many signals , PSNR is defined in terms of logarithmic decibel scale. The index is defined as

$$PSNR(I,R) = 10\log_{10}\frac{\max^2 I}{MSE(I,R)} \text{ such that}$$
$$MSE(I,R) = \frac{1}{MND}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{D}(I(i,j,k) - R(i,j,k))^2,$$

where I is a reference image, R is the approximate image of I, I is a image of size $N \times M$ with D channel, and R is a image of size $N \times M$ with D channel. If $I = R$, then we defined $PSNR(I,R) = \infty$. It satisfies the following properties:

(1) unboundedness: There is a $PSNR(I,R) = \infty$ when $I = R$,
(2) unique maximum: $PSNR(I,R) = \infty$ if and only if $I = R$.
(3) continuous: $PSNR(I,R)$ at point (I,R) except $I \neq R$ or $I = 0$.

Code of Relative Entropy

Code 5.1. Example code RE index

```
clc
clear
O = imread('O1.jpg');
B = imread('B1.jpg');
R = imread('R1.jpg');
RE(1) = REindex(B,O);
RE(2) = REindex(R,O);
Result = [RE;]
```

Code 5.2. sub program

```
function RE = REindex(x,y)
%x is approximate
%y is solution
[nmd] = size(x);
p = 1/(n*m*d)*ones(n,m,d);
x = double(x);
y = double(y);
anglex = (atan(x)+pi/2)/2;
angley = (atan(y)+pi/2)/2;
px(:,:,:,1) = p.*(cos(anglex).^2);
px(:,:,:,2) = p.*(sin(anglex).^2);
py(:,:,:,1) = p.*(cos(angley).^2);
py(:,:,:,2) = p.*(sin(angley).^2);
%D(P||Q) = sumP(x)log_2(P(x)/Q(x))
RE = sum(sum(sum(sum(px.*log2(px./py)))));
```

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