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On E-Statistical Summability of Bounded Sequence

Sabiha Tabassum* **and Ruqaiyya Fatma**

Department of Applied Mathematics, Zakir Husain College of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India
e-mail : sabiha.math08@gmail.com, sabiha.am@amu.ac.in (S. Tabassum);
ruqaiyyafatma@rediffmail.com (R. Fatma)

Abstract A summability method is said to be regular if it preserves the limit, i.e., it sums all convergent series to its Cauchy's sum. In this paper we have introduced the sequence space $X(E)$ defined by Euler matrix for the spaces $X = l_\infty$, c, c_0 and l_p, $(1 \leq p < \infty)$. Some fundamental properties and relation related to these spaces are examined. A new regular statistical summability method (Euler statistical summability) is given. The graph for Euler and Euler statistical summability are given by using MATLAB(2018a).

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1. Introduction

In 1951, Fast [1] and Steinhaus [2] introduced the concept of Statistical convergence, later developed by Schoenberg [3]. Over the last few decades several authors had explored statistical convergence in various directions involving several spaces, (see [4–7] and more).

A single sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to a number L, if for a given $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bar indicates the number of elements.

Let w, l_∞, c, c_0 and l_p, $(1 \leq p < \infty)$ denote the set of all real, bounded, convergent, null and p-summable sequences. Statistically convergence is deeply connected to the strongly Cesro summability and uniform summability [8].

The study of summability theory comes to existence in attempt to sum up some non-convergent series. Euler [9] try to sum up the divergent series. He believe that each series can be summed. But mathematician find some absurd result on dealing with those and for some time after this, divergent series were mostly excluded from mathematics. Cesro [10] was the first one who give a rigorous definition for such summation in 1890. Due to

*Corresponding author.

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its application in different area (such as series, integrals, physics and others), summability theory plays an important role in modern Mathematics.

We can use the summability technique by a linear transformation T and if T can be expressed as an infinite matrix, then we say T is a matrix transformation and the method is known as matrix summability. Toeplitz [11] gives the following conditions for an infinite matrix to be regular.

Theorem 1.1. *A matrix $A = (a_{mn})$ is regular if and only if the following holds,*

(i) $\lim\limits_{m\to\infty} a_{mn} = 0,\ n = 0, 1, 2, ...;$

(ii) $\lim\limits_{m\to\infty} \left(\sum\limits_{n=0}^{\infty} a_{mn}\right) = 1;$

(iii) $\sup\limits_{m} \sum\limits_{n=0}^{\infty} |a_{mn}| < \infty.$

Definition 1.2. The Euler matrix of order r (or (E, r) method), $r \in \mathbb{C}\backslash\{0, 1\}$ is given by the infinite matrix $\left(e^{(r)}_{mn}\right)_{m,n\in\mathbb{N}\cup\{0\}}$, where

$$e^{(r)}_{mn} = \begin{cases} \binom{m}{n} r^n (1-r)^{m-n}, & n \le m; \\ 0, & n > m. \end{cases}$$

Also, for $r = \{0, 1\}$

$$e^{(1)}_{mn} = \begin{cases} 1, & n \le m; \\ 0, & n > m. \end{cases}$$

$$e^{(0)}_{mn} = 0,\ m = 0, 1, 2, ...;\ n = 0, 1, 2, ...;$$
$$e^{(0)}_{m0} = 0,\ m = 0, 1, 2, ...$$

Aasma et al. [12] (Chapter 3, Theorem 3.7) shows that the (E, r) method is regular if and only if r is real and $0 < r \le 1$.

Definition 1.3. Let $K \subseteq \mathbb{N}$, then A-density of K is given by $\delta_A(K) = \lim\limits_{m\to\infty} \sum\limits_{n\in K} a_{mn}$, if the limit exists.

Definition 1.4. A sequence $x = (x_n)$ is said to be A-statistically convergent to $L \in \mathbb{R}$ (or A-summable to L), if for every $\epsilon > 0$, $\delta_A(\{n \le m : |x_n - L| \ge \epsilon\}) = 0$. It is denoted by $x_m \to L(A - st)$.

Theorem 1.5. *Let A be an infinite matrix and $x = (x_n)$ be a bounded sequence of real numbers, then x is statistically converges to L if and only if the transformed sequence $Ax = (Ax)_m = (Ax)_m = \sum\limits_{n=0}^{\infty} a_{mn}x_n$ converges to L for every $A \in T$, where*

$$T = \{\ A = (a_{mn}) : A \textit{ is triangular, non-negative, } \sum_{n=0}^{\infty} a_{mn} = 1 \textit{ for each } m \in \mathbb{N}$$
$$\textit{and } \lim_{m\to\infty} \sum_{n\in K} a_{mn} = 0, \delta(K) = 0\ \}.$$

In present paper we give definition of statistical convergence with the help of Euler matrix and discussed some properties. We connect it to statistical summability for (E, r) methods. This paper also consist the comparison of usual, Euler and E-statistical summability for two series (one being convergent and other non-convergent in conventional sense).

2. Some Topological Properties of E-Statistical Convergent Sequence

Let E be a regular Euler matrix given by Definition 1.2 and $X = \{c,\ c_0,\ l_p,\ 1 \le p \le \infty\}$. We give the matrix space $X(E) = X_E$ as

$$X(E) = \left\{ x = (x_n) \in w : (Ex)_m = \sum_{n=0}^{\infty} e_{mn}^{(r)} x_n \in X \right\}.$$

Similar to Kara [13] (Theorem 2.2), we can say that X_E is a Banach space with the norm given by

$$\|x\|_{X(E)} = \|Ex\|_X = \|y\|_X = \begin{cases} \sup_n |y_n|, & if X \in \{l_\infty, c, c_0\}; \\ \left(\sum_{n=0}^{\infty} |y_n|^p\right)^{\frac{1}{p}}, & if X = l_p;\ 1 \le p < \infty. \end{cases}$$

Lemma 2.1. *If X is a subset of Y, then $X(E)$ is a subset of $Y(E)$.*

Proof. The proof of this lemma is trivial. ■

Theorem 2.2. *Let X be a Banach space. If A is a closed subset of X, then $A(E)$ is also closed in $X(E)$.*

Proof. As A is a closed subset of X, therefore $\bar{A} = A$ and $A(E) \subset X(E)$, (by Lemma 2.1). To show $A(E)$ is closed in $X(E)$, we need to show that $\overline{A(E)} = A(E)$.

Let $x \in \overline{A(E)}$, there exists a sequence (x^m) in $A(E)$ such that $x^m \longrightarrow x$, i.e., $\|x^m - x\|_{A(E)} \longrightarrow 0$ (Theorem 1.4-6 in [14]). Thus, $\|E(x^m - x)\|_A \longrightarrow 0$ or $\|E(x_n^m) - E(x)\|_A \longrightarrow 0$. Further using Theorem 1.4-6 in [14], $E(x) \in \bar{A}$. Hence $x \in \bar{A}(E)$ or $x \in A(E)$, since A is closed.

Conversely, suppose that $x \in A(E)$, then $x \in \overline{A(E)}$ as $A(E) \subseteq \overline{A(E)}$. Therefore, $\overline{A(E)} = A(E)$. ■

Corollary 2.3. *If X is a separable space, then $X(E)$ is also separable.*

Proof. Using the definition of separable space and Theorem 2.2, one can easily prove the result. ■

In view of Definition 1.4 we now give definition of statistical convergence for Euler Matrix and call it Euler type statistical convergence.

Definition 2.4. Let $x = (x_n)$ be a sequence in X. The sequence x is said to be Euler type statistical convergent or E-statistically convergent to L if for a given $\epsilon > 0$,

$$\delta(\{m : m \le t \text{ and } |(Ex)_m - L| \ge \epsilon\}) = 0,$$

where Ex be a sequence and $Ex = Ex_n = (Ex)_m = \sum_{n=0}^{\infty} e_{mn}^{(r)} x_n$.

The set of all such sequence is denoted by $S(E)$. We write $x_n \to L(S(E))$.

Definition 2.5. A sequence $x = (x_n)$ is said to E-statistically Cauchy if for a given $\epsilon > 0$, there exists a positive number $N = N(\epsilon)$ such that

$$\delta(\{m : m \le t \text{ and } |(Ex)_m - (Ex)_N| \ge \epsilon\}) = 0.$$

Definition 2.6. A sequence $x = (x_n)$ is said to be E-bounded if for every $m \in \mathbb{N}$,

$$\sup_m |(Ex)_m| < \infty.$$

The set of all such sequence is denoted by $l_\infty(E)$.

The set of all E-bounded statistically convergent sequence is denoted by $m_0(E)$. It is quite self-explanatory that $m_0(E) \subset l_\infty(E)$.

Definition 2.7. A sequence $x = (x_n)$ is said to be E-statistically bounded if there exists some $H \geq 0$ such that $\delta(\{m : |(Ex)_m| > H\}) = 0$.

Theorem 2.8. *If $x = (x_n)$ is an E-statistically convergent sequence then it is also E-statistically Cauchy.*

Proof. Let $x = (x_n)$ be an E-statistically convergent sequence converges to the limit L. Then for a given $\epsilon > 0$, $|(Ex)_m - L| < \frac{\epsilon}{2}$ for almost all $m \in \mathbb{N}$. If N is chosen so that $|(Ex)_N - L| < \frac{\epsilon}{2}$, then we have $|(Ex)_m - (Ex)_N| < |(Ex)_m - L| + |(Ex)_N - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for almost all $m \in \mathbb{N}$. Hence x is E-statistically Cauchy. ∎

Theorem 2.9. *If x is an E-statistically convergent sequence and there exists a sequence y such that $(Ey)_m = (Ex)_m$ for almost all m, then y is also E-statistically convergent.*

Proof. Given $x = (x_n)$ is an E-statistically convergent sequence, then

$$\delta(\{m : m \leq t \text{ and } |(Ex)_m - L| \geq \epsilon\}) = 0.$$

Also

$$\{m : m \leq t \text{ and } |(Ey)_m - L| \geq \epsilon\} \subseteq \{m : m \leq t \text{ and } (Ex)_m \neq (Ey)_m\} \cup \{m : m \leq t \text{ and } |(Ex)_m - L| \geq \epsilon\}.$$

Thus

$$\begin{aligned}\delta(\{m : m \leq t \text{ and } |(Ey)_m - L| \geq \epsilon\}) &\leq \delta(\{m : m \leq t \text{ and } (Ex)_m \neq (Ey)_m\}) \\ &\quad + \delta(\{m : m \leq t \text{ and } |(Ex)_m - L| \geq \epsilon\}) \\ &\leq 0,\end{aligned}$$

as $(Ey)_m = (Ex)_m$ for almost all m and x is E-statistically convergent to L. ∎

Theorem 2.10. *Every E-statistical convergent sequence is E-statistically bounded.*

Proof. Let x be an E-statistically convergent sequence convergent to L. Then,

$$\delta(\{m : m \leq t \text{ and } |(Ex)_m - L| \geq \epsilon\}) = 0.$$

If we choose a number H so that $H = \epsilon + L$, then $|(Ex)_m| < \epsilon + L$ for almost all m. Hence,

$$\delta(\{m : |(Ex)_m| > H\}) = 0.$$

Therefore x is E-statistically bounded. ∎

Proposition 2.11. *The set $m_0(E)$ and $l_\infty(E)$ are linear spaces and $m_0(E) \subseteq l_\infty(E)$.*

Proposition 2.12. [15] *m_0 is closed subspace of l_∞.*

Lemma 2.13. *The set $m_0(E)$ is a closed linear space of $l_\infty(E)$.*

Proof. Using Proposition 2.12 in Theorem 2.2, one can get the result. ∎

Theorem 2.14. *The space $m_0(E)$ is nowhere dense in $l_\infty(E)$.*

Proof. Every closed linear subspace of a linear normed space, different from the space itself, is nowhere dense in it. Using Lemma 2.13, it suffices to proof that $m_0(E) \neq l_\infty(E)$. Consider the sequence

$$x_n = \begin{cases} 1/(n+1), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Then $(Ex_n)_m \leq 1$, $\forall\ m$. It is clear that x_n is not a statistically convergent sequence. Thus $x_n \in l_\infty(E)$ but not in $m_0(E)$. ∎

Let s denote the Frchet metric space of all real sequences with the metric d_m,

$$d_m = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \tag{2.1}$$

where $x = (x_n), y = (y_n) \in m, \ \forall \ n \in \mathbb{N}$.

Theorem 2.15. *The set of all E-statistically convergent sequence is dense in s.*

Lemma 2.16. [15] *Let g_n $(n = 0, 1, 2, ...)$ be a complex valued continuous functions on $\mathbb{R}$ and c_1, c_2 are two distinct complex numbers such that for each sufficiently large n, we have c_1, $c_2 \in g_n(\mathbb{R})$. Let (a_{mn}) be a triangular matrix with the following properties:*

(P_1) *For each fixed n, we have* $\lim_{m\to\infty} a_{mn} = 0$;

(P_2) $\lim_{m\to\infty} \sum_{n=0}^{m} a_{mn} = 1$,

then the set s_1 of all such $x = (\xi_n) \in s$ for which there exists a finite limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_{mn} g_n(\xi_n)$ *is a set of the first Baire category in s.*

Now we are in a position to prove Theorem 2.15.

Proof. Let $S(E)$ be the set of all E-statistically convergent sequence, for all $(x_n) \in S(E)$, $\lim_{m\to\infty} \sum_{n=0}^{m} e_{mn}^{(r)} x_n$ gives finite. Let $y = (y_n)_{n\in\mathbb{N}}$ be the sequence of real number differs from x only in a finite number of terms. Obviously $y \in S(E)$. Take $g_n = x_n$ and $(a_{mn}) = (e_{mn}^{(r)})$ in Lemma 2.16. Since Euler matrix satisfies property (P_1) and (P_2), $S(E)$ is a set of the first Baire category in s. Let $x = (x_n) \in S(E)$ (for all n) and the sequence of real number $y = (y_n)$ (for all n) differs from x only in a finite number of terms. Obviously $y \in S(E)$. As s is a complete metric space with respect to the translation-invariant metric given by Equation (2.1), $S(E)$ is dense in s. ■

3. Euler Statistical Summability

In this section we show the statistical summability of a sequence through Euler matrix using MATLAB(2018a).

The sequence $x = (x_n)$ is said to be statistically E-summable to L if the sequence $(Ex)_m$ converges statistically or it is E-statistically convergent (Definition 2.4).

Using the definition of Euler matrix we get ,

$(Ex)_m = \sum_{n=0}^{\infty} e_{mn}^{(r)} x_n = \sum_{n=0}^{m} \binom{m}{n} r^n (1-r)^{m-n} x_n.$

3.1. Euler Statistical Summability of a Convergent Series

If we take a statistical convergent series, using the Euler matrix we can converge it to its usual sum as it is a regular matrix method. Consider the series $\sum_{n=0}^{\infty} e^{-(n+1)}$, it is a geometric series converges to 0.5820. We will compare its usual, Euler and Euler statistical sum through graph.

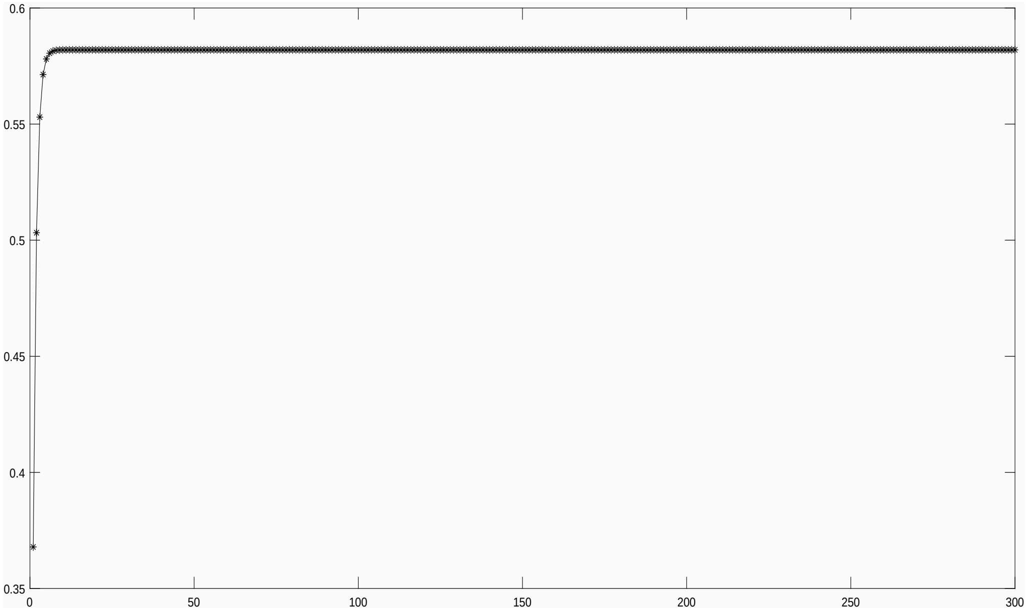

FIGURE 1. Usual convergence for $\sum_{n=0}^{\infty} e^{-(n+1)}$

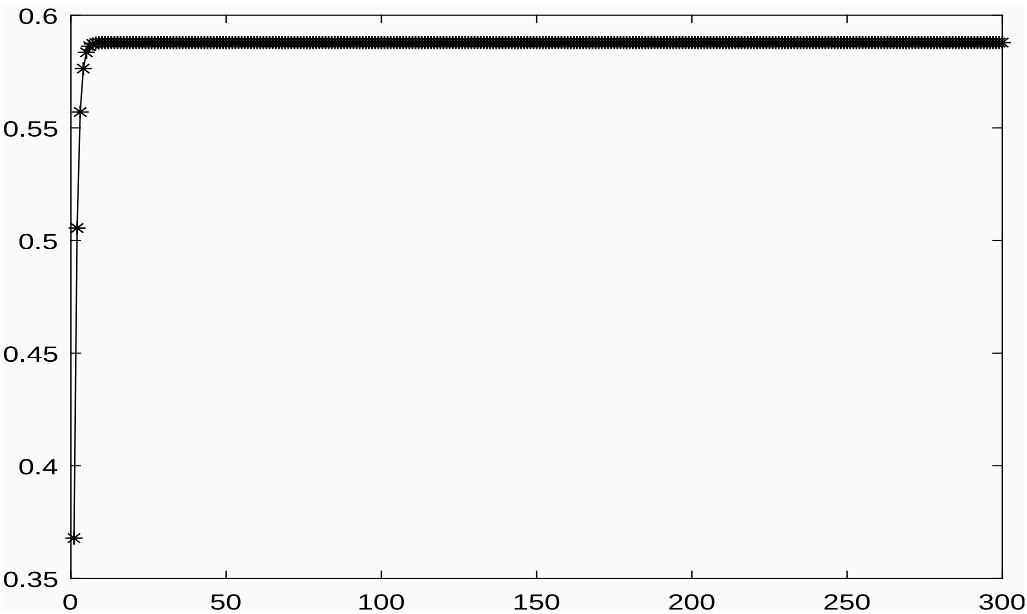

FIGURE 2. Euler summation of $\sum_{n=0}^{\infty} e^{-(n+1)}$

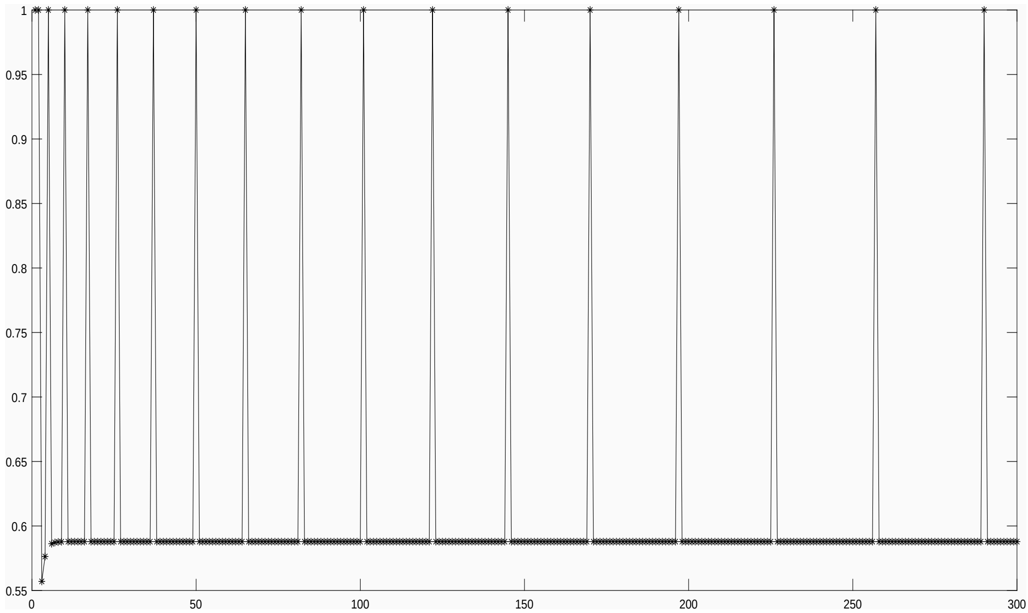

FIGURE 3. Euler statistical summation of $\sum_{n=0}^{\infty} e^{-(n+1)}$

Figure 1 shows the usual sum for the series $\sum_{n=0}^{\infty} e^{-(n+1)}$ as expected it is converging to .5820.

Figure 2 shows the convergence of sequence of partial sum $(Ex)_m = \sum_{n=0}^{m} \binom{m}{n} r^n (1-r)^{m-n} e^{-(n+1)}$, i.e., on applying Euler matrix we can still converge to the same limit, which is a good thing as it is not changing its usual sum.

In Figure 3 we give the E-statistical summability for the same. Again the limit is same except for the set $K \subset \mathbb{N}$ (the set of perfect square) but its density is zero.

3.2. Euler Statistical Summability of a Non-Convergent Series

Divergent series was the motivating factor for the introduction of the summability theory. In this section we will take a non-convergent series $\sum_{n=0}^{\infty} (-1)^n$. Grandi [16] show that this series can be summable to 1/2 and it is gradually accepted as the best sum of this series. Using our matrix we will show that it is summable to 1/2 and compare it with Euler statistical convergence through graph.

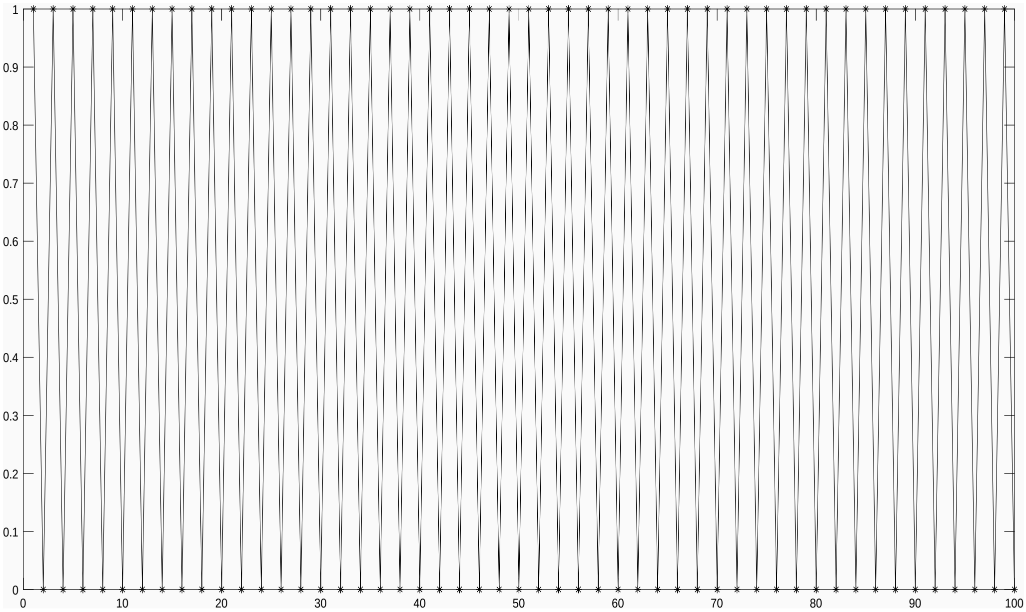

FIGURE 4. Graph representation of $\sum_{n=0}^{m}(-1)^n,\ m \leq 400$

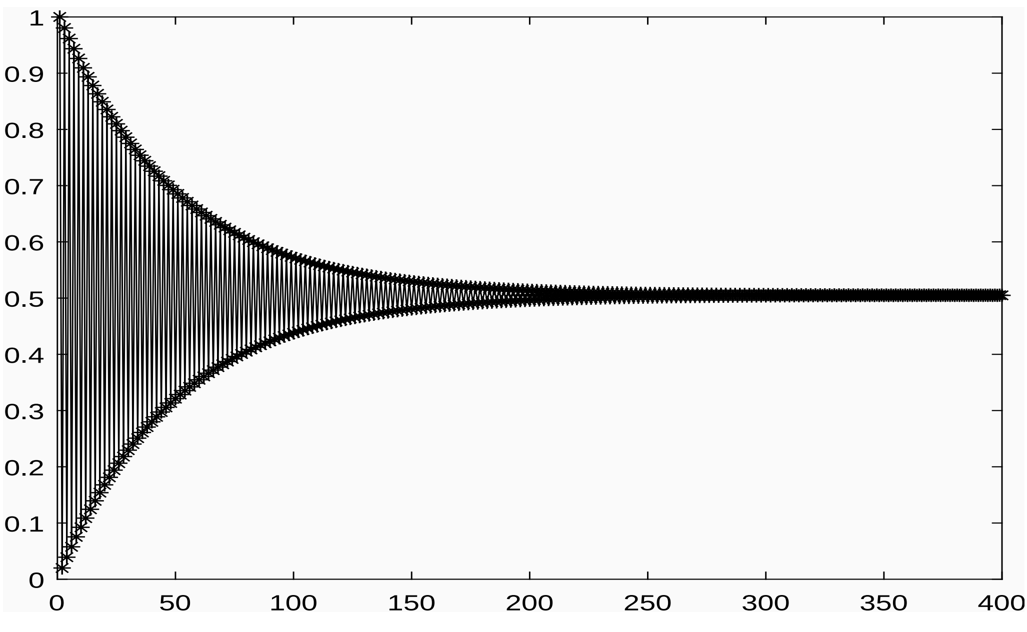

FIGURE 5. Euler convergence for $(-1)^n$

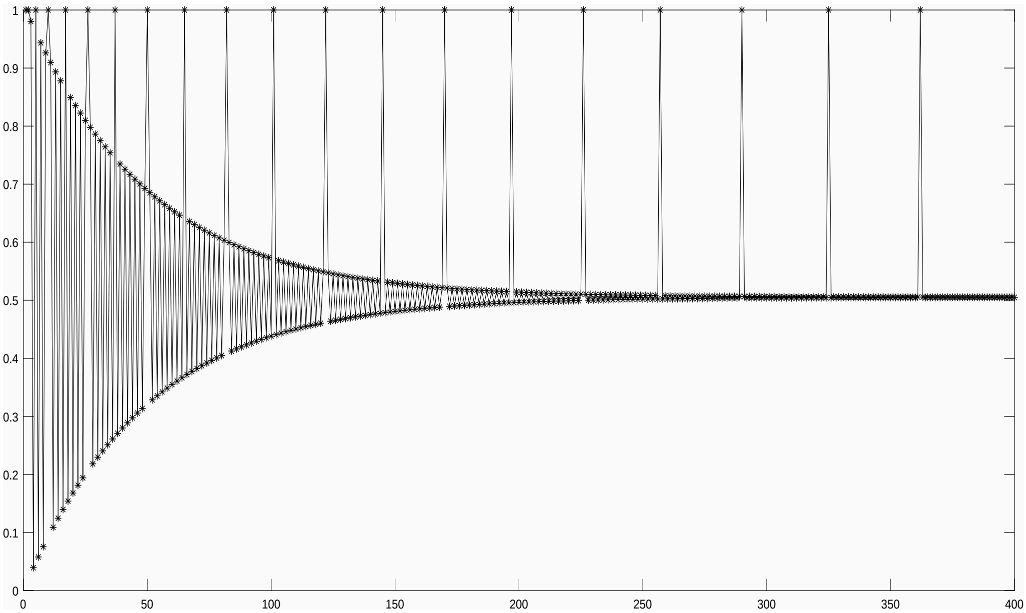

FIGURE 6. Euler statistical convergence for $(-1)^n$

Figure 4 shows that the series $\sum_{n=0}^{\infty}(-1)^n$ cannot be summed to a uniquely as $\sum_{n=0}^{m}(-1)^n$ gives 0, if m is even and 1, if m is odd. The $'*'$ shows the sum at each $m \in \mathbb{N}$.

Figure 5 shows the graph representation after applying Euler matrix on it, i.e., $(Ex)_m = \sum_{n=0}^{m}\binom{m}{n}r^n(1-r)^{m-n}(-1)^n$ for $m = 1$ to 400. One can clearly see that $'*'$ (which represent $(Ex)_m$ for different m) starts moving upwards and downwards from both sides as m increases and finally converges to 0.5, which is same as Grandi's sum. So this matrix can give a sum of some series which are not convergent.

Figure 6 shows the graph representation of E-statistical convergence of $(-1)^n$. As we can see that this also converges to 0.5 with some disturbance point$(m's)$ otherwise. But one can see that the quantity of such points are negligible, which will turned out to be of zero density when m approaches to infinity.

REFERENCES

[1] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (3) (1951) 241–244.

[2] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1) (1951) 73–74.

[3] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly. 66 (5) (1959) 361–375.

[4] O.H.H. Edely, M. Mursaleen, A. Khan, Approximation for periodic functions via weighted statistical convergence., Appl. Math. Comput. 219 (15) (2013) 8231–8236.

[5] A. Khan, V. Sharma, Statistical approximation by (p;q)-analogue of bernstein-stancu operators, Azerb. J. Math. 8 (2) (2018) 100–121.

[6] V.A. Khan, Q.M. Lohani, Statistically pre-cauchy sequences and Orlicz functions, Southeast Asian Bull. Math. 31 (6) (2007) 1107–1112.

[7] S.A. Mohiuddine, A. Alotaibi, M. Mursaleen, Statistical convergence of double sequences in locally solid riesz spaces, In: Abstr. Appl. Anal. Hindwai. 2012 (2012) 719–729.

[8] K.H. Indlekofer, Cesaro means of additive functions, Analysis. 6 (1) (1986) 1–24.

[9] L. Euler, De seriebus divergentibus, Novi Commentarii Academiae Scientiarum Petropolitanae 5 (1760) 205–237.

[10] E. Cesaro, Sur la multiplication des séries, Gauthier-Villars, 1890.

[11] O. Toeplitz, Über allgemeine lineare mittelbildungen, Prace matematyczno-zyczne 22 (1) (1911) 113–119.

[12] A. Aasma, H. Dutta, P.N. Natarajan, An Introductory Course in Summability Theory, Wiley Online Library, 2017.

[13] E.E. Kara, B. Metin, An application of Fibonacci numbers into infnite Toeplitz matrices, Casp. J. Math. Sci. 1 (1) (2012) 43–47.

[14] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley New York, 1978.

[15] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca. 30 (2) (1980) 139–150.

[16] G. Grandi, Quadratura Circuli: et Hyperbolae per Infinitas Hyperbolas, & Parabolas Geometricè Exhibita, Ex Typographia Francisci Bindi, 1703.