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Fixed Point Theorems for Generalized Contraction Mappings in Fuzzy Cone Normed Linear Space

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Abstract In this paper, some fixed point results for generalized contraction mappings in fuzzy cone normed linear space are established and some results are justified by suitable examples.

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1. Introduction

It was Zadeh [1] who introduced fuzzy set in 1965 and A.K. Katsaras [2] who while studying fuzzy topological vector space introduced the idea of fuzzy norm on a linear space in 1984. A different approach on fuzzy norm was brought forward in 1992 by C.Felbin [3] with an associated metric of the Kaleva and Seikkala type [4]. Further developement on the notion of fuzzy norm took place in 1994 when Cheng and Mordeson [5] gave the idea of fuzzy norm having a corresponding metric of the Kramosil and Michalek [6] type. Following the definition of fuzzy norm given by Cheng and Mordeson [5], Bag and Samanta [7] introduced the concept of fuzzy norm in a linear space. On the other hand, several authors generalized the concept of metric space in different approaches. In 2007, Long-Guang et al. [8] introduced the concept of cone metric space where the set of real numbers is replaced by a real Banach space. With the idea of cone metric space given by Long-Guang et.al [8], Bag [9] introduced the concept of fuzzy cone normed linear space (Felbin's type). In 2017, Tamang and Bag [10] extended the concept of fuzzy cone normed linear space and established some basic results. In [11], Tamang and Bag established some fixed point results for well known Banach, Kannan and Chatterjee type contraction in fuzzy cone normed linear space. In this paper, we proved some fixed point results for generalized contraction mappings in fuzzy cone normed linear spaces and some results

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are justified by suitable examples. In this context, it is worth mentioning some papers ([12–15]) used to develope the quality of our manuscript.

2. Preliminaries

Throughout the paper, θ_E denotes the zero element in Banach space E and $\bigwedge$ denotes the infimum.

Definition 2.1. [8] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{\theta_E\}$;
(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P\ \Rightarrow x = \theta_E$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in \mathrm{Int}P$, where $\mathrm{Int}P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, with $\theta_E \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.
The least positive number satisfying above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.
Equivalently, the cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular cone is a normal cone.

Definition 2.2. [16] The cone P is called strongly minihedral if every subset of E which is bounded above via the partial ordering obtained by P, must have a least upper bound. Hence, every subset which is bounded below must have greatest lower bound.

Definition 2.3. [17] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $a * 1 = a\ \forall a \in [0, 1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

If $*$ is continuous then it is called continuous t-norm. The following are examples of some t-norms that are frequently used and defined for all $a, b \in [0, 1]$.

(i) Standard intersection: $a * b = min(a, b)$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = max(0, a + b - 1)$.

(iv) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4. [11] Let X be a linear space over the field K and E be a real Banach space with cone P, $*$ is a t-norm. A fuzzy subset $N_c : X \times E \longrightarrow [0,1]$ is said to be a fuzzy cone norm if

FCN1 $\forall\ t \in E$ with $t \preceq \theta_E,\ N_c(x,t) = 0$
FCN2 ($\forall\ \theta_E \prec t,\ N_c(x,t) = 1$) iff $x = \theta_X$; (θ_X denotes the zero element of X)
FCN3 $\forall\ \theta_E \prec t$ and $0 \neq c \in K,\ N_c(cx,t) = N_c(x, \frac{t}{|c|})$;
FCN4 $\forall\ x, y \in X$ and $s, t \in E,\quad N_c(x+y, s+t) \geq N_c(x,s) * N_c(y,t)$;
FCN5 $N_c(x,t) = 1$ if $s \prec t\ \forall\ s \in P$.

Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t. E.

Definition 2.5. [11] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0,1)$. A sequence $\{x_n\}$ is said to be α-fuzzy convergent and converges to x if $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E,\ t \in E$.
If $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$, $t \in E, \forall\alpha \in (0,1)$, then $\{x_n\}$ is said to be l-fuzzy convergent and converges to x.

Definition 2.6. [11] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0,1)$. A sequence $\{x_n\}$ is said to be α-fuzzy Cauchy sequence if $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E,\ t \in E$, for each $p = 1, 2, 3, \ldots$.
If $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E$,$t \in E\ \forall\alpha \in (0,1)$ and for each $p = 1, 2, 3, \ldots$, then $\{x_n\}$ is said to be l-fuzzy Cauchy sequence.

Definition 2.7. [11] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0,1)$. Then X is said to be α-fuzzy complete if every α-fuzzy Cauchy sequence is α-fuzzy convergent to some element in X.

Definition 2.8. [11] Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0,1)$. Then X is said to be l-fuzzy complete if every α-fuzzy Cauchy sequence is α-fuzzy convergent $\forall\alpha \in (0,1)$.

Example 2.9. [11] Let $(X, \|\ \|_c)$ be a cone normed linear space and take $E = R^2$. Then $P = \{(t_1, t_2) : t_1, t_2 \geq 0\} \subset E$ is a strongly minihedral normal cone with normal constant 1. Define a function $N_c : X \times E \longrightarrow [0,1]$ by

$$\begin{aligned} N_c(x,t) &= 1\ if \|x\|_c \prec t \\ &= 0\ if t \preceq \|x\|_c \end{aligned}$$

If we choose $* = min$, Then $(X, N_c, *)$ is a fuzzy cone normed linear space. If we take $X = R$, then $(X, N_c, *)$ is an l-fuzzy complete fuzzy cone normed linear space.

Proof. (i) $\forall\ t \in E$ with $t \preceq \theta_E$, we have by definition, $N_c(x,t) = 0$ for all $x \in X$.
Thus $(FCN1)$ holds.
(ii) $\forall\ t \in E$ with $\theta_E \prec t$,
$N_c(x,t) = 1$
$\Rightarrow \|x\|_c \prec t\ \forall t \succ \theta_E$
$\Rightarrow \|\|x\|_c\| \leq \|t\|\ \forall t \succ \theta_E$ (Since P is normal cone with normal constant 1)
$\Rightarrow \|\|x\|_c\| = 0.$
$\Rightarrow \|x\|_c = \theta_E.$
$\Rightarrow x = \theta_X$ (θ_X denotes the zero element of X)
Again $x = \theta_X$
$\Rightarrow \|x\|_c = \theta_E.$
$\Rightarrow \|\theta_X\|_c \prec t\ \forall t \succ \theta_E$
$\Rightarrow N_c(x,t) = 1.$
So $(FCN2)$ holds.
(iii) For all $t \in E$ with $\theta_E \prec t$ and $0 \neq c \in K$
Let $N_c(cx,t) = 0$
$\Rightarrow t \preceq \|cx\|_c$
$\Rightarrow t \preceq |c|\|x\|_c$
$\Rightarrow \frac{t}{|c|} \preceq \|x\|_c \Rightarrow N_c(x, \frac{t}{|c|}) = 0.$
Let $N_c(cx,t) = 1$
$\Rightarrow \|cx\|_c \preceq t$
$\Rightarrow |c|\|x\|_c \preceq t$
$\Rightarrow \|x\|_c \preceq \frac{t}{|c|}$
$\Rightarrow N_c(x, \frac{t}{|c|}) = 1.$
So $(FCN3)$ holds.
(iv) We have to show that
$N_c(x+u, s+t) \geq min\{N_c(x,s), N_c(u,t)\}\ \forall x,y \in X$ and $s,t \in E$
If $N_c(x+u, s+t) = 0$
Then $s+t \preceq \|x+u\|_c \preceq \|x\|_c + \|u\|_c$
$\Rightarrow \|x\|_c + \|u\|_c - (s+t) \in P$
$\Rightarrow \|u\|_c - t - (s - \|x\|_c) \in P$
$\Rightarrow s - \|x\|_c \preceq \|u\|_c - t$ (2.9.1)
If $\|x\|_c \prec s$ i.e, $\theta_E \prec s - \|x\|_c$, then from (2.9.1)
$\theta_E \prec \|u\|_c - t$
$\Rightarrow t \prec \|u\|_c$
So if $\|x\|_c \prec s$, then $t \prec \|u\|_c$
So, $N_c(x,s) = 1$ and $N_c(u,t) = 0$.
Similarly, if $\|u\|_c \prec t$, then $s \prec \|x\|_c$
So $N_c(u,t) = 1$ and $N_c(x,s) = 0$.
So in both cases,
$N_c(x+u, s+t) \geq min\{N_c(x,s), N_c(u,t)\} = 0.$
If $N_c(x+u, s+t) = 1$
Then $N_c(x+u, s+t) \geq min\{N_c(x,s), N_c(u,t)\}$
So $(FCN4)$ holds.
(v) If $s \prec t$ for every $s \in E$, then by definition $N_c(x,t) = 1$
So $(FCN5)$ holds.

We now prove that $(X, N_c, *)$ is an l-fuzzy complete cone normed linear space.
Let $\{x_n\}$ be a α-Cauchy sequence in $(X, N_c, *)$ for $\alpha \in (0, 1)$.
Then $\bigwedge\{t \succ \theta_E : N_c(x_n - x_m, t) \geq \alpha\} = \theta_E$ as $m, n \to \infty$
Choose $\epsilon \succ \theta_E$ arbitrarily, then there exists a natural number p such that $\bigwedge\{t \succ \theta_E : N_c(x_n - x_m, t) \geq \alpha\} \prec \epsilon\ \forall t \succ \theta_E$ and $m, n \geq p$.
$\Rightarrow N_c(x_n - x_m, \epsilon) \geq \alpha > 0\ \forall \epsilon \succ \theta_E$ and $m, n \geq p$.
$\Rightarrow \|x_n - x_m\|_c \prec \epsilon\ \forall \epsilon \succ \theta_E$ and $m, n \geq p$. (by the definition of N_c)
$\Rightarrow \|\|x_n - x_m\|_c\| \leq \|\epsilon\|\ \forall \epsilon \succ \theta_E$ (Since P is normal cone with normal constant 1)
$\Rightarrow \|x_n - x_m\|_c \to \theta_E$ as $m, n \to \infty$
$\Rightarrow |x_n - x_m| \to 0$ as $m, n \to \infty$
$\Rightarrow \{x_n\}$ is a Cauchy sequence in R. Since R is complete, $\exists\ x \in R$ such that $x_n \to x$ as $n \to \infty$
$\Rightarrow x_n - x \to 0$ as $n \to \infty$
$\Rightarrow \|x_n - x\|_c \to \theta_E$ as $n \to \infty$
$\Rightarrow$ there exists a natural number $n_0(t)$ such that $\|x_n - x\|_c \prec t\ \forall\ t \succ \theta_E$ and $n \geq n_0(t)$.
$\Rightarrow N_c(x_n - x, t) = 1\ \forall\ t \succ \theta_E$ and $n \geq n_0(t)$.
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$ as $n \to \infty$
$\Rightarrow \{x_n\}$ is α-convergent to x.
Since $\alpha \in (0, 1)$ is arbitrary, every α-Cauchy sequence is α-convergent. So $(X, N_c, *)$ is an l-fuzzy complete fuzzy cone normed linear space. ■

3. Main Results

In this section, we revised the definition of fuzzy cone norm given in [11] to the following form and assumed the condition C1 given below to establish some fixed point results.

Definition 3.1. Let X be a linear space over the field K and E be a real Banach space with cone P, $*$ is a t-norm. A fuzzy subset $N_c : X \times E \longrightarrow [0, 1]$ is said to be a fuzzy cone norm if

(FCN1) $\forall\ t \in E$ with $t \preceq \theta_E,\ N_c(x, t) = 0$;
(FCN2) ($\forall\ \theta_E \prec t,\ N_c(x, t) = 1$) iff $x = \theta_X$;(θ_X denotes the zero element of X)
(FCN3) $\forall\ \theta_E \prec t$ and $0 \neq c \in K,\ N_c(cx, t) = N_c(x, \frac{t}{|c|})$;
(FCN4) $\forall\ x, y \in X$ and $s, t \in E,\quad N_c(x + y, s + t) \geq N_c(x, s) * N_c(y, t)$;
Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t. E.

Remark 3.2. [10] $N_c(x, .)$ is non-decreasing w.r.t. E.

C1: Assume that for a subset $A \subset E$, if inf$A$ exists say α, then for each $c \succ \theta_E$ there exists $t_c \in A$ such that $t_c \prec \alpha + c$.

Example 3.3. Let $(X, \|\ \|_c)$ be a cone normed linear space and take $E = R^2$. Then $P = \{(t, 0) : t \geq 0\} \subset E$ is a strongly minihedral normal cone with normal constant 1. Define a function $N_c : X \times E \longrightarrow [0, 1]$ by

$$\begin{aligned} N_c(x, t) &= 1 if \|x\|_c \prec t \\ &= 0 if t \preceq \|x\|_c \end{aligned}$$

If we choose $* = min$, Then $(X, N_c, *)$ is a fuzzy cone normed linear space satisfying condition C1. If we take $X = R$, then $(X, N_c, *)$ is an l-fuzzy complete fuzzy cone normed linear space.

Theorem 3.4. *Let $(X, N_c, *)$ be an l-fuzzy complete cone normed linear space satisfying C1 where $*$=min and P be a stronghly minihedral normal cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the condition*
$\bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Ty - y, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}\ \forall x, y \in X$ *and* $\forall \alpha \in (0, 1)$ *(3.4.1)*
where $1 \leq k < 5$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$. Define a sequence $\{x_n\}$ in X in the following way:
$x_{n+1} = \frac{x_n + Tx_n}{2}, n = 0, 1, 2, ...$
Then $x_n - Tx_n = 2(x_n - x_{n+1})$
Now we have,
$\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} = \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, \frac{t}{2}) \geq \alpha\}$
$= 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
First we show that $\{x_n\}$ is α-Cauchy sequence for all $\alpha \in (0, 1)$.
Put $x = x_{n-1}, y = x_n$ in (3.4.1), we have for $\alpha \in (0, 1)$,
$\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$ (3.4.2)

Now,
$\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_{n-1}, t) \geq \alpha\}$
$\succeq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n + x_n - Tx_{n-1}, t) \geq \alpha * \alpha = \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_{n-1}, t) \geq \alpha\}$ (3.4.3)
$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$
Again, $x_n - Tx_{n-1} = \frac{x_{n-1} - Tx_{n-1}}{2} = x_{n-1} - x_n$ (3.4.4)

Using (3.4.3) and (3.4.4), we have
$\bigwedge\{t \succ \theta_E : N_c(2(x_n - x_{n+1}), t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$
i.e,
$2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$ (3.4.5)

Using (3.4.2) in (3.4.5), we have
$2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
$\preceq k \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow 4 \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\preceq k \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow 4 \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
$\preceq (k - 1) \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$

$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \frac{(k-1)}{4} \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \delta \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
where $\delta = \frac{(k-1)}{4}$, $0 \leq \delta < 1$.
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \delta^n \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\| \leq M\delta^n \| \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\}\|$
$\Rightarrow \lim_{n\to\infty} \| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\| = 0.$ $(0 \leq \delta < 1)$

$$\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} = \theta_E \tag{3.4.6}$$

Now for $p \geq 1$ we have,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, \frac{t}{p}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}) \geq \alpha\} + ... + \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, \frac{t}{p}) \geq \alpha\} \succeq \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * ... * \alpha = \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \preceq p \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\} + ... + p \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \leq pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\}\| + pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\}\| + ... + pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\|$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \to 0$ as $n \to \infty$ for $p = 1, 2, 3, ...$using (3.4.6)
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \to \theta_E$ as $n \to \infty$ for $p = 1, 2, 3, ...$
$\Rightarrow \{x_n\}$ is an α- Cauchy sequence $\forall \alpha \in (0, 1)$.
Since X is l-fuzzy complete, thus $\exists z \in X$ such that

$$\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - z, t) \geq \alpha\} = \theta_E \ \forall \alpha \in (0, 1). \tag{3.4.7}$$

Now put $x = z, y = x_n$ in (3.1.1), we get

$$\bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} \tag{3.4.8}$$

Now, $\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - z, t) \geq \alpha\}$
i.e,

$$\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(Tx_n - z, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} \tag{3.4.9}$$

Using (3.4.8) in (3.4.9), we get
$\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(Tx_n - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - z, t) \geq \alpha\}$

$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - z, t) \geq \alpha\}$
$\Rightarrow 2 \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} \preceq (k + 1) \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}$
Using normality and (3.4.7), taking limit as $n \to \infty$ we have
$\|2 \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq 0$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| = 0 \ \forall \ \alpha \in (0, 1).$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} = \theta_E \ \forall \ \alpha \in (0, 1).$

Thus for each $c \succ \theta_E, \exists\, t_c$ such that $t_c \prec \theta_E + c$ and $N_c(Tz - z, t_c) \geq \alpha\ \forall\ \alpha \in (0,1)$. (by condition C1)
i.e, for each $c \succ \theta_E, N_c(Tz - z, c) \geq \alpha\ \forall\ \alpha \in (0,1)$ (Since $N_c(x, .)$ is non-decreasing)
Hence for each $c \succ \theta_E, N_c(Tz - z, c) = 1$.
So $Tz - z = \theta_X$ by $(FCN2)$
$\Rightarrow Tz = z$.
Thus T has a fixed point. Thus T has a fixed point. ■

Theorem 3.5. *Let $(X, N_c, *)$ be an l-fuzzy complete cone normed linear space satisfying C1 where $*$=min and P be a stronghly minihedral normal cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the condition*
$\bigwedge\{t \succ \theta_E : N_c(x - Tx, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - Ty, t) \geq \alpha\} \preceq q \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}\ \forall x, y \in X$ *and* $\forall \alpha \in (0,1)$ *(3.5.1)*
where $2 \leq q < 4$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$. Define a sequence $\{x_n\}$ in X in the following way:
$x_{n+1} = \frac{x_n + Tx_n}{2}, n = 0, 1, 2, ...$
Then $x_n - Tx_n = 2(x_n - x_{n+1})$
Now we have,
$\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} = \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, \frac{t}{2}) \geq \alpha\}$
$= 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
First we show that $\{x_n\}$ is α-Cauchy sequence for all $\alpha \in (0,1)$.
Put $x = x_{n-1}, y = x_n$ in (3.5.1), we have for $\alpha \in (0,1)$,
$\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - Tx_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
$\preceq q \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow 2 \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
$\preceq q \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} \preceq \frac{q-2}{2} \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$= \delta \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
for $n = 0, 1, 2, ...$ where $0 \leq \delta < 1$.
From above we have,
$\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} \leq \delta^n \bigwedge\{t \succ \theta_E : N_c(x_0 - x_1, t) \geq \alpha\}$
$\forall \alpha \in (0,1)$.
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
$\leq M\delta^n \| \bigwedge\{t \succ \theta_E : N_c(x_0 - x_1, t) \geq \alpha\}\|\ \forall \alpha \in (0,1)$.
$\Rightarrow \lim_{n\to\infty} \| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\| = 0\ \forall \alpha \in (0,1)$. $(0 \leq \delta < 1)$
$\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} = \theta_E\ \ \forall \alpha \in (0,1)$. (3.5.2)
Now for $p \geq 1$ we have,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, \frac{t}{p}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}) \geq \alpha\} + ... + \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, \frac{t}{p}) \geq \alpha\} \succeq \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * ... * \alpha = \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \preceq p \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\} + ... + p \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \leq pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\}\| + pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\}\| + ... + pM \| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq$

$\alpha\}\|$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \to 0$ as $n \to \infty$ for $p = 1, 2, 3, ...$using (3.5.2)
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \to \theta_E$ as $n \to \infty$ for $p = 1, 2, 3, ...$
$\Rightarrow \{x_n\}$ is an α- Cauchy sequence $\forall \alpha \in (0, 1)$.
Since X is l-fuzzy complete, thus $\exists z \in X$ such that
$$\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - z, t) \geq \alpha\} = \theta_E \ \forall \alpha \in (0, 1). \quad (3.5.3)$$
Put $x = z$ and $y = x_n$ in (3.5.1) we get,
$\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
$\preceq q \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} \ \forall \alpha \in (0, 1)$.
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} + 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
$\preceq q \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} \ \forall \alpha \in (0, 1)$.
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\}\| \leq M\|q \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} - 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(z-Tz, t) \geq \alpha\}\| \leq Mq\| \bigwedge\{t \succ \theta_E : N_c(z-x_n, t) \geq \alpha\}\| + 2M\| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\| \ \forall \alpha \in (0, 1)$.
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\}\| = 0 \ \forall \alpha \in (0, 1)$. using (3.5.2) and (3.5.3)
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} = \theta_E \ \forall \ \alpha \in (0, 1)$.
Thus for each $c \succ \theta_E, \exists \ t_c$ such that $t_c \prec \theta_E + c$ and $N_c(z - Tz, t_c) \geq \alpha \ \forall \ \alpha \in (0, 1)$. (by condition C1)
i.e, for each $c \succ \theta_E, N_c(z - Tz, c) \geq \alpha \ \forall \ \alpha \in (0, 1)$ (Since $N_c(x, .)$ is non-decreasing)
Hence for each $c \succ \theta_E, N_c(z - Tz, c) = 1$.
So $z - Tz = \theta_X$ by $(FCN2)$
$\Rightarrow Tz = z$.
Thus T has a fixed point. ■

Theorem 3.6. *Let $(X, N_c, *)$ be an l-fuzzy complete cone normed linear space satisfying C1 where $*$=min and P be a stronghly minihedral normal cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the condition*
$$a \bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(x - Tx, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - Ty, t) \geq \alpha\}] \preceq s \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} \ \forall x, y \in X \text{ and } \forall \alpha \in (0, 1) \quad (3.6.1)$$
where $0 \leq s + |a| - 2b < 2(a + b)$. Then T has a fixed point in X. If in addition $|a| > M|s|$, then the fixed point is unique.

Proof. Choose $x_0 \in X$. Define a sequence $\{x_n\}$ in X in the following way:
$x_{n+1} = \frac{x_n + Tx_n}{2}, n = 0, 1, 2, ...$
Then $x_n - Tx_n = 2(x_n - x_{n+1})$
Now we have,
$\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} = \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, \frac{t}{2}) \geq \alpha\}$
$= 2 \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
First we show that $\{x_n\}$ is α-Cauchy sequence for all $\alpha \in (0, 1)$.
If $a \geq 0$ putting $x = x_{n-1}, y = x_n$ in (3.6.1), we get for $\alpha \in (0, 1)$,
$a \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - Tx_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}] \preceq s \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
i.e,
$$a \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} + 2b[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq s \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} \quad (3.6.2)$$

Again, $x_n - Tx_{n-1} = \frac{x_{n-1}+Tx_{n-1}}{2} - Tx_{n-1} = \frac{x_{n-1}-Tx_{n-1}}{2} = x_{n-1} - x_n$
Now,
$\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_{n-1}, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$
$\Rightarrow 2\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$ (3.6.3)

Since $a \geq 0$, from (3.6.2) and (3.6.3) we get
$2a\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} - a\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2b[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$ (3.6.4)

Now,
$\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
If $a < 0$ then,
$a[\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}] \preceq a\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$
i.e,
$a[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq a\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - Tx_n, t) \geq \alpha\}$ (3.6.5)

From (3.6.2) and (3.6.5), we get
$2a\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} + a\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2b[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$ (3.6.6)

Combining (3.6.4) and (3.6.6), we have
$2a\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} - |a|\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + 2b[\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow 2(a+b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} \preceq (s+|a|-2b)\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} \preceq \frac{(s+|a|-2b)}{2(a+b)}\bigwedge\{t \succ \theta_E : N_c(x_{n-1} - x_n, t) \geq \alpha\}$ for $n = 0, 1, 2, ...$
Let $\frac{(s+|a|-2b)}{2(a+b)} = \delta$. Then $0 \leq \delta < 1$.
From above we have,
$\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} \leq \delta^n\bigwedge\{t \succ \theta_E : N_c(x_0 - x_1, t) \geq \alpha\}$
$\forall \alpha \in (0, 1)$.
Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
$\leq M\delta^n\| \bigwedge\{t \succ \theta_E : N_c(x_0 - x_1, t) \geq \alpha\}\| \; \forall \alpha \in (0, 1)$.
$\Rightarrow \lim_{n\to\infty} \| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\| = 0 \; \forall \alpha \in (0, 1)$. $(0 \leq \delta < 1)$
$\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} = \theta_E \;\; \forall \alpha \in (0, 1)$. (3.6.7)

Now for $p \geq 1$ we have,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, \frac{t}{p}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}) \geq \alpha\} + ... + \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, \frac{t}{p}) \geq \alpha\} \succeq \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * ... * \alpha = \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \preceq p\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\} + p\bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\} + ... + p\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\|\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \leq pM\|\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\}\| + pM\|\bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\}\| + ... + pM\|\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\|$
$\Rightarrow \|\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \to 0$ as $n \to \infty$ for $p = 1, 2, 3, ...$using (3.6.7)
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \to \theta_E$ as $n \to \infty$ for $p = 1, 2, 3, ...$
$\Rightarrow \{x_n\}$ is an α- Cauchy sequence $\forall \alpha \in (0, 1)$.
Since X is l-fuzzy complete, thus $\exists z \in X$ such that

$$\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - z, t) \geq \alpha\} = \theta_E \ \forall \alpha \in (0, 1). \tag{3.6.8}$$

Now we will show that z is a fixed point of T.
Case I. When $a \geq 0$.
Put $x = z$ and $y = x_n$ in (3.6.1), we get

$$a\bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} \tag{3.6.9}$$

Now,
$\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_n - Tz, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} - \bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tx_n - Tz, t) \geq \alpha\}$

$$\Rightarrow a\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} - a\bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\} \preceq a\bigwedge\{t \succ \theta_E : N_c(Tx_n - Tz, t) \geq \alpha\} \tag{3.6.10}$$

From (3.6.9) and (3.6.10), we get
$a\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} - a\bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}$
$\Rightarrow (a + b)\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} \preceq s\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} - b\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} + a\bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\}$
$\Rightarrow (a + b)\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} \preceq s\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} - b\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\} + a\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + a\bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
$= (s + a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + 2(a - b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\|(a + b)\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\}\| \leq M\|(s + a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + 2(a - b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
$\Rightarrow \|(a + b)\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\}\| \leq M\|(s + a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}\| + M\|2(a - b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
Using (3.6.7) and (3.6.8), we get
$\|(a + b)\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq 0$
Case II. When $a < 0$.
Now,
$\bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - Tx_n, t) \geq \alpha\}$

$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - Tx_n, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + 2\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
$\Rightarrow a[\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + 2\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq a\bigwedge\{t \succ \theta_E : N_c(Tz - Tx_n, t) \geq \alpha\}$ $(a < 0)$ (3.6.11)

From (3.6.9) and (3.6.11), we get
$a\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} + a\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} + 2a\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} + 2\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}$
$\Rightarrow (a + b)\bigwedge\{t \succ \theta_E : N_c(z - Tz, t) \geq \alpha\} \preceq (s - a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} - 2(a + b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\|(a + b)\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq M\|(s - a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\} - 2(a + b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
$\Rightarrow \|(a + b)\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq M\|(s - a)\bigwedge\{t \succ \theta_E : N_c(z - x_n, t) \geq \alpha\}\| + 2M\|(a + b)\bigwedge\{t \succ \theta_E : N_c(x_n - x_{n+1}, t) \geq \alpha\}\|$
Using (3.6.7) and (3.6.8), we get
$\|(a + b)\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq 0$
Thus in both cases, we get
$\|(a + b)\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| \leq 0$
$\Rightarrow \|\bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\}\| = 0 \ \forall \alpha \in (0, 1)$ since $(a + b) > 0$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tz - z, t) \geq \alpha\} = \theta_E \ \forall\ \alpha \in (0, 1)$.
Thus for each $c \succ \theta_E, \exists\ t_c$ such that $t_c \prec \theta_E + c$ and $N_c(Tz - z, t_c) \geq \alpha \ \forall\ \alpha \in (0, 1)$. (by condition C1)
i.e, for each $c \succ \theta_E, N_c(Tz - z, c) \geq \alpha \ \forall\ \alpha \in (0, 1)$ (Since $N_c(x, .)$ is non-decreasing)
Hence for each $c \succ \theta_E, N_c(Tz - z, c) = 1$.
So $Tz - z = \theta_X$ by $(FCN2)$
$\Rightarrow Tz = z$.
Thus T has a fixed point.
Uniqueness: If there exists $x, y \in X$ such that $Tx = x$ and $Ty = y$.
Since $Tx = x$ and $Ty = y$, $N_c(x - Tx, t) = 1, N_c(y - Ty, t) = 1$ by $(FCN2)$
Thus $\bigwedge\{t \succ \theta_E : N_c(x - Tx, t) \geq \alpha\} = \theta_E$ and $\bigwedge\{t \succ \theta_E : N_c(y - Ty, t) \geq \alpha\} = \theta_E$. (3.6.12)

From (3.6.1) we have,
$a\bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(x - Tx, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - Ty, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}$
i.e,
$a\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} + b[\bigwedge\{t \succ \theta_E : N_c(x - Tx, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - Ty, t) \geq \alpha\}] \preceq s\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}$
Using (3.6.12) and the normality condition , from above we have,
$(|a| - M|s|)\|\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}\| \leq 0$
$\Rightarrow \|\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}\| = 0$ since $(|a| > M|s|)$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} = \theta_E \ \forall\ \alpha \in (0, 1)$.
Thus for each $c \succ \theta_E, \exists\ t_c$ such that $t_c \prec \theta_E + c$ and $N_c(x - y, t_c) \geq \alpha \ \forall\ \alpha \in (0, 1)$. (by condition C1)

i.e, for each $c \succ \theta_E, N_c(x-y,c) \geq \alpha \ \forall \ \alpha \in (0,1)$ (Since $N_c(x,.)$ is non-decreasing)
Hence for each $c \succ \theta_E, N_c(x-y,c) = 1$
So $x - y = \theta_X$ by $(FCN2)$
$\Rightarrow x = y$. ∎

Theorem 3.7. *Let $(X, N_c, *)$ be an l-fuzzy complete cone normed linear space satisfying C1 where $*$=min and P be a stronghly minihedral normal cone with normal constant M. Suppose the mappings $f, g : X \longrightarrow X$ satisfies the condition*
$\bigwedge\{t \succ \theta_E : N_c(fx-gy,t) \geq \alpha\} \preceq p\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x-fx,t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y-gy,t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x-gy,t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y-fx,t) \geq \alpha\}] \ \forall x, y \in X$ *and* $\forall \alpha \in (0,1)$ *(3.7.1)*
where $p, q, r \geq 0$ and $p + 2q + 2r < 1$. Then f and g have a unique common fixed point in X.

Proof. Choose $x_0 \in X$. Define a sequence $\{x_n\}$ in X in the following way:
$x_{2n+1} = fx_{2n}, \ x_{2n+2} = gx_{2n+1}, \ n = 0, 1, 2, 3, ...$
Now,
$\bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(fx_{2n} - gx_{2n+1}, t) \geq \alpha\}$
$\preceq p\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - fx_{2n}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - gx_{2n+1}, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - gx_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - fx_{2n}, t) \geq \alpha\}]$
$= p\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+2}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+1}, t) \geq \alpha\}]$
$\preceq p\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\}] + \theta_E$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\} \preceq (\frac{p+q+r}{1-q-r})\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\}$
Let $\delta = \frac{p+q+r}{1-q-r}$. Then $0 \leq \delta < 1$.
Thus we have,
$\bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\} \preceq \delta\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\}$
Similarly it can be shown that,
$\bigwedge\{t \succ \theta_E : N_c(x_{2n+2} - x_{2n+3}, t) \geq \alpha\} \preceq \delta\bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - x_{2n+2}, t) \geq \alpha\}$
Thus for all n,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \delta^n\bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\} \ \forall\alpha \in (0,1)$.
Since P is a normal cone with normal constant M, from above we have,
$\|\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\|$
$\leq M\delta^n\|\bigwedge\{t \succ \theta_E : N_c(x_0 - x_1, t) \geq \alpha\}\| \ \forall\alpha \in (0,1)$.
$\Rightarrow \lim_{n\to\infty} \|\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\| = 0 \ \forall\alpha \in (0,1)$. $(0 \leq \delta < 1)$
$\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} = \theta_E \ \ \forall\alpha \in (0,1)$. (3.7.2)
Now for $p \geq 1$ we have,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, \frac{t}{p}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}) \geq \alpha\} + ... + \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, \frac{t}{p}) \geq \alpha\} \succeq \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * ... * \alpha = \alpha\}$
i.e,
$\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \preceq p\bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\} + p\bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\} + ... + p\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$

Since P is a normal cone with normal constant M, from above we have,
$\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \leq pM\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_{n+p-1}, t) \geq \alpha\}\| + pM\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p-1} - x_{n+p-2}, t) \geq \alpha\}\| + ... + pM\| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\|$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\}\| \to 0$ as $n \to \infty$ for $p = 1, 2, 3, ...$using (3.7.2)

$$\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E \text{ for } p = 1, 2, 3, ... \tag{3.7.3}$$

$\Rightarrow \{x_n\}$ is an α- Cauchy sequence $\forall \alpha \in (0, 1)$.
Since X is l-fuzzy complete, thus $\exists z \in X$ such that

$$\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - z, t) \geq \alpha\} = \theta_E \ \forall \alpha \in (0, 1). \tag{3.7.4}$$

Now,
$\bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_{2n+1} - gz, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(fx_{2n} - gz, t) \geq \alpha\}$
$\preceq \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - fx_{2n}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - gz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}]$
$\preceq \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}]$
$\Rightarrow (1-q-r) \bigwedge\{t \succ \theta_E : N_c(z-gz, t) \geq \alpha\} \preceq \bigwedge\{t \succ \theta_E : N_c(z-x_{2n+1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + q \bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + r \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + r \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\|(1 - q - r) \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}\| \leq M\| \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\} + p \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + q \bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\} + r \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\} + r \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}\|$
$\Rightarrow \|(1 - q - r) \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}\| \leq M\| \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}\| + M\|p \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\}\| + M\|q \bigwedge\{t \succ \theta_E : N_c(x_{2n} - x_{2n+1}, t) \geq \alpha\}\| + M\|r \bigwedge\{t \succ \theta_E : N_c(x_{2n} - z, t) \geq \alpha\}\| + M\|r \bigwedge\{t \succ \theta_E : N_c(z - x_{2n+1}, t) \geq \alpha\}\|$
Using (3.7.3) and (3.7.4), we get
$\|(1 - q - r) \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}\| \leq 0$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}\| \leq 0$ since $1 - q - r > 0$
$\Rightarrow \| \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\}\| = 0 \ \forall \alpha \in (0, 1)$.
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\} = \theta_E \ \forall \ \alpha \in (0, 1)$.
Thus for each $c \succ \theta_E, \exists \ t_c$ such that $t_c \prec \theta_E + c$ and $N_c(z - gz, t_c) \geq \alpha \ \forall \ \alpha \in (0, 1)$. (by condition C1)
i.e, for each $c \succ \theta_E, N_c(z - gz, c) \geq \alpha \ \forall \ \alpha \in (0, 1)$ (Since $N_c(x, .)$ is non-decreasing)
Hence for each $c \succ \theta_E, N_c(z - gz, c) = 1$.
So $z - gz = \theta_X$ by $(FCN2)$
$\Rightarrow gz = z$.
Thus g has a fixed point.
Since $gz = z, N_c(gz - z, t) = 1$ by $(FCN2)$
Thus, $\bigwedge\{t \succ \theta_E : N_c(z - gz, t) \geq \alpha\} = \theta_E$.
Now,
$\bigwedge\{t \succ \theta_E : N_c(fz - z, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(fz - gz, t) \geq \alpha\}$
$\preceq p \bigwedge\{t \succ \theta_E : N_c(z - z, t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(z - fz, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E :$

$N_c(z-gz,t)\geq\alpha\}]+r[\bigwedge\{t\succ\theta_E:N_c(z-gz,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(z-fz,t)\geq\alpha\}]$
$\Rightarrow (1-q-r)\bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}\preceq p\bigwedge\{t\succ\theta_E:N_c(z-z,t)\geq\alpha\}+(q+r)\bigwedge\{t\succ\theta_E:N_c(z-gz,t)\geq\alpha\}$
Since P is a normal cone with normal constant M, from above we have,
$\|(1-q-r)\bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}\|\leq M\|p\bigwedge\{t\succ\theta_E:N_c(z-z,t)\geq\alpha\}+(q+r)\bigwedge\{t\succ\theta_E:N_c(z-gz,t)\geq\alpha\}\|$
$\Rightarrow \|(1-q-r)\bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}\|\leq 0$
$\Rightarrow \|\bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}\|\leq 0$ since $1-q-r>0$
$\Rightarrow \|\bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}\|=0\ \forall\alpha\in(0,1)$.
$\Rightarrow \bigwedge\{t\succ\theta_E:N_c(fz-z,t)\geq\alpha\}=\theta_E\ \forall\ \alpha\in(0,1)$.
Thus for each $c\succ\theta_E,\exists\ t_c$ such that $t_c\prec\theta_E+c$ and $N_c(fz-z,t_c)\geq\alpha\ \forall\ \alpha\in(0,1)$. (by condition C1)
i.e, for each $c\succ\theta_E, N_c(fz-z,c)\geq\alpha\ \forall\ \alpha\in(0,1)$ (Since $N_c(x,.)$ is non-decreasing)
Hence for each $c\succ\theta_E, N_c(fz-z,c)=1$.
So $fz-z=\theta_X$ by $(FCN2)$
$\Rightarrow fz=z$.
Thus f has a fixed point.
Uniqueness: If there exists $z,y\in X$ such that $fy=y, gy=y$ and $fz=z, gz=z$.
Then
$\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}$
$\bigwedge\{t\succ\theta_E:N_c(fz-gy,t)\geq\alpha\}$
$\preceq p\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}+q[\bigwedge\{t\succ\theta_E:N_c(z-fz,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(y-gy,t)\geq\alpha\}]+r[\bigwedge\{t\succ\theta_E:N_c(z-gy,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(y-fz,t)\geq\alpha\}]$
$=p\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}+q[\bigwedge\{t\succ\theta_E:N_c(z-fz,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(y-gy,t)\geq\alpha\}]+r[\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(y-z,t)\geq\alpha\}]$
Since P is a normal cone with normal constant M, from above we have,
$\|(1-p-2r)\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}\|\leq 0$
$\Rightarrow \|\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}\|\leq 0$ since $p+2r<1$
$\Rightarrow \|\bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}\|=0\ \forall\alpha\in(0,1)$.
$\Rightarrow \bigwedge\{t\succ\theta_E:N_c(z-y,t)\geq\alpha\}=\theta_E\ \forall\ \alpha\in(0,1)$.
Thus for each $c\succ\theta_E,\exists\ t_c$ such that $t_c\prec\theta_E+c$ and $N_c(z-y,t_c)\geq\alpha\ \forall\ \alpha\in(0,1)$. (by condition C1)
i.e, for each $c\succ\theta_E, N_c(z-y,c)\geq\alpha\ \forall\ \alpha\in(0,1)$ (Since $N_c(x,.)$ is non-decreasing)
Hence for each $c\succ\theta_E, N_c(z-y,c)=1$
So $z-y=\theta_X$ by $(FCN2)$
$\Rightarrow z=y$. ■

Example 3.8. Consider a fuzzy cone normed linear space $(X,N_c,*)$ where $X=R$, $E=R^2$, $P=\{(t,0):t\geq 0\}$ and $N_c:X\times E\longrightarrow[0,1]$ defined by

$$\begin{aligned} N_c(x,t) &= 1 if \|x\|_c\prec t \\ &= 0 if t\preceq\|x\|_c \end{aligned}$$

where $\|\ \|_c:X\longrightarrow E$ is a cone norm. If we take $*=min$, then $(X,N_c,*)$ is an l-fuzzy complete fuzzy cone normed linear space satisfying C1. Let $T:X\longrightarrow X$ be given by $Tx=\frac{x}{3}$. Take $a=10, b=-1$, and $s=4$.
Then $s+|a|-2b=4+10+2=16<2(a+b)=18$
Now, $s\bigwedge\{t\succ\theta_E:N_c(x-y,t)\geq\alpha\}-b[\bigwedge\{t\succ\theta_E:N_c(Tx-x,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:$

$N_c(y - Ty, t) \geq \alpha\}]$
$= 4\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + [\bigwedge\{t \succ \theta_E : N_c(\frac{x}{3} - x, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - \frac{y}{3}, t) \geq \alpha\}]$
$\succeq \bigwedge\{t \succ \theta_E : N_c(x-y, \frac{t}{4}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(\frac{x}{3} - x + y - \frac{y}{3}, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(4(x-y), t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(\frac{2y}{3} - \frac{2x}{3}, t) \geq \alpha\}$
$\succeq \bigwedge\{t \succ \theta_E : N_c(4x - \frac{2x}{3} - 4y + \frac{2y}{3}), t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(\frac{10x}{3} - \frac{10y}{3}), t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(\frac{x}{3} - \frac{y}{3}), \frac{t}{10}) \geq \alpha\}$
$= 10\bigwedge\{t \succ \theta_E : N_c(\frac{x}{3} - \frac{y}{3}), t) \geq \alpha\}$
$= a\bigwedge\{t \succ \theta_E : N_c(Tx - Ty), t) \geq \alpha\}.$
Thus T satisfies the condition of the Theorem 3.6. We see that 0 is the unique fixed point of T.

Example 3.9. Consider a fuzzy cone normed linear space $(X, N_c, *)$ where $X = R$, $E = R^2$, $P = \{(t,0) : t \geq 0\}$ and $N_c : X \times E \longrightarrow [0,1]$ defined by

$$\begin{aligned} N_c(x,t) &= 1 if \|x\|_c \prec t \\ &= 0 if t \preceq \|x\|_c \end{aligned}$$

where $\| \|_c : X \longrightarrow E$ is a cone norm. If we take $* = min$, then $(X, N_c, *)$ is an l-fuzzy complete fuzzy cone normed linear space satisfying C1. Let $f, g : X \longrightarrow X$ be given by $fx = \frac{x}{3}$ and $gx = \frac{x}{7}$. Take $p = \frac{1}{4}, q = r = \frac{1}{16}$.
Then $p + 2q + 2r = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} < 1$
Now,
$p\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + q[\bigwedge\{t \succ \theta_E : N_c(x - fx, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - gy, t) \geq \alpha\}] + r[\bigwedge\{t \succ \theta_E : N_c(x - gy, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - fx, t) \geq \alpha\}]$
$= \frac{1}{4}\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + \frac{1}{16}[\bigwedge\{t \succ \theta_E : N_c(x - \frac{x}{3}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - \frac{y}{7}, t) \geq \alpha\}] + \frac{1}{16}[\bigwedge\{t \succ \theta_E : N_c(x - \frac{y}{7}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y - \frac{x}{3}, t) \geq \alpha\}]$
$\succeq \frac{1}{4}\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + \frac{1}{16}\bigwedge\{t \succ \theta_E : N_c(\frac{2x}{3} + \frac{6y}{7}, t) \geq \alpha\} + \frac{1}{16}\bigwedge\{t \succ \theta_E : N_c(\frac{2x}{3} + \frac{6y}{7}, t) \geq \alpha\}$
$= \frac{1}{4}\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + \frac{1}{8}\bigwedge\{t \succ \theta_E : N_c(\frac{2x}{3} + \frac{6y}{7}, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(x-y, \frac{t}{\frac{1}{4}}) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(\frac{2x}{3} + \frac{6y}{7}, \frac{t}{\frac{1}{8}}) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(\frac{1}{4}(x-y), t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(\frac{1}{8}(\frac{2x}{3} + \frac{6y}{7}), t) \geq \alpha\}$
$\succeq \bigwedge\{t \succ \theta_E : N_c(\frac{1}{4}(x-y) + \frac{1}{4}(\frac{x}{3} + \frac{3y}{7}), t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(\frac{1}{4}(\frac{4x}{3} - \frac{4y}{7}), t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(\frac{x}{3} - \frac{y}{7}, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(fx - gy, t) \geq \alpha\}.$
Thus f and g satisfies the condition of the Theorem 3.7. Consequently 0 is the unique common fixed point of f and g.

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