Thai Journal of Mathematics
Volume 20 Number 4 (2022)
Pages 1721–1733
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Existence and Uniqueness of Solutions of Fractional Integrodifferential Equations with Deviating Arguments under Integral Boundary Conditions

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Abstract In this paper, we study the existence, uniqueness and boundedness of solutions of following Rieman-Liouville fractional integrodifferential equations with deviating arguments under integral boundary conditions via monotone interative technique by introducing upper and lower solutions:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right), \ t \in J = [0,T], \\ \qquad u(0) = \lambda \int_0^T u(s)ds + d, \ d \in \mathbb{R}. \end{cases}$$

MSC: 26A33; 34A08; 34A12; 34K20; 37C25

Keywords: fractional integrodifferential equations with deviating arguments; integral boundary conditions; existence of solutions

Submission date: 02.06.2020 / Acceptance date: 19.03.2022

1. Introduction

The investigation of the theory of fractional differential and integral equations has started quite recently. One can see the monographs of Kilbas et al. [1], Podlubny [2], etc. The study of integrodifferential equations is linked to the wide applications of calculus in physics, mechanics, signal processing, electromagnetics, biology, economics and many more.

Integral boundary conditions are encountered in population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc. They come up when values of the function on the boundary are connected to its values inside the domain, they have physical significations such as total mass, moments, etc. Sometimes it is better to impose integral conditions because they lead to more

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precise measures than those proposed by a local condition. Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the fractional integrodifferential equations (1.1), see ([3–35]) and some of the references cited therein.

In [13], Jankowski considered the existence and uniqueness of solutions by using Banach fixed point theorem and monotone iterative method of the following initial value problem for nonlinear Riemann-Liouville fractional differential equations with deviating arguments:

$$\begin{cases} D^q_{0+}x(t) = f\left(t, x(t), x(\alpha(t))\right), t \in J = [0,T],\ T > 0, \\ \left[x(t)t^{1-q}\right]\Big|_{t=0} = x_0. \end{cases}$$

So, the motivation for the elaboration of this paper is to study the following problem for Riemann Liouville's fractional integrodifferential equations with deviating arguments under the integral boundary conditions :

$$\begin{cases} D^\alpha_{0+}u(t) = f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right), t \in J = [0,T] \\ u(0) = \lambda \int_0^T u(s)ds + d, d \in \mathbb{R} \end{cases} \tag{1.1}$$

where $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, $\theta \in C(J,J)$, $\theta(t) \leq t$, $t \in J$, $\lambda \geq 0$, $0 < \alpha < 1$, where $K : J \times J \to \mathbb{R}$ and $K_T = sup\{|K(t,s)| : 0 \leq t, s \leq T\}$.

The rest of this paper is arranged as follows: some basic definitions and results are introduced in section 2. Section 3 devoted to discuss the existence and uniqueness of a solution for the problem given by (1.1) using Banach fixed point theorem. In section 4, we develop the monotone method and apply it to obtain existence and uniqueness results for Riemann-Liouville fractional integrodifferential equations with deviating arguments under the integral boundary conditions.

2. Preliminaries

Before proceeding to the statement of our main results, we setforth definitions, preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. ([1],[2]) The Riemann-Liouville fractional integral of order α, is defined by

$$I^\alpha_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds,\ \alpha > 0, \tag{2.1}$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. ([1],[2]) The Riemann-Liouville fractional derivative of order α $(n-1 < \alpha < n)$is defined as

$$D^\alpha_{0+}u(t) = \left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}_{0+}u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}u(s)\, ds,\ t > 0. \tag{2.2}$$

We need the following results in our subsequent discussion.

Lemma 2.3 ([1]). *Let $u \in C^n[0,T], \alpha \in (n-1,n), n \in \mathbb{N}$. Then for $t \in J$,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^k(0). \tag{2.3}$$

Consider the space $C_{1-\alpha}(J,\mathbb{R}) = \{u \in C((0,T],\mathbb{R}) : t^{1-\alpha}u(t) \in C(J,\mathbb{R})\}$.

Lemma 2.4 ([10]). *Let $m \in C_{1-\alpha}(J,\mathbb{R})$ where for some $t_1 \in (0,T]$, $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then $D^{\alpha} m(t_1) \geq 0$.*

Lemma 2.5. *Let $f \in C(J \times \mathbb{R}^3, \mathbb{R})$. A function $u \in C_{1-\alpha}(J,\mathbb{R})$ is a solution of the problem (1.1) if and only if u is a solution of the integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right) ds$$
$$+ \lambda \int_0^T u(s)ds + d. \tag{2.4}$$

Proof. Suppose that u is a solution of (1.1). Then from Lemma 2.3, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^k(0)$$

As $u \in C_{1-\alpha}(J,\mathbb{R})$ so $n = 1$, on using $n = 1$ in equation above equation, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - u(0) \tag{2.5}$$

From first equation of problem (1.1), we have

$$D_{0+}^{\alpha} u(t) = f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right),$$

using it in (2.5), we get

$$I_{0+}^{\alpha}\left[f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right)\right] = u(t) - u(0) \tag{2.6}$$

From equations (2.1) in above equation, we get equation (2.4).
Conversely, suppose that u satisfies equation (2.4). On taking Riemann-Liouville derivative of order α of both sides, we get first equation of the problem (1.1). On putting $t = 0$ in equation (2.4) we get the integral boundary condition

$$u(0) = \lambda \int_0^T u(s)ds + d.$$

This completes the proof. ∎

Lemma 2.6. *Suppose that $\{u_\epsilon\}$ is a family of continuous functions defined on J, for each $\epsilon > 0$, which satisfies*

$$\begin{cases} D_{0+}^{\alpha} u_\epsilon(t) = f\left(t, u_\epsilon(t), u_\epsilon(\theta(t)), \int_0^t K(t,s)u_\epsilon(s)ds\right), t \in J = [0,T] \\ u_\epsilon(0) = \lambda \int_0^T u_\epsilon(s)ds + d, d \in \mathbb{R} \end{cases}$$

where $\left|f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right)\right| \leq M \ \forall \ t \in J$. *Then the family* $\{u_\epsilon\}$ *is equicontinuous on* J.

Proof. For $0 \leq t_1 < t_2 \leq T$, using equation (2.4), consider

$$\begin{aligned}
&|u_\epsilon(t_1) - u_\epsilon(t_2)| \\
&= \left|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right) ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right) ds\right| \\
&\leq \frac{M}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - s)^{\alpha-1} ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} ds \right| \\
&= \frac{M}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - s)^{\alpha-1} ds - \left[\int_0^{t_1} (t_1 - s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds\right] \right| \\
&= \frac{M}{\alpha\Gamma(\alpha)} |2(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha| \\
&\leq \frac{2M}{\Gamma(\alpha+1)} |(t_2 - t_1)^\alpha| < \epsilon
\end{aligned}$$

Choose $\delta = \sqrt[\alpha]{\dfrac{\epsilon\Gamma(\alpha+1)}{2M}}$ Then for $|t_2 - t_1| < \delta$, we have $|u_\epsilon(t_1) - u_\epsilon(t_2)| < \epsilon$.
The proof is completed. ∎

3. Uniqueness of Solutions

Theorem 3.1. *Suppose that*

(1) $f \in C(J \times \mathbb{R}^3, \mathbb{R}),\ \theta(t) \in C(J, J),\ \theta(t) \leq t,\ t \in J$
(2) there exists non-negative constants $M,\ N,\ L$ *such that the function satisfies* $|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq M|u_1 - v_1| + N|u_2 - v_2| + L|u_3 - v_3| \quad \forall\ t \in J, u_i, v_i \in \mathbb{R}, i = 1, 2, 3.$

If

$$\lambda < \frac{\Gamma(\alpha+1)\Gamma(\alpha+2) - \Gamma(\alpha+2)T^\alpha(M+N) - \Gamma(\alpha+1)LK_T T^{\alpha+1}}{T\ \Gamma(\alpha+1)\Gamma(\alpha+2)}$$

then the problem (1.1) has unique solution.

Proof. Define an operator T given by

$$\begin{aligned}
(Tu)(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right) ds \\
&+ \lambda \int_0^T u(s)ds + d.
\end{aligned}$$

Now, we prove that $T : C_{1-\alpha}(J,\mathbb{R}) \to C_{1-\alpha}(J,\mathbb{R})$ is a contraction operator. Consider for any $u,\ v \in C_{1-\alpha}(J,\mathbb{R})$,

$$
\begin{aligned}
&||Tu - Tv||\\
&= \max_{t\in J}\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right) ds\\
&\quad + \lambda\int_0^T u(s)ds + d\\
&- \left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, v(s), v(\theta(s)), \int_0^s K(s,\tau)v(\tau)d\tau\right) ds + \lambda\int_0^T v(s)ds + d\right\}\Bigg|\\
&\le \max_{t\in J}\lambda\int_0^T |u(s)-v(s)|ds +\\
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Bigg|\Bigg[f\left(s, u(s), u(\theta(s)), \int_0^s K(s,\tau)u(\tau)d\tau\right)\\
&- f\left(s, v(s), v(\theta(s)), \int_0^s K(s,\tau)v(\tau)d\tau\right)\Bigg]\Bigg|ds\\
&\le \max_{t\in J}\lambda\int_0^T |u(s)-v(s)|ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Bigg[M|u(s)-v(s)| + N|u(\theta(s)) - v(\theta(s))|\\
&\quad + L\left[\int_0^s |K(s,\tau)|\ |u(\tau)-v(\tau)|d\tau\right]\\
&\le ||u-v||_C \max_{t\in J}\left[\lambda\int_0^T ds + \frac{1}{\Gamma(\alpha)}[M+N]\int_0^t (t-s)^{\alpha-1}ds\right.\\
&\quad \left. + \frac{LK_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s\ ds\right]\\
&\le ||u-v||_C\left[\lambda T + \frac{1}{\Gamma(\alpha)}[M+N]\frac{T^\alpha}{\alpha} + \frac{LK_T}{\Gamma(\alpha)}\frac{T^{\alpha+1}}{(\alpha+1)\alpha}\right]\\
&= ||u-v||_C\left[\lambda T + [M+N]\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{LK_T T^{\alpha+1}}{\Gamma(\alpha+2)}\right]
\end{aligned}
$$

This proves that T is a contraction map. Then by the Banach fixed point theorem, the operator T has a unique fixed point which implies that the problem (1.1) has unique solution. This completes the proof. ■

Corollary 3.2. *Let M, N, L be constants, $\sigma \in C_{1-\alpha}(J,\mathbb{R})$. Then the linear problem*

$$
\begin{cases}
D^\alpha_{0+}u(t) + Mu(t) + Nu(\theta(t)) + L\displaystyle\int_0^t K(t,s)u(s)ds = \sigma(t),\ 0<\alpha<1,\ t\in J,\\
u(0) = \lambda\displaystyle\int_0^T u(s)ds + d, d\in\mathbb{R}
\end{cases}
\tag{3.1}
$$

has a unique solution.

Proof. The proof follows from the Theorem 3.1. ∎

4. Monotone Iterative Method

In this section, we prove the existence and uniqueness of solution for the problem (1.1) by using monotone iterative method together with the method of upper and lower solutions.
Following theorem is a comparison result which will be used in proving more results.

Lemma 4.1. *Let $\theta \in C(J, J)$ where $\theta(t) \leq t$ on J. Assume that $p \in C_{1-\alpha}(J, \mathbb{R})$ satisfies the inequalities*

$$\begin{cases} D^{\alpha}_{0+}p(t) \leq -Mp(t) - Np(\theta(t)) - L\displaystyle\int_0^t K(t,s)p(s)ds \equiv Fp(t), \\ p(0) \leq 0, \end{cases} \tag{4.1}$$

where $M,\ N,\ L$ are constants. If

$$-(1+T^{\alpha})[M+N] < \Gamma(\alpha+1) \tag{4.2}$$

then $p(t) \leq 0$ for all $t \in J$.

Proof. Define a function $p_\epsilon(t) = p(t) - \epsilon(1+t^{\alpha}),\ \epsilon > 0$. Then

$$\begin{aligned} D^{\alpha}_{0+}p_\epsilon(t) &= D^{\alpha}_{0+}p(t) - D^{\alpha}_{0+}\epsilon(1+t^{\alpha}) \\ &\leq Fp(t) - \frac{\epsilon}{t^{\alpha}\Gamma(\alpha-1)} - \epsilon\Gamma(\alpha+1) \\ &= F[p_\epsilon(t) + \epsilon(1+t^{\alpha})] - \epsilon\left[\frac{1}{t^{\alpha}\Gamma(\alpha-1)} + \Gamma(\alpha+1)\right] \\ &= Fp_\epsilon(t) + \epsilon\Bigg[-M(1+t^{\alpha}) - N(1+t^{\alpha}) - L\int_0^t K(t,s)(1+s^{\alpha})ds \\ &\quad - \frac{1}{t^{\alpha}\Gamma(\alpha-1)} - \Gamma(\alpha+1)\Bigg] \\ &< Fp_\epsilon(t) \end{aligned}$$

and $p_\epsilon(0) = p(0) - \epsilon < 0$.
We prove that $p_\epsilon(t) < 0$ on J. On the contrary, assume that $p_\epsilon(t) \not< 0$ on J. Hence, there exists a $t_1 \in (0, T]$ such that $p_\epsilon(t_1) = 0$ and $p_\epsilon(t) < 0,\ t \in (0, t_1)$. Then by the Lemma 2.4, we must have $D^{\alpha}_{0+}p_\epsilon(t_1) \geq 0$. Further, as $p_\epsilon(t_1) = 0$, it follows that

$$0 < Fp_\epsilon(t_1) = -Np_\epsilon(\theta(t_1)) - L\int_0^{t_1} K(t_1,s)p_\epsilon(s)ds \tag{4.3}$$

If $N = 0, L = 0$, we get $0 < 0$ which is absurd.
If $N = 0, L < 0$, then we must have $K(t_1, s) < 0$ which is not possible.
If $N < 0, L = 0$, then we must have $p_\epsilon(\theta(t_1)) > 0$ which is impossible.
If $N < 0, L < 0$, then we get the right hand side of equation (4.3) negative, so again a contradiction. Hence, we must have $p_\epsilon(t) < 0$ on J. Then $p(t) - \epsilon(1+t^{\alpha}) < 0$ on J. On taking $\epsilon \to 0$, we get $p(t) \leq 0$ on J. ∎

Definition 4.2. A pair of functions $[u_0, w_0]$ in $C_{1-\alpha}(J, \mathbb{R})$ are called lower and upper solutions of the problem (1.1) if

$$D^{\alpha}_{0+}v_0(t) \leq f\left(t, v_0(t), v_0(\theta(t)), \int_0^t K(t,s)v_0(s)ds\right), \ v_0(0) \leq \int_0^T v_0(s)ds + d \tag{4.4}$$

$$D^{\alpha}_{0+}w_0(t) \geq f\left(t, w_0(t), w_0(\theta(t)), \int_0^t K(t,s)w_0(s)ds\right), \ w_0(0) \geq \int_0^T w_0(s)ds + d \tag{4.5}$$

Theorem 4.3. *Suppose that*

(1) $f \in C(J \times \mathbb{R}^3, \mathbb{R}),\ \theta(t) \in C(J, J),\ \theta(t) \leq t,\ t \in J$,
(2) functions v_0 and w_0 in $C_{1-\alpha}(J, \mathbb{R})$ are lower and upper solutions of the problem (1.1) such that $v_0(t) \leq w_0(t)$ on J,
(3) there exists nonnegative constants $M,\ N,\ L$ such that the function f satisfies the condition

$$f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3) \geq -M(u_1 - v_1) - N(u_2 - v_2) - L(u_3 - v_3), \tag{4.6}$$

for

$$v_0(t) \leq v_1 \leq u_1 \leq w_0(t),\ v_0(\theta(t)) \leq v_2 \leq u_2 \leq w_0(\theta(t)),$$

$$\int_0^t K(t,s)v_0(s)ds \leq v_3 \leq u_3 \leq \int_0^t K(t,s)w_0(s)ds$$

then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J, \mathbb{R})$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$ for all $t \in J$, where u and w are minimal and maximal solutions of the problem (1.1) respectively and $v(t) \leq u(t) \leq w(t)$ on J.

Proof. For any $\eta \in C_{1-\alpha}(J, \mathbb{R})$ such that $\eta \in [v_0, w_0]$, we consider the following linear problem :

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\left(t, \eta(t), \eta(\theta(t)), \displaystyle\int_0^t K(t,s)\eta(s)ds\right) \\ \qquad\qquad + M[\eta(t) - u(t)] + N[\eta(\theta(t)) - u(\theta(t))] \\ \qquad\qquad + L\left[\displaystyle\int_0^t K(t,s)\eta(s)ds - \int_0^t K(t,s)u(s)ds\right] \\ \quad u(0) = \displaystyle\int_0^T u(s)ds + d \end{cases} \tag{4.7}$$

By the Corollary 3.2, the linear problem (4.7), has a unique solution $u(t)$.
Now we define the iterates as follows and construct the sequences $\{v_n\}$, $\{u_n\}$ by

$$\begin{cases} D^{\alpha}_{0+}v_{n+1}(t) = f\left(t, v_n(t), v_n(\theta(t)), \int_0^t K(t,s)v_n(s)ds\right) \\ \qquad\qquad - M[v_{n+1}(t) - v_n(t)] - N[v_{n+1}(\theta(t)) - v_n(\theta(t))] \\ \qquad\qquad - L\left[\int_0^t K(t,s)v_{n+1}(s)ds - \int_0^t K(t,s)v_n(s)ds\right] \\ v_{n+1}(0) = \int_0^T v_n(s)ds + d \end{cases} \tag{4.8}$$

$$\begin{cases} D^{\alpha}_{0+}w_{n+1}(t) = f\left(t, w_n(t), w_n(\theta(t)), \int_0^t K(t,s)w_n(s)ds\right) \\ \qquad\qquad - M[w_{n+1}(t) - w_n(t)] - N[w_{n+1}(\theta(t)) - w_n(\theta(t))] \\ \qquad\qquad - L\left[\int_0^t K(t,s)w_{n+1}(s)ds - \int_0^t K(t,s)w_n(s)ds\right] \\ w_{n+1}(0) = \int_0^T w_n(s)ds + d \end{cases} \tag{4.9}$$

The existence of the problems (4.8) and (4.9) respectively follows from the arguments made above. For $n = 0$ in the problems (4.8) and (4.9), we get the existence of solutions v_1 and w_1 respectively. Now, we prove that $v_0(t) \le v_1(t) \le w_1(t) \le w_0(t)$.
Define $p(t) = v_1(t) - v_0(t)$. Since $v_0(t)$ is the lower solution of the problem (4.8), we have

$$\begin{aligned} D^{\alpha}_{0+}p(t) &= D^{\alpha}_{0+}v_1(t) - D^{\alpha}_{0+}v_0(t) \\ &\ge f\left(t, v_0(t), v_0(\theta(t)), \int_0^t K(t,s)v_0(s)ds\right) \\ &- M[v_1(t) - v_0(t)] - N[v_1(\theta(t)) - v_0(\theta(t))] \\ &- L\left[\int_0^t K(t,s)v_1(s)ds - \int_0^t K(t,s)v_0(s)ds\right] \\ &- f\left(t, v_0(t), v_0(\theta(t)), \int_0^t K(t,s)v_0(s)ds\right) \\ &= -Mp(t) - Np(\theta(t)) - L\int_0^t K(t,s)p(s)ds \end{aligned}$$

and $p(0) = v_1(0) - v_0(0) \ge \int_0^T v_0(s)ds + d - \int_0^T v_0(s)ds - d = 0$. From Lemma 4.1, we have $p(t) \ge 0$ which implies that $v_1(t) \ge v_0(t)$ on J. Similarly, one can prove that $v_1(t) \le w_1(t)$ and $w_1(t) \le w_0(t)$ on J. Therefore, we obtain $v_0(t) \le v_1(t) \le w_1(t) \le w_0(t)$.
Assume that the result is true for $k > 1$ i.e., $v_{k-1}(t) \le v_k(t) \le w_k(t) \le w_{k-1}(t)$ on J.
Claim : $v_k(t) \le v_{k+1}(t) \le w_{k+1}(t) \le w_k(t)$ on J.

Define $p(t) = v_{k+1} - v_k(t)$. Then

$$\begin{aligned}
D^{\alpha}_{0+}p(t) &= D^{\alpha}_{0+}v_{k+1}(t) - D^{\alpha}_{0+}v_k(t) \\
&\geq f\left(t, v_k(t), v_K(\theta(t)), \int_0^t K(t,s)v_k(s)ds\right) \\
&- M[v_{k+1}(t) - v_k(t)] - N[v_{k+1}(\theta(t)) - v_k(\theta(t))] \\
&- L\left[\int_0^t K(t,s)v_{k+1}(s)ds - \int_0^t K(t,s)v_k(s)ds\right] \\
&- f\left(t, v_{k-1}(t), v_{k-1}(\theta(t)), \int_0^t K(t,s)v_{k-1}(s)ds\right) \\
&+ M[v_k(t) - v_{k-1}(t)] + N[v_k(\theta(t)) - v_{k-1}(\theta(t))] \\
&+ L\left[\int_0^t K(t,s)v_k(s)ds - \int_0^t K(t,s)v_{k-1}(s)ds\right] \\
&\geq -Mp(t) - Np(\theta(t)) - L\int_0^t K(t,s)p(s)ds
\end{aligned}$$

$$\begin{aligned}
p(0) = v_{k+1}(0) - v_k(0) &= \int_0^T v_k(s)ds + d - \int_0^T v_{k-1}(s)ds - d \\
&\geq \int_0^T [v_k(s) - v_k(s)]ds = 0.
\end{aligned}$$

By the Lemma 4.1, we obtain $p(t) \geq 0$, implying that $v_{k+1}(t) \geq v_k(t)$ for all k on J. Similarly, we can prove that $v_{k+1}(t) \leq w_{k+1}(t)$ and $w_{k+1}(t) \leq w_k(t)$ for all t on J. Hence, by the principle of mathematical induction, we have

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq w_k \leq \cdots \leq w_2 \leq w_1 \leq w_0$$

on J.

Therefore, the sequences $\{v_n\}$ and $\{w_n\}$ are monotonic and uniformly bounded. Further, also observe that $\{D^{\alpha}_{0+}v_n\}$ and $\{D^{\alpha}_{0+}w_n\}$ are also uniformly bounded on J in view of relations (4.8) and (4.9). By applying the Lemma 2.4. we can conclude that the sequences $\{v_n\}$ and $\{w_n\}$ are equicontinuous. Therefore, by the Arzela-Ascoli theorem the sequences $\{v_n\}$ and $\{w_n\}$ must converge uniformly to v and w on J respectively.

Now, we prove that v and w are the minimal and maximal solutions of the problem (1.1). Let u be any solution of (1.1) different from v and w. So there exists a positive integer k such that $v_k(t) \leq u(t) \leq w_k(t)$ on J. Define $p(t) = u(t) - v_{k+1}(t)$. Then we have,

$$\begin{aligned}
D_{0+}^{\alpha} p(t) &= D_{0+}^{\alpha} u(t) - D_{0+}^{\alpha} v_{k+1}(t) \\
&\geq f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right) \\
&\quad - f\left(t, v_0(t), v_0(\theta(t)), \int_0^t K(t,s)v_0(s)ds\right) \\
&\quad + M[v_{k+1}(t) - v_k(t)] + N[v_{k+1}(\theta(t)) - v_k(\theta(t))] \\
&\quad + L\left[\int_0^t K(t,s)v_{k+1}(s)ds - \int_0^t K(t,s)v_k(s)ds\right] \\
&\geq -M[u(t) - v_{k+1}(t)] - N[u(\theta(t)) - v_{k+1}(\theta(t))] \\
&\quad - L\left[\int_0^t K(t,s)u(s)ds - \int_0^t K(t,s)v_{k+1}(s)ds\right] \\
&\geq -Mp(t) - Np(\theta(t)) - L\left[\int_0^t K(t,s)p(s)ds\right]
\end{aligned}$$

and

$$p(0) = u(0) - v_{k+1}(0) = \int_0^T [u(s) - v_k(s)]ds \geq 0$$

By the Lemma 4.1, we obtain $p(t) \geq 0$, implying that $u(t) \geq v_{k+1}(t)$ for all k on J. Similarly, we can prove that $u(t) \leq w_{k+1}(t)$ for all k on J. Since, $v_0(t) \leq u(t) \leq u_0(t)$ on J. By induction, it follows that $v_k(t) \leq u(t)$ and $u(t) \leq w_k(t)$ for all k. Therefore, $v_k(t) \leq u(t) \leq w_k(t)$ on J. On taking limit as $k \to \infty$, we obtain $v(t) \leq u(t) \leq w(t)$ on J. Hence, the functions $v(t), w(t)$ are the minimal and maximal solutions of the problem (1.1). This completes the proof. ■

Theorem 4.4. *Assume that*

(1) All the conditions of the Theorem 4.3 hold,
(2) there exists nonnegative constants M, N, L such that the function f satisfies the condition

$$f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3) \leq M(u_1 - v_1) + N(u_2 - v_2) + L(u_3 - v_3), \quad (4.10)$$

for

$$v_0(t) \leq v_1 \leq u_1 \leq w_0(t), v_0(\theta(t)) \leq v_2 \leq u_2 \leq w_0(\theta(t)),$$

$$\int_0^t K(t,s)v_0(s)ds \leq v_3 \leq u_3 \leq \int_0^t K(t,s)w_0(s)ds$$

Then the problem (1.1) has a unique solution.

Proof. We know that $v(t) \leq w(t)$ on J. It is sufficient to prove that $v(t) \geq w(t)$ on J. Consider $p(t) = w(t) - v(t)$. Then consider

$$\begin{aligned} &D^{\alpha}_{0+}p(t) \\ &= D^{\alpha}_{0+}w(t) - D^{\alpha}_{0+}u(t) \\ &= f\left(t, w(t), w(\theta(t)), \int_0^t K(t,s)w(s)ds\right) - f\left(t, u(t), u(\theta(t)), \int_0^t K(t,s)u(s)ds\right) \\ &\leq -M[v(t) - w(t)] - N[v(\theta(t)) - w(\theta(t))] \\ &\qquad - L\left[\int_0^t K(t,s)v(s)ds - \int_0^t K(t,s)w(s)ds\right] \\ &= -M'p(t) - N'p(\theta(t)) - L'\left[\int_0^t K(t,s)p(s)ds\right] \end{aligned}$$

where $M' = -M, N' = -N, L' = -L$. Also, consider

$$p(0) = \int_0^T w(s)ds + d - \int_0^T v(s)ds - d = \int_0^T [w(s) - v(s)]ds = \int_0^T p(s)ds \leq 0.$$

Hence, by the Lemma 4.1, we must have $p(t) \leq 0.$ on J. i.e. $w(t) - v(t) \leq 0$, on J, i.e.$w(t) \leq v(t)$ on J. Hence, we get $v(t) = w(t)$ on J. Hence, the problem (1.1) has a unique solution. ■

Acknowledgements

The authors are grateful to the referee for his/her valuable suggestions to improve the paper.

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