Thai Journal of Mathematics
Volume 20 Number 4 (2022)
Pages 1707–1719

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Rough Statistical Convergence in Probabilistic Normed Spaces

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Abstract The main purpose of this work is to define rough statistical convergence in probabilistic normed spaces. We have proved some basic properties as well as some examples which shows this idea of convergence in probabilistic normed spaces is more generalized as compared to the rough statistical convergence in normed linear spaces. Further, we have shown the results on sets of statistical limit points and sets of cluster points of rough statistically convergent sequences in these spaces.

MSC: 40A05; 26E50; 40G99

Keywords: statistical convergence; rough statistical convergence; probablistic normed space

Submission date: 17.03.2020 / Acceptance date: 25.08.2020

1. Introduction

In 1951, Fast[1] presented a new idea of convergence named as statistical convergence that is more generalized than the usual convergence for the sequences.

Definition 1.1. [1] A sequence $x = \{x_k\}$ of numbers is said to be statistically convergent to ξ if for every $\epsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n}|M(x,\epsilon)| = \delta(M(x,\epsilon)) = 0$, where $|M(x,\epsilon)|$ represents the order of the enclosed set $M(x,\epsilon) = \{k \leq n : |x_k - \xi| \geq \epsilon\}$.

An interesting generalization of usual convergence named as rough convergence was initially introduced by Phu[2] for the sequences on finite dimensional normed linear spaces and later on introduced on infinite dimensional normed linear spaces[3]. He mainly worked on rough limits, roughness degree, rough continuity of linear operators and also introduced rough Cauchy sequences.

Definition 1.2. [2] A sequence $x = \{x_k\}$ in a normed linear space $(\mathbb{X}, \|.\|)$ is said to be rough convergent to $\xi \in \mathbb{X}$ for some non-negative number r if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\|x_k - \xi\| < r + \epsilon$ for all $k \geq k_0$.

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Aytar[4] extended the rough convergence to rough statistical convergence like usual convergence is extended to statistical convergence with the help of natural density.

Definition 1.3. [4] A sequence $x = \{x_k\}$ in a normed linear space $(\mathbb{X}, \|.\|)$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ for some non-negative number r if for every $\epsilon > 0$ we have

$$\delta(\{k \in \mathbb{N} : \|x_k - \xi\| \geq r + \epsilon\}) = 0,$$

and ξ is known as r-St-limit of sequence $x = \{x_k\}$.

Aytar[5] also examined some criteria associated with the convexity and closeness of the set of rough statistical limit points and rough cluster points of a sequence.

Inspired by the work of Aytar[4], Maity[6] presented rough statistical convergence of order $\alpha (0 < \alpha \leq 1)$ in normed linear spaces and explained some important results for the set of rough statistical limit points of order α. The idea of pointwise rough statistical convergence and rough statistical Cauchy sequences for real valued functions was introduced in [7]. The rough convergence has been defined for double sequences in normed linear spaces by Malik and Maity in [8] and after that the authors extended this idea in [9] and defined rough statistical convergence for double sequences.

This idea has motivated many authors to use the concepts of ideals also. Pal et al.[10] introduced rough I-convergence with the help of ideals of $\mathbb{N}$. Later, Malik et al. in [11] extended this concept of rough I-convergence to rough I-statistical convergence and described some topological properties of the set of all rough I-statistical limits of sequences in normed linear spaces. More investigations, generalizations and applications of the rough convergence can be further revealed using statistical convergence as well as generalized statistical convergence in different settings [12–20].

In this paper, we are introducing the concept of rough statistical convergence in the probabilistic normed linear spaces. The probabilistic normed space is an important family of probabilistic metric spaces which was defined by Serstnev[21]. Further extensively studied and redefined by Schweizer, Sklar and Alsina [22–24]. The basic terms related to probabilistic normed spaces are elaborated as:

Definition 1.4. [25] A map $\phi : \mathbb{R} \to R_0^+$ is said to be the *distribution function* if it is non-decreasing and left continuous. Also $inf_x \phi(x) = 0$ and $sup_x \phi(x) = 1$ for $x \in \mathbb{R}$. The set of all distribution functions is represented by $\mathcal{F}$.

Example 1.5. Unit step function $\phi(x)$ is a distribution function, which is defined as

$$\phi(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Definition 1.6. [24] A t-norm is a map $* : [0,1] \times [0,1] \to [0,1]$ which is non-decreasing, continuous, commutative, associative and has 1 as identity.

Example 1.7. $x * y = min\{x, y\}$ and $x * y = max\{x + y - 1, 0\}$ on [0,1] are t-norms.

Definition 1.8. [25] Let $\mathbb{X}$ be a real linear space, $*$ be a t-norm and $\mathcal{F}$ be the collection of distribution functions. Consider a map $\wp : \mathbb{X} \to \mathcal{F}$ such that $\wp(x; t)$ is the value of $\wp(x)$ at $t \in \mathbb{R}$. If the following properties are satisfied, then $\wp$ and $(\mathbb{X}, \wp, *)$ are known as probabilistic norm and probabilistic normed space(PN-Space) respectively.

(1) $\wp(x; 0) = 0$,
(2) $\wp(x; t) = 1$ for all $t > 0$ if and only if $x = 0$,

(3) $\wp(x\alpha;t) = \wp\left(x;\frac{t}{|\alpha|}\right)$ where $\alpha \neq 0$ and $\alpha \in \mathbb{R}$,

(4) $\wp(x+y;s+t) \geq \wp(x;s) * \wp(y;t)$ for all $x,y \in \mathbb{X}$ and $s,t \in \mathbb{R}_0^+ = [\,0,\infty)$.

Example 1.9. For a real normed space $(\mathbb{X}, \|.\|)$, we define probabilistic norm $\wp$ for $x \in \mathbb{X}$ and $t \in \mathbb{R}$ as

$$\wp(x;t) = \begin{cases} \mu\left(\frac{t}{\|x\|}\right) & x \neq 0 \\ h_0(t) & x = 0 \end{cases}$$

where μ is a distribution function as $\mu(0) = 0$ and $\mu \neq h_0$. Also

$$h_0(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

Then $(\mathbb{X}, \wp, *)$ be a PN-Space under the t-norm $*$ which is defined as $x * y = \min\{x,y\}$. For example, define the function μ as

$$\mu(t) = \begin{cases} 0 & t \leq 0 \\ \frac{x}{1+x} & t > 0 \end{cases}$$

Then, we obtain probabilistic norm as

$$\wp(x;t) = \begin{cases} \frac{t}{t+\|x\|} & x \neq 0 \\ h_0(t) & x = 0 \end{cases}$$

Definition 1.10. [26] Let $(\mathbb{X}, \wp, *)$ be a PN-Space. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $k_0 \in \mathbb{N}$ such that

$$\wp(x_k - \xi; \epsilon) > 1 - \lambda \text{ for all } k \geq k_0.$$

It is denoted by $\wp - \lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{\wp} \xi$.

Example 1.11. [26] For a real normed space $(\mathbb{X}, \|.\|)$, we define the probabilistic norm $\wp$ for $x \in \mathbb{X}$, $t \in \mathbb{R}$ as $\wp(x;t) = \frac{t}{t+\|x\|}$. Then $(\mathbb{X}, \wp, *)$ be a PN-Space under the t-norm $*$ which is defined as $x * y = \min\{x,y\}$.
Also, $x_k \xrightarrow{\wp} x$ if and only if $x_k \xrightarrow{\|\cdot\|} x$.

Karakus[27] introduced statistical convergence of sequences in PN-space.

Definition 1.12. [27] Let $(\mathbb{X}, \wp, *)$ be a PN-Space. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$ we have

$$\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; \epsilon) \leq 1 - \lambda\}) = 0.$$

It is denoted by $St_\wp - \lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{St_\wp} \xi$.

2. Main Results

In this section, we first define the rough convergence and rough statistical convergence in probabilistic normed spaces as follows:

Definition 2.1. Let $(\mathbb{X}, \wp, *)$ be a PN-Space. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ if for every $\epsilon > 0$, $\lambda \in (0,1)$ and some non-negative number r there exists $k_0 \in \mathbb{N}$ such that

$$\wp(x_k - \xi; r + \epsilon) > 1 - \lambda \text{ for all } k \geq k_0.$$

It is denoted by $r_\wp - \lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r_\wp} \xi$.

Definition 2.2. Let $(\mathbb{X}, \wp, *)$ be a PN-Space. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ if for every $\epsilon > 0$, $\lambda \in (0,1)$ and some non-negative number r,

$$\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) \leq 1 - \lambda\}) = 0.$$

It is denoted by $r\text{-}St_\wp\text{-}\lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r\text{-}St_\wp} \xi$.

Remark 2.3. For the case $r = 0$, the notion rough statistical convergence with respect to the norm $\wp$ agrees with the statistical convergence with respect to the norm $\wp$.

The $r\text{-}St_\wp$-limit of a sequence may be not unique. So we consider $r\text{-}St_\wp$-limit set of a sequence $x = \{x_k\}$ as $St_\wp\text{-}LIM_x^r = \{\xi : x_k \xrightarrow{r\text{-}St_\wp} \xi\}$. The sequence $x = \{x_k\}$ is $r_\wp$-convergent if $LIM_x^{r_\wp} \neq \phi$ where $LIM_x^{r_\wp} = \{\xi^* \in \mathbb{X} : x_k \xrightarrow{r_\wp} \xi^*\}$. For unbounded sequence $LIM_x^{r_\wp}$ is always empty.
But in case of rough statistical convergence in PN-Space $(\mathbb{X}, \wp, *)$, we have $St_\wp\text{-}LIM_x^r \neq \phi$ even though sequence may be unbounded. For this we have given the next example.

Example 2.4. For a real normed space $(\mathbb{X}, \|.\|)$, we define the probabilistic norm $\wp$ for $x \in \mathbb{X}$, $t \in \mathbb{R}$ as $\wp(x;t) = \frac{t}{t+\|x\|}$. Then $(\mathbb{X}, \wp, *)$ be a PN-Space under the t-norm $*$ which is defined as $x * y = \min\{x, y\}$. Then, define a sequence

$$x_k = \begin{cases} (-1)^k & k \neq n^2 \\ k & \text{otherwise} \end{cases}$$

Then

$$St_\wp\text{-}LIM_x^r = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise} \end{cases}$$

and $St_\wp\text{-}LIM_x^r = \phi$ for all $r \geq 0$. Thus, this sequence is divergent in ordinary sense as it is unbounded. Also, the sequence is not rough convergent in PN-Space for any r.

With the help of statistically cluster points defined by Fridy[28], we are giving following definition as follows:

Definition 2.5. Let $(\mathbb{X}, \wp, *)$ be a PN-Space. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough statistically bounded with respect to the norm $\wp$ for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists a real number $M > 0$ such that

$$\delta(\{k \in \mathbb{N} : \wp(x_k; M) \leq 1 - \lambda\}) = 0.$$

In view of above definitions, we obtained the following interesting results on rough statistical convergence in PN-Spaces.

Theorem 2.6. *A sequence $x = \{x_k\}$ is statistically bounded in a PN-Space $(\mathbb{X}, \wp, *)$ if and only if $St_\wp\text{-}LIM_x^r \neq \phi$ for some $r > 0$.*

Proof. Necessary part:
Let the sequence $x = \{x_k\}$ is statistically bounded in a PN-Space $(\mathbb{X}, \wp, *)$. Then, for every $\epsilon > 0$, $\lambda \in (0, 1)$ and some $r > 0$, there exists a real number $M > 0$ such that

$$\delta(\{k \in \mathbb{N} : \wp(x_k; M) \leq 1 - \lambda\}) = 0.$$

Let $K = \{k \in \mathbb{N} : \wp(x_k; M) \leq 1 - \lambda\}$. For $k \in K^c$ we have $\wp(x_k; M) > 1 - \lambda$.
Also

$$\begin{aligned} \wp(x_k; r + M) &\geq \wp(0; r) * \wp(x_k; M) \\ &> 1 * (1 - \lambda) \\ &= 1 - \lambda. \end{aligned}$$

Hence, $0 \in St_\wp\text{-}LIM_x^r$. Therefore, $St_\wp\text{-}LIM_x^r \neq \phi$.

Sufficient Part:
Let $St_\wp\text{-}LIM_x^r \neq \phi$ for some $r > 0$. Then there exists $\xi \in \mathbb{X}$ such that $\xi \in St_\wp\text{-}LIM_x^r$. For every $\epsilon > 0$ and $\lambda \in (0, 1)$ we have

$$\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) \leq 1 - \lambda\}) = 0.$$

Therefore, almost all x_k's are contained in some ball with center ξ which implies that sequence $x = \{x_k\}$ is statistically bounded in a PN-Space $(\mathbb{X}, \wp, *)$. ■

Next, we discuss the algebraic characterization of rough statistically convergent sequences in PN-Spaces.

Theorem 2.7. *Let $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences in a PN-Space $(\mathbb{X}, \wp, *)$. Then for some non-negative number r the following holds*

(1) If $x_k \xrightarrow{r\text{-}St_\wp} x_0$ and $\alpha \in \mathbb{N}$ then $\alpha x_k \xrightarrow{r\text{-}St_\wp} \alpha x_0$,
(2) If $x_k \xrightarrow{r\text{-}St_\wp} x_0$ and $y_k \xrightarrow{r\text{-}St_\wp} y_0$ then $(x_k + y_k) \xrightarrow{r\text{-}St_\wp} (x_0 + y_0)$.

Proof. Proof of above results are obvious so we are omitting them. ■

If $x' = \{x_{k_i}\}$ be a subsequence of $x = \{x_k\}$ in a PN-Space $(\mathbb{X}, \wp, *)$ then $LIM_{x_k}^{r_\wp} \subset LIM_{x_{k_i}}^{r_\wp}$. But this fact does not hold in case of statistical convergence. This can be justified by the following example.

Example 2.8. For real normed space $(\mathbb{X}, \|.\|)$, we define the probabilistic norm $\wp$ for $x \in \mathbb{X}$, $t \in \mathbb{R}$ as $\wp(x; t) = \frac{t}{t + \|x\|}$. Then $(\mathbb{X}, \wp, *)$ be a PN-Space under the t-norm $*$ which is defined by $x * y = \min\{x, y\}$. Then the sequence

$$x_k = \begin{cases} k & k \neq n^2 \\ 0 & \text{otherwise} \end{cases}$$

have $St_\wp\text{-}LIM_x^r = [-r, r]$. And its subsequence $x' = \{1, 4, 9, ...\}$ have $St_\wp\text{-}LIM_{x'}^r = \phi$.

But this fact is true for nonthin subsequences of the rough statistical convergent sequence in a PN-Space which is explained by the next result.

Theorem 2.9. *If $x' = \{x_{k_i}\}$ be a nonthin subsequence of $x = \{x_k\}$ in a PN-Space $(\mathbb{X}, \wp, *)$ then $St_\wp\text{-}LIM_x^r \subset St_\wp\text{-}LIM_{x'}^r$.*

Proof. Proof of above result is obvious so we are omitting it. ■

Theorem 2.10. *The set $St_\wp\text{-}LIM_x^r$ of a sequence $x = \{x_k\}$ in a PN-Space $(\mathbb{X}, \wp, *)$ is a closed set.*

Proof. We have nothing to prove as $St_{\wp}\text{-}LIM_x^r = \phi$.
Let $St_{\wp}\text{-}LIM_x^r \neq \phi$ for some $r > 0$ and consider $y = \{y_k\}$ be a convergent sequence in $St_{\wp}\text{-}LIM_x^r$ with respect to the norm $\wp$ to $y_0 \in \mathbb{X}$.
For $t \in (0,1)$ choose $\lambda \in (0,1)$ such that $(1-\lambda) * (1-\lambda) > 1-t$. Then for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists a $k_1 \in \mathbb{N}$ such that

$$\wp\left(y_k - y_0; \frac{\epsilon}{2}\right) > 1-\lambda \text{ for all } \ k \geq k_1.$$

Let us choose $y_m \in St_{\wp}\text{-}LIM_x^r$ with $m > k_1$ such that

$$\delta\left(\{k \in \mathbb{N} : \wp\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1-\lambda\}\right) = 0. \tag{2.1}$$

For $j \in \{k \in \mathbb{N} : \wp\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda\}$ we have $\wp\left(x_j - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda$. Then, we have

$$\begin{aligned}\wp(x_j - y_0; r+\epsilon) &\geq \wp\left(x_j - y_m; r + \frac{\epsilon}{2}\right) * \wp\left(y_m - y_0; \frac{\epsilon}{2}\right)\\ &> (1-\lambda) * (1-\lambda)\\ &> 1-t.\end{aligned}$$

Hence, $j \in \{k \in \mathbb{N} : \wp(x_k - y_0; r+\epsilon) > 1-t\}$. Now we have the following inclusion

$$\{k \in \mathbb{N} : \wp\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda\} \subseteq \{k \in \mathbb{N} : \wp(x_k - y_0; r+\epsilon) > 1-t\}$$

i.e.

$$\{k \in \mathbb{N} : \wp(x_k - y_0; r+\epsilon) \leq 1-t\} \subseteq \{k \in \mathbb{N} : \wp\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1-\lambda\}$$

Then

$$\delta(\{k \in \mathbb{N} : \wp(x_k - y_0; r+\epsilon) \leq 1-t\}) \leq \delta\left(\{k \in \mathbb{N} : \wp\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1-\lambda\}\right)$$

Using (2.1) we get

$$\delta(\{k \in \mathbb{N} : \wp(x_k - y_0; r+\epsilon) \leq 1-t\}) = 0$$

Therefore, $y_0 \in St_{\wp}\text{-}LIM_x^r$. ∎

In next result, we are proving the convexity of the set $St_{\wp}\text{-}LIM_x^r$.

Theorem 2.11. *Let $x = \{x_k\}$ be a sequence in a PN-Space $(\mathbb{X}, \wp, *)$. Then, rough statistical limit set $St_{\wp}\text{-}LIM_x^r$ with respect to the norm $\wp$ is convex for some non-negative number r.*

Proof. Let $\xi_1, \xi_2 \in St_{\wp}\text{-}LIM_x^r$. For the convexity of the set $St_{\wp}\text{-}LIM_x^r$, we have to show that $[(1-\beta)\xi_1 + \beta\xi_2] \in St_{\wp}\text{-}LIM_x^r$ for some $\beta \in (0,1)$.
For $t \in (0,1)$ choose $\lambda \in (0,1)$ such that $(1-\lambda) * (1-\lambda) > 1-t$. Now for every $\epsilon > 0$ and $\lambda \in (0,1)$, we define

$$M_1 = \{k \in \mathbb{N} : \wp\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \leq 1-\lambda\},$$

$$M_2 = \{k \in \mathbb{N} : \wp\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \leq 1-\lambda\}.$$

As $\xi_1, \xi_2 \in St_{\wp}\text{-}LIM_x^r$, we have $\delta(M_1) = \delta(M_2) = 0$. For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}\wp(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) &\geq \wp((1-\beta)(x_k - \xi_1) + \beta(x_k - \xi_2); r+\epsilon) \\ &\geq \wp\left((1-\beta)(x_k-\xi_1); \frac{r+\epsilon}{2}\right) * \wp\left(\beta(x_k-\xi_2); \frac{r+\epsilon}{2}\right) \\ &\geq \wp\left(x_k-\xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) * \wp\left(x_k-\xi_2; \frac{r+\epsilon}{2\beta}\right) \\ &> (1-\lambda) * (1-\lambda) \\ &> 1-t.\end{aligned}$$

Thus, $\delta(\{k \in \mathbb{N} : \wp(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) \leq 1-t\}) = 0$.
Hence, $[(1-\beta)\xi_1 + \beta\xi_2] \in St_{\wp}\text{-}LIM_x^r$ *i.e.* $St_{\wp}\text{-}LIM_x^r$ is a convex set. ■

Theorem 2.12. *A sequence $x = \{x_k\}$ in a PN-Space $(\mathbb{X}, \wp, *)$ is rough statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ for some non-negative number r if there exists a sequence $y = \{y_k\}$ in $\mathbb{X}$, which is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ and for every $\lambda \in (0,1)$ have $\wp(x_k - y_k; r) > 1-\lambda$ for all $k \in \mathbb{N}$.*

Proof. Let $\epsilon > 0$ and $\lambda \in (0,1)$. Consider $y_k \xrightarrow{St_{\wp}} \xi$ and $\wp(x_k - y_k; r) > 1-\lambda$ for all $k \in \mathbb{N}$. For given $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t)*(1-t) > 1-\lambda$. Define

$$A = \{k \in \mathbb{N} : \wp(y_k - \xi; \epsilon) \leq 1-t\}$$
$$B = \{k \in \mathbb{N} : \wp(x_k - y_k; r) \leq 1-t\}$$

Clearly, $\delta(A) = 0$ and $\delta(B) = 0$. For $k \in A^c \cap B^c$ we have

$$\begin{aligned}\wp(x_k - \xi; r+\epsilon) &\geq \wp(x_k - y_k; r) * \wp(y_k - \xi; \epsilon) \\ &> (1-t)*(1-t) \\ &> 1-\lambda.\end{aligned}$$

Then $\wp(x_k - \xi; r+\epsilon) > 1-\lambda$ for all $k \in A^c \cap B^c$.
This implies that $\{k \in \mathbb{N} : \wp(x_k - \xi; r+\epsilon) \leq 1-\lambda\} \subseteq A \cup B$.
Then, $\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r+\epsilon) \leq 1-\lambda\}) \leq \delta(A) + \delta(B)$.
Hence, we get $\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r+\epsilon) \leq 1-\lambda\}) = 0$. Therefore, $x_k \xrightarrow{r\text{-}St_{\wp}} \xi$. ■

Theorem 2.13. *Let $x = \{x_k\}$ be a sequence in a PN-Space $(\mathbb{X}, \wp, *)$ then there does not exist elements $y, z \in St_{\wp}\text{-}LIM_x^r$ for some $r > 0$ and every $\lambda \in (0,1)$ such that $\wp(y - z; mr) \leq 1-\lambda$ for $m > 2$.*

Proof. We prove this result by contradiction. Assume there exists elements $y, z \in St_{\wp}\text{-}LIM_x^r$ such that

$$\wp(y - z; mr) \leq 1-\lambda \text{ for } m > 2 \tag{2.2}$$

As $y, z \in St_{\wp}\text{-}LIM_x^r$.
For given $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t)*(1-t) > 1-\lambda$. Then for every $\epsilon > 0$ and $t \in (0,1)$ we have $\delta(K_1) = \delta(K_2) = 0$ where $K_1 = \{k \in \mathbb{N} : \wp\left(x_k - y; r + \frac{\epsilon}{2}\right) > 1-t\}$ and $K_2 = \{k \in \mathbb{N} : \wp\left(x_k - z; r + \frac{\epsilon}{2}\right) > 1-t\}$. For $k \in K_1^c \cap K_2^c$ we have

$$\begin{aligned}\wp(y - z; 2r+\epsilon) &\geq \wp\left(x_k - z; r + \frac{\epsilon}{2}\right) * \wp\left(x_k - y; r + \frac{\epsilon}{2}\right) \\ &> (1-t)*(1-t) \\ &> 1-\lambda.\end{aligned}$$

Hence,

$$\wp(y-z;2r+\epsilon)>1-\lambda. \tag{2.3}$$

Then, from (2.3) we have

$$\wp(y-z;mr)>1-\lambda \text{ for } m>2.$$

which is a contradiction to (2.2). Therefore, there does not exists elements $y,z\in St_{\wp}\text{-}LIM_x^r$ such that $\wp(y-z;mr)\le 1-\lambda$ for $m>2$. ∎

Next, we define statistical cluster point of a sequence in PN-Space and establish some results related to it.

Definition 2.14. Let $(\mathbb{X},\wp,*)$ be a PN-Space. Then $\gamma\in\mathbb{X}$ is called rough statistical cluster point of the sequence $x=\{x_k\}$ in $\mathbb{X}$ with respect to the norm $\wp$ if for every $\epsilon>0,\ \lambda\in(0,1)$ and some non-negative number r,

$$\delta(\{k\in\mathbb{N}:\wp(x_k-\gamma;r+\epsilon)>1-\lambda\})>0,$$

i.e.

$$\delta(\{k\in\mathbb{N}:\wp(x_k-\gamma;r+\epsilon)>1-\lambda\})\neq 0.$$

In this case, γ is known as r-$St_{\wp}$-cluster point of a sequence $x=\{x_k\}$.

Let $\Gamma_{\wp}^r(x)$ denotes the set of all r-$St_{\wp}$-cluster points of a sequence $x=\{x_k\}$. If $r=0$ then we get ordinary statistical cluster point defined by Karakus[27] *i.e.* $\Gamma_{\wp}^r(x)=\Gamma_{\wp}(x)$.

Theorem 2.15. *Let $(\mathbb{X},\wp,*)$ be a PN-Space. Then, set $\Gamma_{\wp}^r(x)$ of any sequence $x=\{x_k\}$ is closed for some non-negative real number r.*

Proof. (i) If $\Gamma_{\wp}^r(x)=\phi$, then we have to prove nothing.
(ii) If $\Gamma_{\wp}^r(x)\neq\phi$. Then, take a sequence $y=\{y_k\}\subseteq\Gamma_{\wp}^r(x)$ such that $y_k\xrightarrow{\wp}y_*$. It is sufficient to show that $y_*\in\Gamma_{\wp}^r(x)$. For $t\in(0,1)$ choose $\lambda\in(0,1)$ such that $(1-\lambda)*(1-\lambda)>(1-t)$
As $y_k\xrightarrow{\wp}y_*$, then for every $\epsilon>0$ and $\lambda\in(0,1)$ there exists $k_\epsilon\in\mathbb{N}$ such that $\wp(y_k-y_*;\frac{\epsilon}{2})>1-\lambda$ for $k\ge k_\epsilon$.
Now choose $k_0\in\mathbb{N}$ such that $k_0\ge k_\epsilon$. Then, we have $\wp(y_{k_0}-y_*;\frac{\epsilon}{2})>1-\lambda$. Again as $y=\{y_k\}\subseteq\Gamma_{\wp}^r(x)$, we have $y_{k_0}\in\Gamma_{\wp}^r(x)$.

$$\Rightarrow\delta\left(\left\{k\in\mathbb{N}:\wp\left(x_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\lambda\right\}\right)>0 \tag{2.4}$$

Choose $j\in\{k\in\mathbb{N}:\wp\left(x_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\lambda\}$, then we have $\wp\left(x_j-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\lambda$.

$$\begin{aligned}\wp(x_j-y_*;r+\epsilon)&\ge\wp\left(x_j-y_{k_0};r+\frac{\epsilon}{2}\right)*\wp\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)\\&>(1-\lambda)*(1-\lambda)\\&>1-t.\end{aligned}$$

Thus, $j\in\{k\in\mathbb{N}:\wp(x_k-y_*;r+\epsilon)>1-t\}$.
Hence

$$\{k\in\mathbb{N}:\wp\left(x_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\lambda\}\subseteq\{k\in\mathbb{N}:\wp(x_k-y_*;r+\epsilon)>1-t\}.$$

Now,

$$\delta(\{k\in\mathbb{N}:\wp\left(x_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\lambda\})\le\delta(\{k\in\mathbb{N}:\wp(x_k-y_*;r+\epsilon)>1-t\}).$$

(2.5)

Using equation (2.4), we obtained that the set on left side of (2.5) has natural density more than 0.

$$\Rightarrow \delta(\{k \in \mathbb{N} : \wp(x_k - y_*; r + \epsilon) > 1 - \lambda\}) > 0.$$

Therefore, $y_* \in \Gamma_\wp^r(x)$. ■

Theorem 2.16. *Let $\Gamma_\wp(x)$ be the set of all statistical cluster points of a sequence $x = \{x_k\}$ in a PN-Space $(\mathbb{X}, \wp, *)$ and r be some non-negative real number. Then, for an arbitrary $\gamma \in \Gamma_\wp(x)$ and $\lambda \in (0,1)$ we have $\wp(\xi - \gamma; r) > 1 - \lambda$ for all $\xi \in \Gamma_\wp^r(x)$.*

Proof. For $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t) * (1-t) > 1 - \lambda$. Let $\gamma \in \Gamma_\wp(x)$. Then, for every $\epsilon > 0$ and $t \in (0,1)$ we have

$$\delta(\{k \in \mathbb{N} : \wp(x_k - \gamma; \epsilon) > 1 - t\}) > 0. \tag{2.6}$$

Now we will show that if for $\xi \in \mathbb{X}$ we have $\wp(\xi - \gamma; r) > 1 - t$ then $\xi \in \Gamma_\wp^r(x)$. Let $j \in \{k \in \mathbb{N} : \wp(x_k - \gamma; \epsilon) > 1 - t\}$ then $\wp(x_j - \gamma; \epsilon) > 1 - t$. Now,

$$\begin{aligned} \wp(x_j - \xi; r + \epsilon) &\geq \wp(x_j - \gamma; \epsilon) * \wp(\xi - \gamma; r) \\ &> (1-t) * (1-t) \\ &> 1 - \lambda. \end{aligned}$$

we have $\wp(x_j - \xi; r + \epsilon) > 1 - \lambda$. Thus $j \in \{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) > 1 - \lambda\}$. Now the next inclusion holds.

$$\{k \in \mathbb{N} : \wp(x_k - \gamma; \epsilon) > 1 - t\} \subseteq \{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) > 1 - \lambda\}.$$

Then

$$\delta(\{k \in \mathbb{N} : \wp(x_k - \gamma; \epsilon) > 1 - t\}) \leq \delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) > 1 - \lambda\})$$

Using equation (2.6) we get $\delta(\{k \in \mathbb{N} : \wp(x_k - \xi; r + \epsilon) > 1 - \lambda\}) > 0$. Therefore, $\xi \in \Gamma_\wp^r(x)$. ■

Theorem 2.17. *If $\overline{B(c, \lambda, r)} = \{x \in \mathbb{X} : \wp(x - c; r) \geq 1 - \lambda\}$ represents the closure of open ball $B(c, \lambda, r) = \{x \in \mathbb{X} : \wp(x - c; r) > 1 - \lambda\}$ for some $r > 0$, $\lambda \in (0,1)$ and fixed $c \in \mathbb{X}$ then $\Gamma_\wp^r(x) = \bigcup_{c \in \Gamma_\wp(x)} \overline{B(c, \lambda, r)}$.*

Proof. For $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t) * (1-t) > 1 - \lambda$. Let $\gamma \in \bigcup_{c \in \Gamma_\wp(x)} \overline{B(c, \lambda, r)}$ then there exists $c \in \Gamma_\wp(x)$ for some $r > 0$ and every $t \in (0,1)$ such that

$$\wp(c - \gamma; r) > 1 - t.$$

Fix $\epsilon > 0$. Since $c \in \Gamma_\wp(x)$ then there exists a set $K = \{k \in \mathbb{X} : \wp(x_k - c; \epsilon) > 1 - t\}$ with $\delta(K) > 0$. Now, for $k \in K$

$$\begin{aligned} \wp(x_k - \gamma; r + \epsilon) &\geq \wp(x_k - c; \epsilon) * \wp(c - \gamma; r) \\ &> (1-t) * (1-t) \\ &> 1 - \lambda. \end{aligned}$$

This implies that $\delta(\{k \in \mathbb{N} : \wp(x_k - \gamma; r + \epsilon) > 1 - \lambda\}) > 0$. Hence, $\gamma \in \Gamma_\wp^r(x)$. Therefore, $\bigcup_{c \in \Gamma_\wp(x)} \overline{B(c, \lambda, r)} \subseteq \Gamma_\wp^r(x)$.

Conversely,
Let $\gamma \in \Gamma_{\wp}^{r}(x)$. Then we have to show that $\gamma \in \bigcup_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$.
Let if possible, $\gamma \notin \bigcup_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$ *i.e.* $\gamma \notin \overline{B(c,\lambda,r)}$ for all $c \in \Gamma_{\wp}(x)$.
Then $\wp(\gamma-c;r) \leq 1-\lambda$ for every $c \in \Gamma_{\wp}(x)$. By Theorem 2.16 for arbitrary $c \in \Gamma_{\wp}(x)$ we have $\wp(\gamma-c;r) > 1-\lambda$ for every $c \in \Gamma_{\wp}^{r}(x)$ which is a contradiction to the assumption. Therefore, $\gamma \in \bigcup_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$. Hence, $\Gamma_{\wp}^{r}(x) \subseteq \bigcup_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$. ■

Theorem 2.18. *Let $x=\{x_k\}$ be a sequence in a PN-Space $(\mathbb{X},\wp,*)$ then for any $\lambda \in (0,1)$,*

(1) If $c \in \Gamma_{\wp}(x)$ then $St_{\wp}$-$LIM_x^r \subseteq \overline{B(c,\lambda,r)}$.
(2) $St_{\wp}$-$LIM_x^r = \bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{\wp}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.

Proof. (1) Let $\epsilon > 0$. For given $\lambda \in (0,1)$ choose $t \in (0,1)$ such that$(1-t)*(1-t) > 1-\lambda$. Consider $\xi \in St_{\wp}$-LIM_x^r and $c \in \Gamma_{\wp}(x)$.
For every $\epsilon > 0$ and $t \in (0,1)$ define sets

$$A = \{k \in \mathbb{N} : \wp(x_k-\xi : r+\epsilon) > 1-t\} \text{ with } \delta(A^c)=0,$$

and

$$B = \{k \in \mathbb{N} : \wp(x_k-c;\epsilon) > 1-t\} \text{ with } \delta(B) \neq 0.$$

Now for $k \in A \cap B$ we have

$$\begin{aligned} \wp(\xi-c;r) &\geq \wp(x_k-c;\epsilon) * \wp(x_k-\xi;r+\epsilon) \\ &> (1-t)*(1-t) \\ &> 1-\lambda. \end{aligned}$$

Therefore, $\xi \in \overline{B(c,\lambda,r)}$. Hence, $St_{\wp}$-$LIM_x^r \subseteq \overline{B(c,\lambda,r)}$.
(2) By previous part we have $St_{\wp}$-$LIM_x^r \subseteq \bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$.

Assume $y \in \bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$ then $\wp(\xi-c;r) \geq 1-\lambda$ for all $c \in \Gamma_{\wp}(x)$. This implies that $\Gamma_{\wp}(x) \subseteq \overline{B(c,\lambda,r)}$, *i.e.* $\bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)} \subseteq \{\xi \in \mathbb{X} : \Gamma_{\wp}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.
Further, let $y \notin St_{\wp}$-LIM_x^r then for $\epsilon > 0$ we have $\delta(\{k \in \mathbb{N} : \wp(x_k-y;r+\epsilon) \leq 1-\lambda\}) \neq 0$, which implies that the existence of a statistical cluster point c of the sequence $x=\{x_k\}$ with $\wp(x_k-y;r+\epsilon) \leq 1-\lambda$. Thus, $\Gamma_{\wp}(x) \nsubseteq \overline{B(y,\lambda,r)}$ and $y \notin \{\xi \in \mathbb{X} : \Gamma_{\wp}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$. This implies that $\{\xi \in \mathbb{X} : \Gamma_{\wp}(x) \subseteq \overline{B(\xi,\lambda,r)}\} \subseteq St_{\wp}$-$LIM_x^r$ and we get $\bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)} \subseteq St_{\wp}$-$LIM_x^r$.
Therefore, $St_{\wp}$-$LIM_x^r = \bigcap_{c\in\Gamma_{\wp}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{\wp}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.

■

Theorem 2.19. *Let $x = \{x_k\}$ be a sequence in a PN-Space $(\mathbb{X}, \wp, *)$ which is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $\wp$ then there exists $\lambda \in (0,1)$ such that $St_{\wp}\text{-}LIM_x^r = \overline{B(\xi, \lambda, r)}$ for some $r > 0$.*

Proof. Let $\epsilon > 0$. For given $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t) * (1-t) > 1-\lambda$. Since $x_k \xrightarrow{St_{\wp}} \xi$ then there is a set $A = \{k \in \mathbb{N} : \wp(x_k - \xi : \epsilon) \leq 1-t\}$ with $\delta(A) = 0$. Consider $y \in \overline{B(\xi, t, r)} = \{y \in \mathbb{X} : \wp(y - \xi; r) \geq 1-t\}$.
For $k \in A^c$

$$\begin{aligned}\wp(x_k - y; r+\epsilon) &\geq \wp(x_k - \xi; \epsilon) * \wp(y - \xi; r) \\ &> (1-t) * (1-t) \\ &> 1-\lambda.\end{aligned}$$

This implies that $y \in St_{\wp}\text{-}LIM_x^r$, *i.e.* $\overline{B(\xi, \lambda, r)} \subseteq St_{\wp}\text{-}LIM_x^r$. Also $St_{\wp}\text{-}LIM_x^r \subseteq \overline{B(\xi, \lambda, r)}$. Hence, $St_{\wp}\text{-}LIM_x^r = \overline{B(\xi, \lambda, r)}$. ∎

Theorem 2.20. *Let $x = \{x_k\}$ be a sequence in a PN-Space $(\mathbb{X}, \wp, *)$ which converges statistically with respect to the norm $\wp$ then $\Gamma_{\wp}^r(x) = St_{\wp}\text{-}LIM_x^r$ for some $r > 0$.*

Proof. Necessary part:
Suppose $x_k \xrightarrow{St_{\wp}} \xi$. Then $\Gamma_{\wp}(x) = \{\xi\}$. By Theorem 2.17 for some $r > 0$ and $\lambda \in (0,1)$ we have $\Gamma_{\wp}^r(x) = \overline{B(\xi, \lambda, r)}$. Also by Theorem 2.19 we get $\overline{B(\xi, \lambda, r)} = St_{\wp}\text{-}LIM_x^r$. Hence, $\Gamma_{\wp}^r(x) = St_{\wp}\text{-}LIM_x^r$.

Sufficient part:
Let $\Gamma_{\wp}^r(x) = St_{\wp}\text{-}LIM_x^r$. By Theorem 2.17 and Theorem 2.18(ii) we have $\bigcup_{c \in \Gamma_{\wp}(x)} \overline{B(c, \lambda, r)} = \bigcap_{c \in \Gamma_{\wp}(x)} \overline{B(c, \lambda, r)}$. This implies that either $\Gamma_{\wp}(x) = \phi$ or $\Gamma_{\wp}(x)$ is a singleton set. Then $St_{\wp}\text{-}LIM_x^r = \bigcap_{c \in \Gamma_{\wp}(x)} \overline{B(c, \lambda, r)} = \overline{B(\xi, \lambda, r)}$ for some $\xi \in \Gamma_{\wp}(x)$, further by Theorem 2.19 we get $St_{\wp}\text{-}LIM_x^r = \{\xi\}$. ∎

Acknowledgements

We express great sense of gratitude and deep respect to the referees of this paper and reviewers for their valuable suggestions.

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