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Fixed Point Theorem Satisfying Generalized Weakly Contractive Condition of Integral Type Using C-Class Functions

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Abstract The aim of this paper is to present sufficient condition for the existence and uniqueness of common fixed points for selfmaps satisfying a generalized $\int_{\varphi}^{\psi}$ weakly contractive condition involving C-class functions in the setting of complete metric spaces. As applications of our results, we obtain several consequence results. At last, an example is given to justify the importance and applicability of our result.

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1. Introduction and Basic Notions

Branciari [1] in 2002, gave one of the real extension and generalization of Banach contraction principle [2] by initiating most promising notion, known as integral type contractions. Before stating Branciari [1] result, first recall the definition of Lebesgue - integrable function.

Notify L as a function defined as: $L = \Big\{ l : R^+ \to R^+$ which is nonnegative, summable on each compact subset of R^+, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} l(m)dm > 0. \Big\}$

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Theorem 1.1. [1] *If a self map $U : P \to P$ on a complete metric space (P, d) satisfying the contraction*

$$\int_0^{d(Uf,Ug)} l(m)dm \leq a \int_0^{d(f,g)} l(m)dm,\ \forall\ f, g \in P \tag{1.1}$$

where $a \in (0, 1)$ and $l \in L$. Then the map U has a unique fixed point.

Next illustration due to Branciari[1] proved that if a map satisfying Branciari integral type contraction, doesn't implies that we always get a fixed point.

Example 1.2. [1] Let d be a Euclidean distance function, and let $P = R_+$.
Define the map $U : P \to P$ and Lebesgue-integrable function l as

$$U(f) = f + 1 \quad and \quad l(m) = -1.$$

Clearly, for some arbitrary $a \in (0, 1)$, all the assumptions of Theorem 1.1 are satisfied. But the map U has no fixed point.

Next example due to Branciari[1] was quite different. Beacuse it proved that his result was a proper generalization of the Banach [2] contraction but conversely it does'nt satisfied.

Example 1.3. [1] Let $P = \left\{\frac{1}{s} | s \in N\right\} \bigcup \{0\}$ with usual metric, then (P, d) is a complete metric spaces. Let $U : P \to P$ be a function defined by

$$Uf = \begin{cases} \frac{1}{s+1} \ if\ f = \frac{1}{s}, s \in N, \\ 0 \qquad if\ f = 0, \end{cases}$$

then it satisfies (1.1) with $l(m) = m^{\frac{1}{m-2}[1-logm]}$ for $m > 0$, $l(0) = 0$, and $a = \frac{1}{2}$. But does not satisfies Banach contraction principal [2].

One of the finest extension of Branciari [1] was $^{\psi}\int_l$ - weakly contractive mapping. Luong and Thuan [3] were the initiator of this type of mappings.

Definition 1.4. [3] Let (P, d) be a metric space. A mapping $U : P \to P$ is said to be $^{\psi}\int_l -$ weakly contractive if for all $f, g \in P$

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \leq \psi\left(\int_0^{d(f,g)} l(m)dm\right) - \delta\left(\int_0^{d(f,g)} l(m)dm\right),$$

where map $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing and continuous, $\delta : [0, \infty) \to [0, \infty)$ is a non-decreasing and lower semi-continuous map, and are such that $\psi(m) = 0 = \delta(m)$ if and only if $m = 0$, and $l \in L$.

Theorem 1.5. [3] *Let (P, d) be a complete metric space and let $U : P \to P$ is a $^{\psi}\int_l$ -weakly contractive mapping. Then there exist a unique fixed point of U in P.*

Another extension of Branciari was proved by Aydi [4] in 2012 as follows:

Theorem 1.6. [4] *Let (P, d) be a complete metric space and $U : P \to P$ be a map satisfying*

$$\psi\Big(\int_0^{d(Uf,Ug)} l(m)dm\Big) \leq \psi\left(\theta\left(f, g\right)\right) - \delta\left(\theta\left(f, g\right)\right),\ \forall\ f, g \in P,$$

where

$$\theta(f,g) = h\int_0^{d(f,Uf)+d(g,Ug)} l(m)dm + q\int_0^{d(f,g)} l(m)dm + c\int_0^{\max\{d(f,Ug),d(g,Uf)\}} l(m)dm,$$

ψ,δ are altering distances, h,q,c are non-negative reals with $2h+q+2c<1$ and $l\in L$. Then U has a unique fixed point in P.

This idea and result of Luong and Thuan [3] was noticed, and generalised by Gupta and Mani [5] for 2 self compatible maps.

Theorem 1.7. [5] *Let (P,d) be a complete metric space and $V,U:P\to P$ be 2 selfmaps satisfying*

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \le \psi\left(\int_0^{E(f,g)} l(m)dm\right) - \delta\left(\int_0^{E(f,g)} l(m)dm\right)$$

for each $f,g\in P$, where

$$E(f,g) = \max\left\{d(Vf,Vg), d(Vf,Uf), d(Vg,Ug), \frac{d(Vf,Ug)+d(Vg,Uf)}{2}\right\},$$

$\psi:[0,+\infty)\to[0,+\infty)$ is a non-decreasing and continuous function, $\delta:[0,+\infty)\to[0,+\infty)$ is a non-decreasing and lower semi continuous function such that $\psi(m)=\delta(m)=0$ if and only if $m=0$ and $l\in L$.
Further, if $U(P)\subset V(P)$ then V and U have a coincidence point in P.
Moreover, if V and U are weakly compatible, then V and U have a unique common fixed point in P.

Branciari [1] as well as Banach [2] result was further extended, unified and generalized by number of authors in different spaces. We are refering here few of them [6–18]

Ansari [19, 20], in 2014, developed the concept of C -class functions as a novel extension of Banach Contraction principle.

Definition 1.8. [19] A family of continuous mappings $J:[0,\infty)^2\to\mathbb{R}$ is called *C-class* function if it satisfies following conditions:

(1) $J(p,m)\le p$;
(2) $J(p,m)=p$ implies that either $p=0$ or $m=0$; for all $p,m\in[0,\infty)$.

For brevity, we denote C-class functions as $\mathcal{C}$.

Indeed, for some J, we have $J(0,0)=0$. We can consider it as an extra condition on J in some particular cases.

Example 1.9. [19] The following functions $J:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathcal{C}$, for all $p,m\in[0,\infty)$:

(1) $J(p,m)=p-m$, $J(p,m)=p\Rightarrow m=0$;
(2) $J(p,m)=ap$, $0<a<1$, $J(p,m)=p\Rightarrow p=0$;
(3) $J(p,m)=\frac{p}{(1+m)^h}$; $h\in(0,\infty)$, $J(p,m)=p\Rightarrow p=0$ or $m=0$;
(4) $J(p,m)=pq(p)$, $q:[0,\infty)\to[0,1)$, $J(p,m)=p\Rightarrow p=0$;

(5) $J(p,m) = p - \frac{m}{a+m}, J(p,m) = p \Rightarrow m = 0$;

(6) $J(p,m) = p - l(p), J(p,m) = p \Rightarrow p = 0$,here $l : [0,\infty) \to [0,\infty)$ is a continuous function such that $l(m) = 0 \Leftrightarrow m = 0$;

This notion was utilised by Ansari et al. [21] in proving coupled fixed point result in partially ordered metric spaces. Recently, Saini et al. [22] and Gupta et al. [23] presented a weak contraction and established some result by using C-class function. All these presented results are weaker than previous findings. Recently, Mani et al. [24] studied some aspects of integral type contractions with help of auxiliary function. In this article, we are going to derive a result for two self weakly compatible maps with the help of C-class function.

2. Main Results: Existence and Uniqueness of Common Fixed Points

In present section, we prove the existence and uniqueness of fixed point for pair of weakly compatible mappings with $\mathcal{C}$ -class function in sense of complete metric spaces. In addition, as applications, some consequence results are derived.

Theorem 2.1. *Let V and U be pair of self mappings on a complete metric space (P, d) satisfying the contraction*

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \le J\left(\psi\left(\int_0^{E(f,g)} l(m)dm\right), \delta\left(\int_0^{E(f,g)} l(m)dm\right)\right) \tag{2.1}$$

for all $f, g \in P$, where $l \in L$, J is a C-class function, $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and continuous function such that $\psi(m) = 0$ if and only if $m = 0$, $\delta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and lower semi continuous such that $\delta(0) \ge 0$, and $\delta(m) > 0$ for all $m > 0$ and

$$E(f,g) = \max\left\{d(Vf,Vg), d(Vf,Uf), d(Vg,Ug), \frac{d(Vf,Ug)+d(Vg,Uf)}{2}\right\}. \tag{2.2}$$

Further, if $U(P) \subset V(P)$, then V and U have a coincidence point in P.
Moreover, if V and U are weakly compatible then they have a unique common fixed point in P.

Proof. Set two initial approximation $f_0, f_1 \in P$ as any arbitrary point in P. Using our assumption $U(P) \subset V(P)$, we can define $Vf_1 = Uf_0$. In general, construct sequence $f_{s+1} \in P$ such that

$$g_{s+1} = Vf_{s+1} = Uf_s \text{ and } g_s = Vf_s = Uf_{s-1},\ \forall\ s = 0, 1, 2 \cdots$$

For each $s \ge 1$, from (2.18) we have

$$\begin{aligned}\psi\left(\int_0^{d(g_s,g_{s+1})} l(m)dm\right) &= \psi\left(\int_0^{d(Uf_{s-1},Uf_s)} l(m)dm\right)\\ &\le J\left\{\begin{array}{c}\psi\left(\int_0^{E(f_{s-1},f_s)} l(m)dm\right),\\ \delta\left(\int_0^{E(f_{s-1},f_s)} l(m)dm\right)\end{array}\right\}.\end{aligned} \tag{2.3}$$

From (2.2), we get

$$\begin{aligned}E(f_{s-1},f_s)&=\max\left\{\begin{array}{c} d(Vf_{s-1},Vf_s),d(Vf_{s-1},Uf_{s-1}),d(Vf_s,Uf_s),\\ \frac{d(Vf_{s-1},Uf_s)+d(Vf_s,Uf_{s-1})}{2}\end{array}\right\}\\ &=\max\left\{\begin{array}{c} d(g_{s-1},g_s),d(g_{s-1},g_s),d(g_s,g_{s+1}),\\ \frac{d(g_s,g_s)+d(g_{s-1},g_{s+1})}{2}\end{array}\right\}\\ &=\max\left\{d(g_{s-1},g_s),d(g_s,g_{s+1}),\frac{d(g_s,g_s)+d(g_{s-1},g_{s+1})}{2}\right\}\\ &=\max\left\{d(g_{s-1},g_s),d(g_s,g_{s+1})\right\}.\end{aligned}$$

Thus from (2.3),

$$\begin{aligned}\psi\left(\int_0^{d(g_s,g_{s+1})}l(m)dm\right)&\leq J\left\{\begin{array}{c}\psi\left(\int_0^{\max\{d(g_{s-1},g_s),d(g_s,g_{s+1})\}}l(m)dm\right),\\ \delta\left(\int_0^{\max\{d(g_{s-1},g_s),d(g_s,g_{s+1})\}}l(m)dm\right)\end{array}\right\}\\ &\leq\psi\left(\int_0^{\max\{d(g_{s-1},g_s),d(g_s,g_{s+1})\}}l(m)dm\right).\end{aligned}\tag{2.4}$$

Suppose $d(g_s,g_{s+1})\geq d(g_{s-1},g_s)$ for some s, then from (2.4), we get a contradiction. Thus $d(g_s,g_{s+1})<d(g_{s-1},g_s)$, and so

$$\psi\left(\int_0^{d(g_s,g_{s+1})}l(m)dm\right)\leq\psi\left(\int_0^{d(g_{s-1},g_s)}l(m)dm\right).$$

This is a monotone decreasing and lower bounded sequence $\left\{\int_0^{d(g_s,g_{s+1})}l(m)dm\right\}$, and so there exist $p\geq 0$ such that

$$\lim_{s\to\infty}\left(\int_0^{d(g_s,g_{s+1})}l(m)dm\right)=p.\tag{2.5}$$

Suppose that $p>0$. On taking limit as $s\to\infty$ in (2.3) and using equation (2.4),(2.5), we get

$$\psi\left(p\right)\leq J\Big(\psi\left(p\right),\delta\left(p\right)\Big)<\psi\left(p\right).$$

This is a contradiction. Therefore $p=0$. Hence from 2.5

$$\lim_{s\to\infty}\left(\int_0^{d(g_s,g_{s+1})}l(m)dm\right)=0.\tag{2.6}$$

Consequently, it gives

$$\lim_{s\to\infty}d(g_s,g_{s+1})=0.\tag{2.7}$$

Next we assert that sequence $\{g_s\}$ is Cauchy.
Assume not. so for an $\epsilon>0$, there exists subsequences $\{g_{w(i)}\}$ and $\{g_{s(i)}\}$ of $\{g_s\}$ with $w(i)<s(i)<w(i+1)$ satisfying

$$d(g_{w(i)},g_{s(i)})\geq\epsilon\text{ and }d(g_{w(i)},g_{s(i-1)})<\epsilon.\tag{2.8}$$

Consider

$$\psi\left(\int_0^{\epsilon} l(m)dm\right) \leq \psi\left(\int_0^{d(g_{w(i)},g_{s(i)})} l(m)dm\right)$$
$$= \psi\left(\int_0^{d(Uf_{w(i)-1},Uf_{s(i)-1})} l(m)dm\right)$$
$$\leq J\left\{\begin{array}{c} \psi\left(\int_0^{E(f_{w(i)-1},f_{s(i)-1})} l(m)dm\right), \\ \delta\left(\int_0^{E(f_{w(i)-1},f_{s(i)-1})} l(m)dm\right) \end{array}\right\}. \quad (2.9)$$

From (2.2),

$$E(f_{w(i)-1},f_{s(i)-1}) = \max\left\{\begin{array}{c} d(Vf_{w(i)-1},Vf_{s(i)-1}), d(Vf_{w(i)-1},Uf_{w(i)-1}), \\ d(Vf_{s(i)-1},Uf_{s(i)-1}), \\ \frac{d(Vf_{w(i)-1},Uf_{s(i)-1})+d(Vf_{s(i)-1},Uf_{w(i)-1})}{2} \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(g_{w(i)-1},g_{s(i)-1}), d(g_{w(i)-1},g_{w(i)}), \\ d(g_{s(i)-1},g_{s(i)}), \\ \frac{d(g_{w(i)-1},g_{s(i)})+d(g_{s(i)-1},g_{w(i)})}{2} \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(g_{w(i)-1},g_{s(i)-1}), d(g_{w(i)-1},g_{w(i)}), \\ d(g_{s(i)-1},g_{s(i)}), a(w,s) \end{array}\right\},$$

where

$$a(w,s) = \frac{d(g_{w(i)-1},g_{s(i)}) + d(g_{s(i)-1},g_{w(i)})}{2}. \quad (2.10)$$

Consider,

$$\int_0^{E(f_{w(i)-1},f_{s(i)-1})} l(m)dm$$
$$= \int_0^{\max\left\{d(g_{w(i)-1},g_{s(i)-1}),d(g_{w(i)-1},g_{w(i)}),d(g_{s(i)-1},g_{s(i)}),a(w,s)\right\}} l(m)dm$$
$$= \max\left\{\begin{array}{c} \int_0^{d(g_{w(i)-1},g_{s(i)-1})} l(m)dm, \int_0^{d(g_{w(i)-1},g_{w(i)})} l(m)dm, \\ \int_0^{d(g_{w(i)-1},g_{w(i)})} l(m)dm, \int_0^{d(g_{s(i)-1},g_{s(i)})} l(m)dm, \int_0^{a(w,s)} l(m)dm \end{array}\right\}.$$

By using (2.8) and triangle inequality, we get

$$d(g_{w(i)-1},g_{s(i)-1}) \leq d(g_{w(i)-1},g_{w(i)}) + d(g_{w(i)},g_{s(i)-1})$$
$$\leq d(g_{w(i)-1},g_{w(i)}) + \epsilon.$$

$$\lim_{i\to\infty} \int_0^{d(g_{w(i)-1},g_{s(i)-1})} l(m)dm \leq \int_0^{\epsilon} l(m)dm \quad (2.11)$$

Also, from (2.10)

$$\begin{aligned} a(w,s) &= \frac{d(g_{w(i)-1}, g_{s(i)}) + d(g_{s(i)-1}, g_{w(i)})}{2} \\ &\leq \frac{d(g_{w(i)-1}, g_{w(i)}) + d(g_{w(i)}, g_{s(i-1)})}{2} \\ &\quad + \frac{d(g_{s(i)-1}, g_{s(i)}) + d(g_{s(i)-1}, g_{w(i)})}{2} \\ &\leq \frac{d(g_{w(i)-1}, g_{w(i)}) + d(g_{s(i)-1}, g_{s(i)})}{2} + \epsilon. \end{aligned} \tag{2.12}$$

Taking $\lim_{i\to\infty}$ and using (2.7), we get

$$\lim_{i\to\infty} \int_0^{a(w,s)} l(m)dm \leq \int_0^{\epsilon} l(m)dm. \tag{2.13}$$

Taking $\lim_{i\to\infty}$ in equality (2.10) and using (2.11),(2.12),(2.13), we get

$$\begin{aligned} \psi\left(\int_0^{\epsilon} l(m)dm\right) &\leq J\left(\psi\left(\int_0^{\epsilon} l(m)dm\right), \delta\left(\int_0^{\epsilon} l(m)dm\right)\right) \\ &\leq \psi\left(\int_0^{\epsilon} l(m)dm\right), \end{aligned}$$

this is a contradiction. Therefore $\{g_s\}$ is a Cauchy sequence. Call the limit as z such that

$$\lim_{s\to\infty} g_s = z$$

$$i.e \qquad \lim_{s\to\infty} Vf_s = \lim_{s\to\infty} Uf_s = z. \tag{2.14}$$

Since $U(P) \subset V(P)$, therefore there exist some $b \in P$ such that $Vb = z$.
Hence $\lim_{s\to\infty} Vf_s = Vb$.
Consider

$$\lim_{s\to\infty} \psi\left(\int_0^{d(Uf_s,Ub)} l(m)dm\right) \leq \lim_{s\to\infty} J\left\{\begin{array}{c} \psi\left(\int_0^{E(f_s,b)} l(m)dm\right), \\ \delta\left(\int_0^{E(f_s,b)} l(m)dm\right) \end{array}\right\} \tag{2.15}$$

where

$$\lim_{s\to\infty} E(f_s,b) = \lim_{s\to\infty} \max\left\{\begin{array}{c} d(Vf_s,Vb), d(Vf_s,Uf_s), d(Vb,Ub), \\ \frac{d(Vf_s,Ub)+d(Vb,Uf_s)}{2} \end{array}\right\},$$

implies

$$\lim_{s\to\infty} E(f_s,b) = d(z,Ub).$$

Hence from (2.15), we get

$$\begin{aligned} \psi\left(\int_0^{d(z,Ub)} l(m)dm\right) &\leq J\left(\psi\left(\int_0^{d(z,Ub)} l(m)dm\right), \delta\left(\int_0^{d(z,Ub)} l(m)dm\right)\right) \\ &\leq \psi\left(\int_0^{d(z,Ub)} l(m)dm\right), \end{aligned}$$

this is a contradiction. This implies $d(z, Ub) = 0$. Thus $Vb = z = Ub$.
This proves that b is the coincidence point of V and U.
Also, weakly compatible property of maps V and U implies that $STu = TSu$. Therefore, $Uz = Vz$.
Next assert that z is a fixed point of U.
Consider,

$$\begin{aligned}\psi\left(\int_0^{d(z,Uz)} l(m)dm\right) &= \psi\left(\int_0^{d(Ub,Uz)} l(m)dm\right)\\ &\leq J\left(\psi\left(\int_0^{E(b,z)} l(m)dm\right), \delta\left(\int_0^{E(b,z)} l(m)dm\right)\right),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}E(b,z) &= \max\left\{d(Vb,Vz), d(Vb,Ub), d(Vz,Uz), \frac{d(Vb,Uz)+d(Vz,Ub)}{2}\right\}\\ &= d(z,Uz).\end{aligned}$$

Hence from (2.16)

$$\begin{aligned}\psi\left(\int_0^{d(z,Uz)} l(m)dm\right) &\leq J\left(\psi\left(\int_0^{d(z,Uz)} l(m)dm\right), \delta\left(\int_0^{d(z,Uz)} l(m)dm\right)\right)\\ &\leq \psi\left(\int_0^{d(z,Uz)} l(m)dm\right).\end{aligned}$$

We arrived at contradiction. Thus $d(z, Uz) = 0$. Therefore z is the fixed point of map U and so the fixed point of V. This proves that z is the common fixed point of V and U.
For Uniqueness, assume that there exist another point e such that $Ve = e = Ue$.
From (2.18), we have

$$\psi\left(\int_0^{d(Uz,Ue)} l(m)dm\right) \leq J\left(\psi\left(\int_0^{E(z,e)} l(m)dm\right), \delta\left(\int_0^{E(z,e)} l(m)dm\right)\right), \tag{2.17}$$

where

$$\begin{aligned}E(z,e) &= \max\left\{d(Vz,Ve), d(Vz,Uz), d(Ve,Ue), \frac{d(Vz,Ue)+d(Ve,Uz)}{2}\right\}\\ &= d(z,e).\end{aligned}$$

This implies that $e = z$ and hence, fixed point of maps are unique.
This accomplished the proof of our result. ■

If we take $V = I$ (Identity mapping) in Theorem 2.1, we have the following consequence result.

Corollary 2.2. *Let U be a selfmap on a complete metric space (P, d) satisfying the contraction*

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \leq J\left(\psi\left(\int_0^{E(f,g)} l(m)dm\right), \delta\left(\int_0^{E(f,g)} l(m)dm\right)\right) \tag{2.18}$$

for all $f, g \in P$, where $l \in L$, J is a C-class function, $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and continuous function such that $\psi(m) = 0$ if and only if $m = 0$, $\delta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and lower semi continuous such that $\delta(0) \geq 0$, and $\delta(m) > 0$ for all $m > 0$ and

$$E(f, g) = \max\left\{d(f, g), d(f, Uf), d(g, Ug), \frac{d(f, Ug) + d(g, Uf)}{2}\right\}.$$

Then U has a unique fixed point in P.

If we take $J(p, m) = \frac{p}{(1+m)^s}$ and assume $s = 1$ in Theorem 2.1, we obtained following result.

Corollary 2.3. *Let V and U be pair of self mappings on a complete metric space (P, d) satisfying the contraction*

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \leq \frac{\psi\left(\int_0^{E(f,g)} l(m)dm\right)}{1 + \delta\left(\int_0^{E(f,g)} l(m)dm\right)}$$

for all $f, g \in P$, where $l \in L$, $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and continuous function such that $\psi(m) = 0$ if and only if $m = 0$, $\delta : [0, +\infty) \to [0, +\infty)$ is a non-decreasing and lower semi continuous such that $\delta(0) \geq 0$, and $\delta(m) > 0$ for all $m > 0$ and

$$E(f, g) = \max\left\{d(Vf, Vg), d(Vf, Uf), d(Vg, Ug), \frac{d(Vf, Ug) + d(Vg, Uf)}{2}\right\}$$

Further, if $U(P) \subset V(P)$, then V and U have a coincidence point in P.
Moreover, if V and U are weakly compatible then they have a unique common fixed point in P.

If we take $J(p, m) = ar$ for $0 < a < 1$ in Theorem 2.1, then we have following corollary.

Corollary 2.4. *Let V and U be pair of self mappings on a complete metric space (P, d) satisfying the contraction*

$$\psi\left(\int_0^{d(Uf,Ug)} l(m)dm\right) \leq a\psi\left(\int_0^{E(f,g)} l(m)dm\right)$$

for all $f, g \in P$, where $l \in L$, $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and continuous function such that $\psi(m) = 0$ if and only if $m = 0$ and

$$E(f, g) = \max\left\{d(Vf, Vg), d(Vf, Uf), d(Vg, Ug), \frac{d(Vf, Ug) + d(Vg, Uf)}{2}\right\}$$

Further, if $U(P) \subset V(P)$, then V and U have a coincidence point in P.
Moreover, if V and U are weakly compatible then they have a unique common fixed point in P.

If we assume that $\psi(m) = m$ in Theorem 2.1, we obtain the following result.

Corollary 2.5. *Let V and U be pair of self mappings on a complete metric space (P, d) satisfying the contraction*

$$\int_0^{d(Uf,Ug)} l(m)dm \leq J\left(\int_0^{E(f,g)} l(m)dm, \delta\left(\int_0^{E(f,g)} l(m)dm\right)\right)$$

for all $f, g \in P$, where $l \in L$, $\delta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and lower semi continuous such that $\delta(0) \geq 0$, and $\delta(m) > 0$ for all $m > 0$ and

$$E(f, g) = \max\left\{d(Vf, Vg), d(Vf, Uf), d(Vg, Ug), \frac{d(Vf, Ug) + d(Vg, Uf)}{2}\right\}$$

Further, if $U(P) \subset V(P)$, then V and U have a coincidence point in P.
Moreover, if V and U are weakly compatible then they have a unique common fixed point in P.

3. Example

In this section, we gave an example to justify the importance of our result.

Example 3.1. Take $P = N - \{\infty\}$ and let metric $d(f, g) = |f - g|$.
Define mappings V and U as

$$Uf = \frac{f}{2} \quad and \quad Vf = f \quad \forall \quad f \in P.$$

Clearly, $U(P) \subset V(P)$. Define a function $J : [0, \infty)^2 \to \mathbb{R}$ as

$$J(p, m) = \frac{p}{2}.$$

Then clearly, J is a C-*class* function (from Example 1.9).
Let us define $\psi, \delta, l : [0, +\infty) \to [0, +\infty)$ as

$$\psi(m) = m, \quad \delta(m) = \frac{m}{2}, \quad l(m) = 2m, \quad \forall \quad m \in [0, +\infty)$$

then for each $\epsilon > 0$,

$$\int_0^{\epsilon} l(m)dm = \epsilon^2.$$

If $f = g$ for all $f, g \in P$, then result holds trivially.
So suppose that $f \neq g$ for all $f, g \in P$. Since d is usual metric for all $f, g \in P$, then we get

$$\text{L.H.V.} = \frac{|f - g|}{4}^2, \quad E(f, g) = |f - g|, \quad \text{R.H.V.} = \frac{|f - g|}{2}^2$$

Then clearly, L.H.V $\leq$ R.H.V for all $f, g \in P$ and hence all conditions of Theorem 2.1 are verified.
Clearly, $0 \in P$ is the unique fixed point of V and U.

4. Conclusion

From above proved results, examples and remarks, we conclude that our result is a new approach in the field of fixed point theory. Our main theorem itself is an innovative idea to find common fixed point by combining three auxiliary functions independent on each others. In support, and for importance of our result, we have solved some new examples. Various corollaries are presented here to demonstrate that our contraction is generalization of various existing results in complete metric spaces.

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