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A Note on Wavelet Approximation of Functions

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Abstract In this paper, a class $Lip_\alpha^{(\omega)}[0,1)$ is introduced. This class of functions is the generalization of known Lipschitz class $Lip_\alpha[0,1), 0<\alpha\leq 1$ of functions. Four new estimators $E_N^{(1)}(f), E_J^{(2)}(f), E_N^{(3)}(f)$ and $E_N^{(4)}(f)$ of functions of $Lip_\alpha[0,1)$ and $Lip_\alpha^{(\omega)}[0,1)$ classes have been obtained. These estimators are new, sharper and best possible in the approximation of functions by wavelet methods.

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1. Introduction

The classical idea of approximation theory is applied in more technical ways in the modern field of wavelet analysis. In wavelet analysis, complicated functions are approximated by their simple expressions in the form of wavelet series. This concept plays a desirable role as applications in many branches of pure as well as applied mathematics.

Thus there is a need to describe wavelets more in words than symbols so that fundamental ideas like removing noise from music recording and storing fingerprints electronically using the approximation of functions can be solved by involving the concept of wavelets. Presently, the wavelet era has been begun and several applications of wavelet analysis and approximation of functions by their wavelet series are found in signal processing, image processing, engineering, and technology.

Several mathematicians like Chui, Daubechies, Debnath, Khalil Ahmad, Meyer, Morlet, Natanson, Lal and Kumari [1–15] have studied some theories of wavelets in detail. In 1950, Natanson [3] initiated a well ordered study of approximation theory in wavelet analysis.

One of the main advantages of wavelet analysis is that a function is approximated by few coefficients of its wavelet expansion. This is good enough approximation in comparison of other techniques. This method is applicable in signal expansion.

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The aims of this paper are mentioned below;
1. To introduce the generalized Haar scaling function $\phi^{(\lambda)}$ and generalized Haar mother wavelet $\psi^{(\lambda)}$.
2. To introduce the expansion of a function f by generalized Haar scaling function and generalized Haar mother wavelet.
3. To introduce the generalized Lipschitz class of function,i.e., $Lip_\alpha^{(\omega)}[0,1)$.
4. The approximation of a function $f \in Lip_\alpha[0,1)$ and $f \in Lip_\alpha^{(\omega)}[0,1)$ by partial sums of the wavelet expansions of associated functions.
5. To generalize the results of Christensen [9] and Walnut [16].

2. Definitions and Preliminaries

Definition 2.1. Function of $Lip_\alpha[0,1)$ class: A function $f \in Lip_\alpha[0,1), 0 < \alpha \leq 1$ if

$$|f(x) - f(y)| = O(|x-y|^\alpha), \forall x, y \in [0,1] \quad \text{(Titchmarsh [17])}.$$

Consider a function $f(x) = x^{2\alpha}, \forall x \in [0,1]$.
For this,

$$\begin{aligned} |f(x) - f(y)| &= |x^{2\alpha} - y^{2\alpha}| \forall x, y \in [0,1] \\ &= |x^\alpha - y^\alpha||x^\alpha + y^\alpha| \\ &\leq 2|x^\alpha - y^\alpha| \\ &\leq 2|x-y|^\alpha, \text{since } |x^\alpha - y^\alpha| \leq |x-y|^\alpha, 0 < \alpha \leq 1. \end{aligned}$$

Thus $f \in Lip_\alpha[0,1)$.

Definition 2.2. Generalized Lipschitz class: A function $f \in Lip_\alpha^{(\omega)}[0,1)$ if

$$|f(x+t) - f(x)| = O(\omega(t)|t|^\alpha); 0 < \alpha < 1, x, t, x+t \in [0,1],$$

where $\omega(t)$ is positive monotonic increasing function of t such that $\omega(t)|t|^\alpha \to 0$ as $t \to 0^+$.
Remark: It is important to note that if $\omega(t) = c$ where c is a positive real number, then $Lip_\alpha^{(\omega)}[0,1)$ reduces to the class $Lip_\alpha[0,1)$.

Definition 2.3. Haar scaling function ϕ (Father wavelet) and Haar wavelet ψ (Mother wavelet): Haar scaling function can be defined as;

$$\phi(t) = \chi_{[0,1)}(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For each $j, k \in \mathbb{Z}, \{\phi_{j,k}(t)\}_{j,k\in\mathbb{Z}} = 2^{j/2}\phi(2^j t - k) = D_{2^j} T_k \phi(t)$, where dilation operator $D_a f(t) = a^{1/2} f(at)$ and the translation of operator $T_k f(t) = f(t-k)$.
If $\psi \in L^2(\mathbb{R})$ satisfies the "admissibility condition", i.e.,

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

then ψ is called a basic wavelet. The basic wavelet is known as the mother wavelet. The father wavelet and mother wavelet are jointly known as parent wavelet. Some different dilations and translations of ψ mother wavelet using the relation

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), j, k \in \mathbb{Z},$$

the basis can be generated. A function defined on the real line $\mathbb{R}$ as;

$$\psi(t) = \begin{cases} 1 & \mathit{for}\ t \in \left[0, \frac{1}{2}\right), \\ -1 & \mathit{for}\ t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

is known as Haar function.
Haar wavelet is discontinuous at $t = 0, 1/2, 1$ and it is very well localized in the time domain.

Definition 2.4. Generalized Haar scaling function $\phi^{(\lambda)}_{N,k}$ and Generalized Haar wavelet $\psi^{(\lambda)}_{j,k}$: For $\lambda = 1, 2, 3....$, generalized Haar scaling function $\phi^{(\lambda)}$ in the interval [0,1] can be defined as follows;

$$\phi^{(\lambda)}_{N,k}(t) = \lambda^{N/2}\phi(\lambda^N t - k) \quad = \quad \begin{cases} \lambda^{N/2} & : \quad \frac{k}{\lambda^N} \le t < \frac{k+1}{\lambda^N} \\ 0 & : \quad \text{otherwise} \end{cases}$$

where $k = 1, 2, ...\lambda^j - 1$.
The generalized Haar wavelet over the interval [0,1] is defined as;

$$\psi^{(\lambda)}_{j,k}(t) = \lambda^{j/2}\psi(\lambda^j t - k) \quad = \quad \begin{cases} \lambda^{j/2} : & \frac{k}{\lambda^j} \le t < \frac{k+1/2}{\lambda^j} \\ -\lambda^{j/2} : & \frac{k+1/2}{\lambda^j} \le t < \frac{k+1}{\lambda^j} \\ 0 \quad : & \text{otherwise.} \end{cases}$$

Definition 2.5. Generalized Haar wavelet series: Let $f \in L^2[0, 1]$. A wavelet series of f using generalized Haar scaling function $\phi^{(\lambda)}$ and Haar mother wavelet $\psi^{(\lambda)}$ is given by

$$\begin{aligned} f(t) &= \sum_{k=0}^{\lambda^N-1} < f, \phi^{(\lambda)}_{N,k} > \phi^{(\lambda)}_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1} < f, \psi^{(\lambda)}_{j,k} > \psi^{(\lambda)}_{j,k}(t) \\ &= (P_N f)(t) + (R_N f)(t), say \end{aligned}$$

where $< f, \phi^{(\lambda)}_{N,k} >= \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} f(t)\phi^{(\lambda)}_{N,k}(t)dt$ and $< f, \psi^{(\lambda)}_{j,k} >= \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} f(t)\psi^{(\lambda)}_{j,k}(t)dt$ and the basis functions $\psi^{(\lambda)}_{j,k}$ are orthonormal signals , each of which is associated by scale $\frac{1}{\lambda^j}$ and position $\frac{k}{\lambda^j}$.

Definition 2.6. Multiresolution analysis: A multiresolution analysis of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces $V^{(G)}_j$ of $L^2(\mathbb{R}), j \in \mathbb{Z}$, with the following properties;

(1) $V^{(G)}_j \subset V^{(G)}_{j+1}$,
(2) $f(t) \in V^{(G)}_j \Leftrightarrow f(\mu t) = V^{(G)}_{j+1}$,
(3) $f(t) \in V^{(G)}_0 \Leftrightarrow f(t+1) \in V^{(G)}_0$,
(4) $\bigcup_{j=-\infty}^{\infty} V^{(G)}_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j=-\infty}^{\infty} V^{(G)}_j = \{0\}$,
(5) Suppose a function $\phi \in V^{(G)}_0$ exists such that the collection $\{\phi(t-k); k \in \mathbb{Z}\}$ is Riesz basis of $V^{(G)}_0$.

If some wavelet $\psi \in L^2(\mathbb{R})$ has to be constructed, then it is advisable to study the structure of the $L^2(\mathbb{R})$ decomposition it generates. As usual, let

$$\psi_{j,k}^{(\lambda)}(t) = \mu^{j/2}\psi(\mu^j t - k)$$

and

$$W_j^{(G)} = clos_{L^2(\mathbb{R})}\left\{\psi_{j,k}^{(\lambda)}; k \in \mathbb{Z}\right\}.$$

Then this family of subspaces of $L^2(\mathbb{R})$ gives a direct sum decomposition of $L^2(\mathbb{R})$ in the sense that every $f \in L^2(\mathbb{R})$ has a unique decomposition

$$f(t) = ... + g_{-2} + g_{-1} + g_0 + g_1 + g_2 + ...$$

where $g_j \in W_j^{(G)}$ for all $j \in \mathbb{Z}$, and we shall describe by writing

$$L^2(\mathbb{R}) = \sum_{j\in\mathbb{Z}} W_j^{(G)} = ... \oplus W_{-2}^{(G)} \oplus W_{-1}^{(G)} \oplus W_0^{(G)} \oplus W_1^{(G)} \oplus W_2^{(G)} \oplus ... \tag{2.3}$$

Being in $W_j^{(G)}$, the component g_j of f has a unique wavelet series representation, where the coefficient sequence gives localized spectral information of f in the j^{th} octave (or frequency band) in terms of the integral wavelet transform of f with dual ψ of ψ as the basis wavelet. Using the decomposition of $L^2(\mathbb{R})$ in Eq.(2.3), we also have a nested sequence of closed subspaces $V_j^{(G)}, j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ defined by

$$V_j^{(G)} = ... \oplus W_{j-2}^{(G)} \oplus W_{j-1}^{(G)}.$$

Let $\phi \in V_0^{(G)}$, since $V_0^{(G)} \subset V_1^{(G)}$, a sequence $\{h_k\} \in l^2(\mathbb{Z})$ exists such that the ϕ function satisfies,

$$\phi(t) = \lambda \sum_{j=-\infty}^{\infty} h_k \phi(\mu t - k). \tag{2.4}$$

This functional equation is known as the refinement equation or the dilation equation or the two-scale difference equation. The collection of functions $\left\{\phi_{j,k}^{(\lambda)}; k \in \mathbb{Z}\right\}$, with $\phi_{j,k}^{(\lambda)}(t) = \lambda^{j/2}\phi(\lambda^j t - k)$, is a Riesz basis of $V_j^{(G)}$. Integrating equation(2.4) and dividing by the (non-vanishing) integral of ϕ, we have

$$\sum_{k=-\infty}^{\infty} h_k = 1.$$

A function $\phi^{(\lambda)} \in L^2(\mathbb{R})$ is called a generalized Haar scaling function, if the subspace

$$V_j^{(G)} = clos_{L^2(\mathbb{R})}\left\{\phi_{j,k}^{(\lambda)}; k \in \mathbb{Z}\right\}, j \in \mathbb{Z}$$

satisfy the properties (1) to (5) stated above in this section. It is important to note that the generalized Haar scaling function $\phi^{(\lambda)}$ generates a multiresolution analysis $\left\{V_j^{(G)}\right\}$ of $L^2(\mathbb{R})$.

Definition 2.7. Projection $P_N(f)$: Let $P_N(f)$ be the orthogonal projection of $L^2(\mathbb{R})$ onto V_N. Then

$$(P_N f)(t) = \sum_{k=0}^{\lambda^N-1} < f, \phi^{(\lambda)}_{N,k} > \phi^{(\lambda)}_{N,k}(t), N = 1, 2, 3..., [18].$$

Definition 2.8. Wavelet approximation: The wavelet approximation $E_N(f)$ of f by $P_N(f)$ under $||\ \ ||_2$ is defined by

$$E_N(f) \quad = \quad min||f - P_N(f)||_2.$$

Let

$$(S_J f)(t) = \sum_{k=0}^{\lambda^N-1} < f, \phi^{(\lambda)}_{N.k} > \phi^{(\lambda)}_{N,k}(t) + \sum_{j=N}^{J} \sum_{k=0}^{\lambda^j-1} < f, \psi^{(\lambda)}_{j,k} > \psi^{(\lambda)}_{j,k}(t).$$

Wavelet approximation of f by $(S_J f)(t)$ is defined by

$$E_J(f) = min||f - S_J(f)||_2.$$

If $E_N(f) \to 0$ as $N \to \infty$ then $E_N(f)$ is called the best approximation of f of order N (Zygmund [19], p.115).

Definition 2.9. Mean value theorem for definite integrals: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then there exists c in $[a, b]$ such that

$$\int_a^b f(t)dt = f(c)(b - a) \quad \text{(Rudin [20])}.$$

3. Theorems

In this paper, we prove the following theorems;

Theorem 3.1. *If a function $f \in L^2[0, 1]$ is continuous in [0,1] and its wavelet expansion by generalized Haar scaling function $\phi^{(\lambda)}$ and generalized Haar wavelet $\psi^{(\lambda)}$ is given by*

$$\begin{aligned} f(t) &= \sum_{k=0}^{\lambda^N-1} < f, \phi^{(\lambda)}_{N,k} > \phi^{(\lambda)}_{N,k}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} < f, \phi^{(\lambda)}_{j,k} > \psi^{(\lambda)}_{j,k}(t) \\ &= (P_N f)(t) + (R_N f)(t), \end{aligned}$$

where $\phi^{(\lambda)}_{N,k}(t) = \lambda^{N/2}\phi(\lambda^N t - k)$ and $\psi^{(\lambda)}_{j,k}(t) = \lambda^{N/2}\psi(\lambda^j t - k), \mu = 1, 2, 3...,$ then the wavelet approximation $E_N(f)$ of f by $P_N(f)$ satisfies;

$$E^{(1)}_N(f) = ||f - P_N(f)||_2 = O\left(\frac{1}{\lambda^{N/2}}\right)$$

and

$$||R_N(f)||_2 = ||\sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^N-1} < f, \psi^{(\lambda)}_{j,k} > \psi^{(\lambda)}_{j,k}|| = O\left(\frac{1}{\lambda^{N/2}}\right).$$

Proof. By defining the error between $f(t)$ and expansion over any subinterval as;

$$(e_k f)(t) =< f, \phi^{(\lambda)}_{N,k} > \phi^{(\lambda)}_{N,k}(t) - f(t), t \in \left[\frac{k}{\lambda^N}, \frac{k+1}{\lambda^N}\right), N = 1, 2, 3...$$

We obtain

$$\begin{aligned}
||e_k(f)||_2^2 &= \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} |e_k f(t)|^2 dt \\
&= \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} | < f, \phi^{(\lambda)}_{N,k} > \phi^{(\lambda)}_{N,k}(t) - f(t)|^2 dt \\
&= \left(< f, \phi^{(\lambda)}_{N,k} > \lambda^{N/2} - f(\zeta_k)\right)^2 \frac{1}{\lambda^N}, \zeta_k \in \left[\frac{k}{\lambda^N}, \frac{k+1}{\lambda^N}\right), \qquad (3.1)
\end{aligned}$$

by first mean value theorem for integrals.

Now,

$$< f, \phi^{(\lambda)}_{N,k} >= \lambda^{N/2} \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} f(t)dt = \frac{1}{\lambda^{N/2}} f(\rho_k), \rho_k \in \left[\frac{k}{\lambda^N}, \frac{k+1}{\lambda^N}\right), \qquad (3.2)$$

by first mean value theorem of integrals.

By Eqs.(3.1) and (3.2), we have

$$||e_k(f)||_2^2 = (f(\rho_k) - f(\zeta_k))^2 \frac{1}{\lambda^N}. \qquad (3.3)$$

Since $f(t)$ is continuous in the closed interval [0,1] and hence it is uniformly continuous in [0,1] and in each sub-interval of [0,1], therefore for $\epsilon = \frac{1}{\lambda^{N/2}} \; \exists \; \delta > 0$ s.t.,

$$|f(\rho_k) - f(\zeta_k)| \leq \frac{1}{\lambda^{N/2}}, \forall \rho_k, \zeta_k \in \left[\frac{k}{\lambda^N}, \frac{k+1}{\lambda^N}\right) < \delta. \qquad (3.4)$$

Then, from Eqs.(3.3) and (3.4), we have

$$||e_k(f)||_2^2 \leq \frac{1}{\lambda^{2N}}, \qquad (3.5)$$

which leads to;

$$\begin{aligned}
(E^{(1)}_N(f))^2 &= \int_0^1 \left(\sum_{k=0}^{\lambda^N-1} (e_k f)(t)\right)^2 dt \\
&= \int_0^1 \left(\sum_{k=0}^{\lambda^N-1} (e_k^2 f)(t)\right) dt + 2 \sum\sum\nolimits_{0\leq k\neq k' \leq \lambda^N-1} \int_0^1 (e_k f)(t)(e'_k f)(t)dt.
\end{aligned}$$

Due to disjointness of the supports of these basis functions, we have;

$$(E^{(1)}_N(f))^2 = \int_0^1 \left(\sum_{k=0}^{\lambda^N-1} e_k^2(f)\right) dt = \sum_{k=0}^{\lambda^N-1} ||e_k(f)||_2^2 \leq \sum_{k=0}^{\lambda^N-1} \left(\frac{1}{\lambda^{2N}}\right) = O\left(\frac{1}{\lambda^N}\right),$$

i.e.,

$$E^{(1)}_N(f) = O\left(\frac{1}{\lambda^{N/2}}\right).$$

Remainder operator $R_N(f)$ is given by

$$(R_N f)(t) = \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t).$$

Thus

$$\begin{aligned}
(R_N f(t))^2 &= \left(\sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) \right)^2 \\
&= \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >^2 \psi_{j,k}^{(\lambda)2}(t) \\
&+ \sum_{j=N}^{\infty} \sum_{0\leq k\neq} \sum_{k'\leq \mu^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j,k'}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t)\psi_{j,k'}^{(\lambda)}(t) \\
&+ \sum_{N\leq j\neq} \sum_{j'<\infty} \sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j',k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t)\psi_{j',k}^{(\lambda)}(t) \\
&+ \sum_{N\leq j\neq} \sum_{j'<\infty} \sum_{0\leq k\neq} \sum_{k'\leq \lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j',k'}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t)\psi_{j',k'}^{(\lambda)}(t).
\end{aligned}$$

Then,

$$\begin{aligned}
||R_N(f)||_2^2 &= \int_0^1 |(R_N f)(t)|^2 dt \\
&= \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^J-1} | < f, \psi_{j,k}^{(\lambda)} > |^2 \int_0^1 \psi_{j,k}^{(\lambda)}(t)^2 dt \\
&+ \sum_{j=N}^{\infty} \sum_{0\leq k\neq} \sum_{k'\leq \lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j,k'}^{(\lambda)} > \int_0^1 \psi_{j,k}^{(\lambda)}(x)\psi_{j,k'}^{(\lambda)}(t)dt \\
&+ \sum_{N\leq j\neq} \sum_{j'<\infty} \sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j',k}^{(\lambda)} > \int_0^1 \psi_{j,k}^{(\lambda)}(t)\psi_{j',k}^{(\lambda)}(t)dt \\
&+ \sum_{N\leq j\neq} \sum_{j'<\infty} \sum_{0\leq k\neq} \sum_{k'\leq \lambda^j-1} < f, \psi_{j,k}^{(\lambda)} >< f, \psi_{j',k'}^{(\lambda)} > \int_0^1 \psi_{j,k}^{(\lambda)}(t)\psi_{j',k'}^{(\lambda)}(t)dt \\
&= \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} | < f, \psi_{j,k}^{(\lambda)} > |^2 ||\psi_{j,k}^{(\lambda)}||^2, \text{by orthogonality of } \psi_{j,k}^{(\lambda)} \\
&= \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} | < f, \psi_{j,k}^{(\lambda)} > |^2. \qquad (3.6)
\end{aligned}$$

Since

$$\begin{aligned}
|< f,\psi_{j,k}^{(\lambda)} >| &= |\int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\mu^j}} f(t)\psi_{j,k}^{(\lambda)}(t)dt| = \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} |f(t)||\psi_{j,k}^{(\lambda)}(t)|dt \\
&\leq \left|f\left(\frac{k+1}{\lambda^j}\right)\right| \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} |\psi_{j,k}^{(\lambda)}(t)|dt, \text{by first mean value theorem for integrals} \\
&= \left|f\left(\frac{k+1}{\lambda^j}\right)\right| \frac{1}{\lambda^{j/2}}.
\end{aligned}$$

Next,

$$\begin{aligned}
\sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1} |< f,\psi_{j,k}^{(\lambda)} >|^2 &\leq \sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1} \left|f\left(\frac{k+1}{\lambda^j}\right)\right|^2 \frac{1}{\lambda^j} = \sum_{j=N}^{\infty}\left(\frac{1}{\lambda^j}\right)\sum_{k=0}^{\lambda^j-1}\left|f\left(\frac{k+1}{\lambda^j}\right)\right|^2 \\
&= \sum_{j=N}^{\infty} O\left(\frac{1}{\lambda^j}\right), \text{since } \sum_{k=0}^{\lambda^j-1}\left|f\left(\frac{k+1}{\lambda^j}\right)\right|^2 = O(1) \\
&= O\left(\frac{\frac{1}{\lambda^N}}{1-\frac{1}{\lambda}}\right) = O\left(\frac{1}{\lambda^N}\right). \qquad (3.7)
\end{aligned}$$

By Eqs.(3.6) and (3.7), we have

$$||R_N(f)||_2^2 = O\left(\frac{1}{\lambda^N}\right).$$

Hence,

$$||R_N(f)||_2 = O\left(\frac{1}{\lambda^{N/2}}\right).$$

Thus the proof of Theorem 3.1 is completely established. ∎

Theorem 3.2. *Let $f \in L^2[0,1]$ is continuous in [0,1] and*

$$\begin{aligned}
(S_J f)(t) &= \sum_{k=0}^{\lambda^N-1} < f,\phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{J}\sum_{k=0}^{\lambda^j-1} < f,\psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) \\
&= (P_N f)(t) + (R_N(f) - R_J(f))(t),
\end{aligned}$$

then the Haar wavelet approximation $E_J(f)$ of f by $S_J(f)$ of its wavelet expansion is given by

$$E_J^{(2)}(f) = min||f - S_J(f)||_2 = O\left(\frac{1}{\lambda^{(J+1)/2}}\right).$$

Proof. Now,

$$\begin{aligned}
f(t) - S_J(f)(t) &= \sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1} < f,\psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) - \sum_{j=N}^{J}\sum_{k=0}^{\lambda^j-1} < f,\psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) \\
&= \sum_{j=J+1}^{\infty}\sum_{k=0}^{\lambda^j-1} < f,\psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t).
\end{aligned}$$

Then,

$$(f(t)-(S_Jf)(t))^2=\left(\sum_{j=J+1}^{\infty}\sum_{k=0}^{\lambda^j-1}<f,\psi_{j,k}^{(\lambda)}>\psi_{j,k}^{(\lambda)}(t)\right)^2.$$

Following the proof of Theorem 3.1, we have

$$||f-S_J(f)||_2^2=\sum_{j=J+1}^{\infty}\sum_{k=0}^{\lambda^j-1}|<f,\psi_{j,k}^{(\lambda)}>|^2=O\left(\frac{1}{\lambda^{J+1}}\right).$$

So

$$E_J^{(2)}(f)=min||f-S_J(f)||=O\left(\frac{1}{\lambda^{(J+1)/2}}\right).$$

Thus the Theorem 3.2 is proved. ■

Theorem 3.3. *$f\in Lip_\alpha[0,1), 0<\alpha\le 1$, i.e.,$|f(x+t)-f(x)|=O(|t|^\alpha)$ then*

$$E_N^{(3)}(f)=O\left(\frac{1}{\lambda^{\alpha N}}\right)\ \ and\ \ ||R_N(f)||_2=O\left(\frac{1}{\lambda^{\alpha N}}\right).$$

Proof. Following the proof of Theorem 3.1,

$$||e_k(f)||_2^2=(f(\rho_k)-f(\zeta_k))^2\,\frac{1}{\lambda^N}. \tag{3.8}$$

Since $f\in Lip_\alpha[0,1)$, therefore

$$|f(\rho_k)-f(\zeta_k)|=O(|\rho_k-\zeta_k|^\alpha),\forall\rho_k,\zeta_k\in\left[\frac{k}{\lambda^N},\frac{k+1}{\lambda^N}\right). \tag{3.9}$$

Then, from Eqs.(3.8) and (3.9), we have

$$||e_k(f)||_2^2=O(|\rho_k-\zeta_k|^\alpha)^2\frac{1}{\lambda^N}=O\left(\frac{1}{\lambda^{2\alpha N}}\right)\frac{1}{\lambda^N}=O\left(\frac{1}{\lambda^{(2\alpha+1)N}}\right),$$

which leads to;

$$(E_N^{(3)}(f))^2=\int_0^1\left(\sum_{k=0}^{\lambda^N-1}(e_kf)(t)\right)^2dt=\sum_{k=0}^{\lambda^N-1}||e_k(f)||_2^2=O\left(\frac{1}{\lambda^{2\alpha N}}\right).$$

Thus, $E_N^{(3)}(f)=O\left(\frac{1}{\lambda^{\alpha N}}\right)$.
Also,

$$||R_N(f)||_2^2=\sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1}|<f,\psi_{j,k}^{(\lambda)}>|^2.$$

Now,

$$\begin{aligned}
| < f,\psi_{j,k}^{(\lambda)} > | &= \left| \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} f(t)\psi_{j,k}^{(\lambda)}(t)dt \right| \\
&= \left| \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \left(f(t) - f\left(\frac{k}{\lambda^j}\right) + f\left(\frac{k}{\lambda^j}\right)\right) \psi_{j,k}^{(\lambda)}(t)dt \right| \\
&\leq \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \left| f(t) - f\left(\frac{k}{\lambda^j}\right)\right| |\psi_{j,k}^{(\lambda)}(t)|dt \\
&+ \left| f\left(\frac{k}{\lambda^j}\right)\right| \left| \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \psi_{j,k}^{(\lambda)}(t)dt \right| \\
&= \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \left| f(t) - f\left(\frac{k}{\lambda^j}\right)\right| |\psi_{j,k}^{(\lambda)}(t)|dt + 0, \text{since } \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \psi_{j,k}^{(\lambda)}dt = 0 \\
&= O\left(\frac{1}{(\lambda^j)^\alpha}\right) \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} |\psi_{j,k}^{(\lambda)}(t)|dt = O\left(\frac{1}{(\lambda^j)^\alpha}\right)\frac{1}{\lambda^{j/2}}.
\end{aligned}$$

Next,

$$||R_N(f)||_2^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} | < f,\psi_{j,k}^{(\lambda)} > |^2 = O\left(\frac{1}{\lambda^{2\alpha N}}\right).$$

Thus, $||R_N f||_2 = O\left(\frac{1}{\lambda^{\alpha N}}\right)$. Thus Theorem 3.3 is completely established. ∎

Theorem 3.4. *Let $\omega(t)$ is a positive monotonic increasing function of t such that*

$$f(x+t) - f(x) = O(\omega(t)|t|^\alpha), 0 < \alpha < 1$$

and $\omega(t)|t|^\alpha \to 0$ as $t \to 0^+$. Then the wavelet approximation $E_N(f)$ of a function $f \in Lip_\alpha^{(\omega)}[0,1)$ satisfies;

$$E_N^{(4)}(f) = min||f - P_N(f)||_2 = O\left(\frac{\omega(1/\lambda^N)}{\lambda^{\alpha N}}\right)$$

and

$$||R_N(f)||_2 = || \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} < f,\psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}||_2 = O\left(\frac{\omega(1/\lambda^N)}{\lambda^{\alpha N}}\right).$$

Proof. Following the proof of Theorem 3.1,

$$||e_k(f)||_2^2 = (f(\rho_k) - f(\zeta_k))^2 \frac{1}{\lambda^N}. \tag{3.10}$$

Since $f \in Lip_\alpha^{(\omega)}[0,1)$, therefore

$$|f(\rho_k) - f(\zeta_k)| = O(\omega(\rho_k - \zeta_k)|\rho_k - \zeta_k|^\alpha), \forall \rho_k, \zeta_k \in \left[\frac{k}{\lambda^N}, \frac{k+1}{\lambda^N}\right). \tag{3.11}$$

Then from Eqs.(3.10) and (3.11), we have

$$||e_k(f)||_2^2 = O(\omega(\rho_k - \zeta_k)|\rho_k - \zeta_k|^\alpha)^2 \frac{1}{\lambda^N} = O\left(\frac{\omega(1/\lambda^N)}{\lambda^{\alpha N}}\right)^2 \frac{1}{\lambda^N},$$

which leads to;

$$(E_N^{(4)}(f))^2 = \sum_{k=0}^{\lambda^N-1} ||e_k(f)||_2^2 = \sum_{k=0}^{\lambda^N-1} O\left(\frac{\omega^2(1/\lambda^N)}{\lambda^{(2\alpha+1)N}}\right) = O\left(\frac{\omega^2(1/\lambda^N)}{\mu^{2\alpha N}}\right).$$

Thus, $E_N^{(4)}(f) = O\left(\frac{\omega(1/\lambda^N)}{\lambda^{\alpha N}}\right)$.
Also,

$$||R_N(f)||_2^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2.$$

Since

$$< f, \psi_{j,k}^{(\lambda)} >= \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} f(x)\psi_{j,k}^{(\lambda)}(t)dt.$$

Then,

$$\begin{aligned} |< f, \psi_{j,k}^{(\lambda)} >| &\leq \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \left|f(x) - f\left(\frac{k}{\lambda^j}\right)\right| |\psi_{j,k}^{(\lambda)}(t)|dt + \left|f\left(\frac{k}{\lambda^j}\right)\right| |\int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} \psi_{j,k}^{(\lambda)}(t)dt| \\ &\leq \left|f\left(\frac{k+1}{\lambda^j}\right) - f\left(\frac{k}{\lambda^j}\right)\right| \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} |\psi_{j,k}^{(\lambda)}(t)|dt, \text{ by first mean value theorem} \\ &= O\left(\frac{\omega(1/\lambda^j)}{\lambda^{\alpha N}}\right) \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} |\psi_{j,k}^{(\lambda)}(t)|dt = O\left(\frac{\omega(1/\lambda^j)}{\lambda^{\alpha j}}\right) \frac{1}{\lambda^{j/2}}. \end{aligned}$$

Next,

$$||R_N(f)||_2^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2 = O\left(\frac{\omega^2\left(\frac{1}{\lambda^N}\right)}{\lambda^{2\alpha N}}\right).$$

Thus,

$$||R_N(f)||_2 = O\left(\frac{\omega(1/\lambda^N)}{\lambda^{\alpha N}}\right).$$

This proves the Theorem 3.4. ∎

4. Corollaries

Following corollaries are deduced from Theorem 3.4;

Corollary 4.1. *If* $\omega(t) = \frac{1}{t^\beta}, 0 < \beta < \alpha \leq 1$, *i.e.,*

$$|f(x+t) - f(x)| = O(|t|^{\alpha-\beta}), 0 < \beta < \alpha \leq 1$$

then the wavelet approximation $E_N(f)$ *of* f *satisfies;*

$$E_N(f) = O\left(\frac{1}{\lambda^{(\alpha-\beta)N}}\right) \quad and \quad ||R_N(f)||_2 = O\left(\frac{1}{\lambda^{(\alpha-\beta)N}}\right).$$

Proof: Proof of the Cor.4.1 can be developed parallel to the proof of Theorem 3.4 by taking $\omega(t) = \frac{1}{t^\beta}, 0 < \beta < \alpha \leq 1, \forall t \in (0,1)$.

Corollary 4.2. *If* $f \in Lip_\alpha^{(\omega)}[0,1)$ *and*

$$f(t) = \sum_{k=0}^{2^N-1} < f, \phi_{N,k} > \phi_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} < f, \psi_{j,k} > \psi_{j,k}(t),$$

where $\phi_{N,k}(t) = 2^{N/2}\phi(2^N t - k)$ *and* $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ *then*

$$E_N(f) = O\left(\frac{\omega(1/2^N)}{2^{\alpha N}}\right) \quad and \quad ||R_N(f)||_2 = O\left(\frac{\omega(1/2^N)}{2^{\alpha N}}\right).$$

Proof: Proof of the Cor.4.2 can be obtained following the same lines of the proof of Theorem 3.4.

Corollary 4.3. *If* $f \in Lip_\alpha[0,1)$ *and*

$$f(t) = \sum_{k=0}^{2^N-1} < f, \phi_{N,k} > \phi_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} < f, \psi_{j,k} > \psi_{j,k}(t),$$

where $\phi_{N,k}$ *and* $\psi_{j.k}$ *are defined as Cor.4.2 then*

$$E_N(f) = O\left(\frac{1}{2^{\alpha N}}\right) \quad and \quad ||R_N(f)||_2 = O\left(\frac{1}{2^{\alpha N}}\right).$$

Proof: Proof of Cor.4.3 is very trivial.

5. Remark

Remark 5.1. Christensen([9], p.126) proved;

Theorem 5.2. *If* $x_0 \in [0,1] - \mathbb{Q}$ *and* $f = \chi_{[0,x_0]}$ *then*

$$||f - < f, \chi_{[0,1)} > \chi_{[0,1)} - \sum_{j=0}^{J-1}\sum_{n=0}^{2^j-1} < f, \psi_{j,n} > \psi_{j,n}||_2 = O\left(\frac{1}{2^{J/2}}\right).$$

Remark 5.3. Walnut([16], Lemma(5.37), p.135) proved the following Lemma;

Theorem 5.4. *Given* $f(x) \in C_c^0$ *on* $\mathbb{R}$*,then* $lim_{N\to\infty}||P_N(f) - f||_2 = 0$.

Remark 5.5. Let $f(t) = t, \forall t \in [0,1)$. Then $f \in Lip_1[0,1]$.

$$||f||_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_0^1 t^2 dt = \frac{1}{3}. \tag{5.1}$$

Thus $f \in L^2[0,1)$. Then wavelet series of f using generalized Haar scaling function $\phi^{(\lambda)}$ and generalized Haar wavelet $\psi^{(\lambda)}$ is given by

$$f(t) = \sum_{k=0}^{\lambda^N-1} < f, \phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{\lambda^j-1} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t).$$

By orthogonality of $\psi_{j,k}^{(\lambda)}$, we have

$$||f||_2^2 = \sum_{k=0}^{\lambda^N-1} |< f, \phi_{N,k}^{(\lambda)} >|^2 + \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2. \tag{5.2}$$

Now,

$$< f, \phi_{N,k}^{(\lambda)} >= \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} f(t)\phi_{N,k}^{(\lambda)}(t)dt = \int_{\frac{k}{\lambda^N}}^{\frac{k+1}{\lambda^N}} t\lambda^{N/2}\phi(\lambda^N t - k)dt = \frac{1}{\lambda^{3N/2}}\left(k + \frac{1}{2}\right).$$

Then,

$$\sum_{k=0}^{\lambda^N-1} |< f, \phi_{N,k}^{(\lambda)} >|^2 = \sum_{k=0}^{\lambda^N-1} \frac{1}{\lambda^{3N}}\left(k + \frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{12}\frac{1}{\lambda^{2N}}. \tag{5.3}$$

Now,

$$< f, \psi_{j,k}^{(\lambda)} >= \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} f(t)\psi_{j,k}^{(\lambda)}(t)dt = \int_{\frac{k}{\lambda^j}}^{\frac{k+1}{\lambda^j}} t\lambda^{j/2}\psi(\lambda^j t - k)dt = -\frac{1}{4}\frac{1}{\lambda^{3j/2}}.$$

Next,

$$\sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2 = \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} \frac{1}{16}\frac{1}{\lambda^{3j}} = \frac{1}{16}\frac{1}{\lambda^{2N}}\left(\frac{\lambda^2}{\lambda^2 - 1}\right). \tag{5.4}$$

Substituting values from Eqs.(5.3) and (5.4) in Eq.(5.2), we have

$$\begin{aligned}
\sum_{k=0}^{\lambda^N-1} |< f, \phi_{N,k}^{(\lambda)} >|^2 + \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2 &= \frac{1}{3} - \frac{1}{12}\frac{1}{\lambda^{2N}} + \frac{1}{16}\left(\frac{\lambda^2}{\lambda^2 - 1}\right)\frac{1}{\lambda^{2N}} \\
&= \frac{1}{3} - \frac{1}{\lambda^{2N}}\left[\frac{1}{12} - \frac{1}{16}\left(\frac{\mu^2}{\lambda^2 - 1}\right)\right] \\
&\le \frac{1}{3} = ||f||_2^2.
\end{aligned}$$

Thus the series

$$\sum_{k=0}^{\lambda^N-1} |< f, \phi_{N,k}^{(\lambda)} >|^2 + \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2$$

is convergent and

$$\sum_{k=0}^{\lambda^N-1} |< f, \phi_{N,k}^{(\lambda)} >|^2 + \sum_{j=N}^{\infty} \sum_{k=0}^{\lambda^j-1} |< f, \psi_{j,k}^{(\lambda)} >|^2 \le ||f||_2^2.$$

6. Conclusions

1. Theorem 5.2 (Christensen[9] p.126) and Theorem 5.4 (Walnut[16] Lemma 5.37, p.135) are particular cases of our Theorem 3.4.

2. $E_N^{(1)}(f) = O\left(\frac{1}{\lambda^{N/2}}\right) \to 0$ as $N \to \infty$,
$E_J^{(2)}(f) = O\left(\frac{1}{\lambda^{(J+1)/2}}\right) \to 0$ as $J \to \infty$,
$E_N^{(3)}(f) = O\left(\frac{1}{\lambda^{\alpha N}}\right) \to 0$ as $N \to \infty$,
$E_N^{(4)}(f) = O\left(\frac{\omega\left(\frac{1}{\lambda^N}\right)}{\lambda^{\alpha N}}\right) \to 0$ as $N \to \infty$.

Then wavelet approximation $E_N^{(1)}(f), E_J^{(2)}(f), E_N^{(3)}(f)$ and $E_N^{(4)}(f)$ are best possible in wavelet analysis(Zygmund[19]).

3. $f(t) = \sum_{k=0}^{\mu^N-1} < f, \phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{\mu^j-1} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t).$

$$f(t) = (P_N f)(t) + (R_N f)(t), \text{i.e.}, f(t) - (P_N f)(t) = (R_N f)(t).$$

Researchers have estimated either $||f - P_N(f)||_2$ or $||R_N(f)||_2$ for wavelet estimation $E_N(f)$ for very easy functions under uncomplicated supposition. In this paper, in Theorems 3.1, 3.3 and 3.4, $||f - P_N(f)||_2$ and $||R_N(f)||_2$ are calculated by slightly different methods and their estimates are same. This is a remarkable achievement of this research paper in approximation of functions in wavelet theory.

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