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Fractional Differential Equations Associated with Generalized Fractional Operators in $F_{p,\nu}$ Space

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Abstract The key aim of the present work is to study the fractional differential equations (FDEs) pertaining to generalized fractional operators in $F_{p,\nu}$ space. First, we derive the Laplace transform of the generalized fractional operators in terms of generalized modified Bessel function type transform $\mathbb{L}^{(\sigma)}_{\alpha,\beta}$ in $F_{p,\nu}$ space. The results obtained are used to solve the FDEs involving constant as well as variable coefficients in $F_{p,\nu}$ space. Due to the general nature of M-S-M integral operators many new and useful special cases of the key results can be obtained.

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1. Introduction and Definitions

It is known that there exists a clear structural and spatial interpretation for the integral derivatives and integrations. However, unlike the integer-order calculus. Fractional calculus represents a rapidly growing field both in theoretical and in applied aspects to scientific and engineering problems, it is not so. Arbitrary ordered derivatives and integrals offer an outstanding instrument for the illustrations of memory effects along with hereditary properties of different materials and processes in this reference one can see the work conducted by Miller and Ross [1], Podlubny [2], Srivastava et al. [3, 4], Singh et al. [5], Choudhary et al. [6], Zhao et al. [7], etc.
Motivated by the well proven ability for their uses in significant investigation areas such as mathematical biology, biomedical engineering, and statistical sciences. The primary aim of this work is to study the applications of fractional differential equations with aid

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of the M-S-M arbitrary-order integral operator.
We recall here fractional integral operators [8, 9] involving Appell's hypergeometric function F_3 (see details, [10, p. 23, Eq. 4]) as defined as below:

$$\left(I_{0,u}^{\alpha,\beta,\sigma,\delta,\eta}\psi\right)(u) = \frac{u^{-\alpha}}{\Gamma(\eta)}\int_0^u (u-v)^{\eta-1}v^{-\beta}F_3\left(\alpha,\beta,\sigma,\delta;\eta;1-\frac{v}{u},1-\frac{u}{v}\right)\psi(v)dv,$$
$$(u>0 \quad \text{and} \quad \alpha,\beta,\sigma,\delta,\eta\in\mathbb{C}, \qquad \Re(\eta)>0) \tag{1.1}$$

and

$$\left(I_{u,\infty}^{\alpha,\beta,\sigma,\delta,\eta}\psi\right)(u) = \frac{u^{-\beta}}{\Gamma(\eta)}\int_u^\infty (v-u)^{\eta-1}v^{-\alpha}F_3\left(\alpha,\beta,\sigma,\delta;\eta;1-\frac{u}{v},1-\frac{v}{u}\right)\psi(v)dv,$$
$$(u>0 \quad \text{and} \quad \alpha,\beta,\sigma,\delta,\eta\in\mathbb{C}, \qquad \Re(\eta)>0) \tag{1.2}$$

given that the above integrand is existing.
We recall here classical definition of Riemann-Liouville (RL) fractional operators [1, 11, 12] defined by:

$$(J_{a+}^{\rho}\psi)(u) = \frac{1}{\Gamma(\rho)}\int_a^u \frac{\psi(t)}{(u-t)^{1-\rho}}\,dt, \qquad \left(\Re(\rho)>0\right) \tag{1.3}$$

and

$$(D_{a+}^{\rho}\psi)(u) = \left(\frac{d}{du}\right)^n (J_{a+}^{n-\rho}\psi)(u), \qquad \left(\Re(\rho)>0;\ n=[\Re(\rho)]+1\right), \tag{1.4}$$

where [.] is the greatest integer function.
Hilfer fractional derivative operator [13] is more generalization of R-L and Caputo operator which is defined by:

$$(D_{a+}^{\rho,\kappa}\psi)(u) = \left(J_{a+}^{\kappa(1-\rho)}\frac{d}{du}\left(J_{a+}^{(1-\kappa)(1-\rho)}\psi\right)\right)(u), \qquad (0<\rho<1, 0\leq\kappa\leq 1). \tag{1.5}$$

If $\kappa=0$ and $\kappa=1$, then (1.5) reduces into R-L operator (1.4) and Caputo Operator [12], respectively.
The following result is recalled (see, for details, [14] and [15]):

$$\mathcal{L}[(D_{a+}^{\rho,\kappa}\psi)(u)](s) = s^{\rho}\mathcal{L}[\psi(u)](s) - s^{-\kappa(1-\rho)}\left(J_{0+}^{(1-\kappa)(1-\rho)}\psi\right)(0+),$$
$$\left(\Re(s)>0;\ 0<\rho<1\right). \tag{1.6}$$

The $F_{p,\nu}$ space [16–18] is defined by

$$F_{p,\nu} = \left\{g\in C_0^\infty(\mathbb{R}^+) : x^k\frac{d^k}{dx^k}(x^{-\nu}g(x))\in L^p(\mathbb{R}^+)(k\in\mathbb{N}_0)\right\}, \tag{1.7}$$

where

$$(\nu\in\mathbb{C}, 1\leq p\leq\infty).$$

The modified Bessel-type integral transform $\mathbb{L}_{\gamma,\sigma}^{(\beta)}$ is given by [19, p.159, Eq.(4.1)]

$$\left(\mathbb{L}_{\alpha,\beta}^{(\sigma)}g\right)(u) = \int_0^\infty \lambda_{\alpha,\beta}^{(\sigma)}(uw)g(w)dw, \tag{1.8}$$

where $\lambda_{\alpha,\beta}^{(\sigma)}(w)$ is an extension of modified Bessel function of the third kind, which was given by Kilbas et al. [20]

$$\lambda_{\alpha,\beta}^{(\sigma)}(w) = \frac{\sigma}{\Gamma(\alpha+1-1/\sigma)} \int_1^{\infty} (t^{\sigma}-1)^{\alpha-1/\sigma} t^{\beta} e^{-wt} dt, \tag{1.9}$$

$$\left(\sigma > 0; \quad \Re(\beta) > \frac{1}{\sigma} - 1; \quad \beta \in \mathbb{R}; \quad \Re(w) > 0\right).$$

Bonilla et al.[21] used this kind of function to give the solutions of certain homogeneous FDEs and Volterra integral equations.
The H-function is given by [22, p. 10]:

$$H_{p,q}^{m,n}[w] = H_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_j, E_j)_{1,p} \\ (f_j, F_j)_{1,q} \end{matrix}\right] = H_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_1, E_1), \cdots, (e_p, E_p) \\ (f_1, F_1), \cdots, (f_q, F_q) \end{matrix}\right]$$
$$= \frac{1}{2\pi i} \int_{\mathfrak{L}} \Theta(s)\, w^{s}\, ds. \tag{1.10}$$

In Eq. (1.10), the value of $\Theta(s)$ is expressed as

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(f_j - F_j s) \prod_{j=1}^{n} \Gamma(1 - e_j + E_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - f_j + F_j s) \prod_{j=n+1}^{p} \Gamma(e_j - E_j s)}, \tag{1.11}$$

and the set of conditions and details are given in the text book [12, 22].

2. Classical Transform of the Generalized Fractional Operator in Terms of Generalized Modified Bessel Function Type Transform

Laplace transforms (LT) of a function $\varphi(z)$ is presented as [23]

$$\mathcal{L}[\varphi(u)](s) = \int_0^{\infty} e^{-su} \varphi(u) du, \qquad \Re(s) > 0, \tag{2.1}$$

provided that the existence conditions for the integral (2.1) are satisfied. Now, we present the integral transform result for the generalized fractional operator in terms of extended modified Bessel function type transform.

Theorem 2.1. *If $u > 0, \Re(s) > 0$, $\alpha, \eta, \beta, \sigma, \rho, \delta \in \mathbb{C}$ and $\Re(\eta) > 0$. Then, the LT of generalized fractional operator in terms of generalized modified Bessel function type transform is given as follows:*

$$\mathcal{L}\left[u^{\rho-1}\left(I_{0,u}^{\alpha,\beta,\sigma,\delta,\eta}\psi\right)(u)\right](s)$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n(\sigma)_m(\delta)_n}{m!n!}(-1)^n s^{\alpha+\beta-\rho-\eta+1}\left(\mathbb{L}_{m-\rho+\alpha,\beta-\eta-m}\psi\right)(s), \tag{2.2}$$

where $\psi \in F_{p,\nu}$.

Proof. To establish (2.2), first, Laplace transform of (1.1), is taken and following (say $\Delta(\mathsf{s})$) is obtained:

$$\Delta(\mathsf{s}) = \int_0^\infty e^{-\mathsf{su}}\mathsf{u}^{\rho-1}\left[\frac{\mathsf{u}^{-\alpha}}{\Gamma(\eta)}\int_0^{\mathsf{u}}(\mathsf{u}-\mathsf{v})^{\eta-1}\mathsf{v}^{-\beta}F_3\left(\alpha,\beta,\sigma,\delta;\eta;1-\frac{\mathsf{v}}{\mathsf{u}},1-\frac{\mathsf{u}}{\mathsf{v}}\right)\psi(\mathsf{v})d\mathsf{v}\right]d\mathsf{u}.$$

Further, Appell's Function F_3 is expressed in series and then the order of series and u,v-integrals is interchanged (the conditions stated permit such an interchange), following is obtained:

$$\Delta(\mathfrak{s}) = \sum_{\mathsf{m},\mathsf{n}=0}^{\infty}\frac{(\alpha)_\mathsf{m}(\beta)_\mathsf{n}(\sigma)_\mathsf{m}(\delta)_\mathsf{n}}{\Gamma(\eta)(\eta)_{\mathsf{m}+\mathsf{n}}\mathsf{m}!\mathsf{n}!}(-1)^\mathsf{n}\int_0^\infty \mathsf{v}^{-\beta-\mathsf{n}}\psi(\mathsf{v}) \times\left[\int_\mathsf{v}^\infty e^{-\mathsf{su}}\mathsf{u}^{\rho-\alpha-\mathsf{m}-1}(\mathsf{u}-\mathsf{v})^{\eta+\mathsf{m}+\mathsf{n}-1}d\mathsf{u}\right]d\mathsf{v}.$$

Next step involves evaluation of the resulting z-integral using [24, p.348, Eq.(4.11)]

$$\Delta(\mathfrak{s}) = \sum_{\mathsf{m},\mathsf{n}=0}^{\infty}\frac{(\alpha)_\mathsf{m}(\beta)_\mathsf{n}(\sigma)_\mathsf{m}(\delta)_\mathsf{n}}{\mathsf{m}!\mathsf{n}!}(-1)^\mathsf{n} \times\int_0^\infty e^{-\frac{\mathsf{sv}}{2}}\mathsf{v}^{-\beta-\mathsf{n}}\psi(\mathsf{v})\mathsf{s}^{-\frac{\rho-\alpha+\eta+\mathsf{n}}{2}}\mathsf{v}^{\frac{\rho-\alpha+\eta+\mathsf{n}-2}{2}}\mathbb{W}_{\frac{\rho-\alpha-\eta-2\mathsf{m}-\mathsf{n}}{2},\frac{1-\rho+\alpha-\eta-\mathsf{n}}{2}}(\mathsf{sv})d\mathsf{v}.$$

Further, using the result [22, p. 18, Eq. (2.6.7)], following is obtained:

$$\Delta(\mathsf{s}) = \sum_{\mathsf{m},\mathsf{n}=0}^{\infty}\frac{(\alpha)_\mathsf{m}(\beta)_\mathsf{n}(\sigma)_\mathsf{m}(\delta)_\mathsf{n}}{\mathsf{m}!\mathsf{n}!}(-1)^\mathsf{n}\mathsf{s}^{\alpha+\beta-\rho-\eta+1} \times\int_0^\infty H^{2,0}_{1,2}\left[\mathsf{sv}\left|\begin{matrix}(\eta-\beta+\mathsf{m},1)\\(-\beta-\mathsf{n},1),(\eta+\rho-\alpha-\beta,1)\end{matrix}\right.\right]\psi(\mathsf{v})d\mathsf{v}.$$

Now, with the aid of result [20, p.155, Eq.(2.6)], following equation is got:

$$\Delta(\mathfrak{s}) = \sum_{\mathsf{m},\mathsf{n}=0}^{\infty}\frac{(\alpha)_\mathsf{m}(\beta)_\mathsf{n}(\sigma)_\mathsf{m}(\delta)_\mathsf{n}}{\mathsf{m}!\mathsf{n}!}(-1)^\mathsf{n}\mathsf{s}^{\alpha+\beta-\rho-\eta+1}\int_0^\infty \lambda_{\mathsf{m}-\rho+\alpha,\beta-\eta-\mathsf{m}}(\mathsf{sv})\psi(\mathsf{v})d\mathsf{v}.$$

Finally, making use of [20, p.151, Eq.(1.1)], the require result is obtained: ∎

3. Solutions of FDEs Involving M-S-M Operator

The FDEs associated with the M-S-M operator (1.1) are given by Theorem 3.1 and Theorem 3.2 and solutions of these theorems are given in $F_{p,\nu}$ space.

Theorem 3.1. *If* $\tau > 0$, $\alpha,\eta,\beta,\sigma,\delta \in \mathbb{C}$ *and* $\Re(\eta) > 0$. *Then the solution of the following FDE with Hilfer fractional derivative*

$$\left(D^{\mu,\nu}y\right)(\tau) = \lambda\tau^{\rho-1}\left(I^{\alpha,\beta,\sigma,\delta,\eta}_{0,\tau}1\right) + \psi(\tau), \tag{3.1}$$

with initial condition (IC)

$$\left(J_{0+}^{(1-\nu)(1-\mu)}y\right)(0+) = C \tag{3.2}$$

is presented as

$$y(\tau) = \frac{C\tau^{\mu+\nu(1-\mu)-1}}{\Gamma(\mu+\nu(1-\mu))} + \lambda\Delta\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-1}}{\Gamma(\mu+\rho+\eta-\alpha-\beta)} + \frac{1}{\Gamma(\mu)}\int\limits_0^\tau(\tau-t)^{\mu-1}\psi(t)dt,$$

where

$$\Delta = \frac{\Gamma(1+\eta-\alpha-\beta-\sigma)\Gamma(1+\delta-\beta)\Gamma(\rho+\eta-\alpha-\beta)}{\Gamma(1+\delta)\Gamma(1+\eta-\alpha-\beta)\Gamma(1+\eta-\beta-\sigma)}. \tag{3.3}$$

Proof. Taking the LT on the both sides of (3.1), we obtain

$$\begin{aligned}\mathfrak{s}^{\mu}y(\mathfrak{s}) - C\mathfrak{s}^{-\nu(1-\mu)} =& \lambda\mathfrak{s}^{-\rho-\eta+\alpha+\beta}\frac{\Gamma(1+\eta-\alpha-\beta-\sigma)\Gamma(1+\delta-\beta)\Gamma(\rho+\eta-\alpha-\beta)}{\Gamma(1+\delta)\Gamma(1+\eta-\alpha-\beta)\Gamma(1+\eta-\beta-\delta)}\\ &+\bar{\psi}(\mathfrak{s})\end{aligned}$$

or

$$y(\mathfrak{s}) = C\mathfrak{s}^{-\mu-\nu(1-\mu)} + \lambda\mathfrak{s}^{-\mu-\rho-\eta+\alpha+\beta}\Delta + \mathfrak{s}^{-\mu}\bar{\psi}(\mathfrak{s}).$$

Now, on taking the inverse LT, we get

$$y(\tau) = \frac{C\tau^{\mu+\nu(1-\mu)-1}}{\Gamma(\mu+\nu(1-\mu))} + \lambda\Delta\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-1}}{\Gamma(\mu+\rho+\eta-\alpha-\beta)} + \frac{1}{\Gamma(\mu)}\int\limits_0^\tau(\tau-t)^{\mu-1}\psi(t)dt,$$

which is the solution of (3.1) for $y \in F_{p,\nu}$. ■

Theorem 3.2. *If $\tau > 0$, $\alpha, \eta, \beta, \sigma, \delta \in \mathbb{C}$ and $\Re(\eta) > 0$. Then, the solution of the following FDE with Hilfer fractional derivative*

$$\tau\left(D_{0+}^{\mu,\nu}y\right)(\tau) = \lambda\tau^{\rho-1}\left(I_{0,\tau}^{\alpha,\beta,\sigma,\delta,\eta}1\right)(\tau), \tag{3.4}$$

with initial condition

$$\left(J_{0+}^{(1-\nu)(1-\mu)}y\right)(0+) = C \tag{3.5}$$

is written as

$$y(\tau) = \frac{C\tau^{\mu+\nu(1-\mu)-1}}{\Gamma(\mu+\nu(1-\mu))} + \lambda\Delta'\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-2}}{\Gamma(\mu+\rho+\eta-\alpha-\beta-1)} + \frac{C_2\tau^{\mu-1}}{\Gamma(\mu)},$$

where

$$\Delta' = \frac{\Gamma(1+\eta-\alpha-\beta-\sigma)\Gamma(1+\delta-\beta)\Gamma(\rho+\eta-\alpha-\beta)}{\Gamma(1+\delta)\Gamma(1+\eta-\alpha-\beta)\Gamma(1+\eta-\beta-\sigma)(-\rho-\eta+\alpha+\beta+1)}. \tag{3.6}$$

Proof. Taking the LT of (3.4), we obtain

$$\frac{d}{ds}\left(\mathfrak{s}^{\mu}y(\mathfrak{s}) - C\mathfrak{s}^{-\nu(1-\mu)}\right) = \lambda\Delta\mathfrak{s}^{-\rho-\eta+\alpha+\beta}.$$

Now integrate on the both side, we get

$$\mathfrak{s}^{\mu}y(\mathfrak{s}) = C\mathfrak{s}^{-\nu(1-\mu)} + \lambda\Delta\frac{\mathfrak{s}^{-\rho-\eta+\alpha+\beta+1}}{(-\rho-\eta+\alpha+\beta+1)} + C_2$$

or

$$y(\mathfrak{s}) = C\mathfrak{s}^{-\mu-\nu(1-\mu)} + \lambda\Delta'\mathfrak{s}^{-\mu-\rho-\eta+\alpha+\beta+1} + C_2\mathfrak{s}^{-\mu}.$$

where Δ' is defined in (3.6) and taking the inversion of LT, we get

$$y(\tau) = \frac{C\tau^{\mu+\nu(1-\mu)-1}}{\Gamma(\mu+\nu(1-\mu))} + \lambda\Delta'\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-2}}{\Gamma(\mu+\rho+\eta-\alpha-\beta-1)} + \frac{C_2\tau^{\mu-1}}{\Gamma(\mu)},$$

which is the solution of (3.4) for $y \in F_{p,\nu}$. ∎

Setting $\nu = 0$ in Eq. (3.1), we find the below result:

Corollary 3.3. *If $\tau > 0$, $\alpha, \eta, \beta, \sigma, \delta \in \mathbb{C}$ and $\Re(\eta) > 0$. Then the solution of the following FDE with Riemann-Liouville (RL) fractional derivative*

$$(D^{\mu}y)(\tau) = \lambda\tau^{\rho-1}\left(I_{0,\tau}^{\alpha,\beta,\sigma,\delta,\eta}1\right)(\tau) + \psi(\tau), \tag{3.7}$$

with initial condition

$$\left(J_{0+}^{(1-\mu)}y\right)(0+) = C \tag{3.8}$$

is given by

$$y(\tau) = \frac{C\tau^{\mu-1}}{\Gamma(\mu)} + \lambda\Delta\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-1}}{\Gamma(\mu+\rho+\eta-\alpha-\beta)} + \frac{1}{\Gamma(\mu)}\int_0^{\tau}(\tau-t)^{\mu-1}\psi(t)dt,$$

where Δ is given in (3.3).

Setting $\nu = 0$ in Eq. (3.4), we find the subsequent result:

Corollary 3.4. *If $\tau > 0$ and $\alpha, \eta, \beta, \sigma, \delta \in \mathbb{C}$, $\Re(\eta) > 0$. Then the solution of the following FDE with RL fractional derivative*

$$\tau\,(D^{\mu}y)(\tau) = \lambda\tau^{\rho-1}\left(I_{0,\tau}^{\alpha,\beta,\sigma,\delta,\eta}1\right)(\tau), \tag{3.9}$$

with initial condition

$$\left(J_{0+}^{(1-\mu)}y\right)(0+) = C \tag{3.10}$$

is given by

$$y(\tau) = \frac{C\tau^{\mu-1}}{\Gamma(\mu)} + \lambda\Delta\frac{\tau^{\mu+\rho+\eta-\alpha-\beta-2}}{\Gamma(\mu+\rho+\eta-\alpha-\beta-1)} + \frac{C_2\tau^{\mu-1}}{\Gamma(\mu)},$$

where Δ' is given in (3.6).

Taking $\nu = 1$ in Eq. (3.1), we achieve the below result:

Corollary 3.5. *If $\tau > 0$, $\alpha, \eta, \beta, \sigma, \delta \in \mathbb{C}$ and $\Re(\eta) > 0$. Then, the solution of the following FDE with Caputo fractional derivative*

$$(D^{\mu}y)(\tau) = \lambda\tau^{\rho-1}\left(I_{0,\tau}^{\alpha,\beta,\sigma,\delta,\eta}1\right)(\tau) + \psi(\tau) \tag{3.11}$$

with initial condition

$$y(0) = C \tag{3.12}$$

is given by

$$y(\tau) = C + \lambda\Delta \frac{\tau^{\mu+\rho+\eta-\alpha-\beta-1}}{\Gamma(\mu+\rho+\eta-\alpha-\beta)} + \frac{1}{\Gamma(\mu)} \int_0^\tau (\tau - t)^{\mu-1} \psi(t)dt,$$

where Δ is given in (3.3).

Taking $\nu = 1$ in Eq. (3.4), we achieve the below result:

Corollary 3.6. *If $\tau > 0$, $\alpha, \eta, \beta, \sigma, \delta \in \mathbb{C}$ and $\Re(\eta) > 0$. Then, the solution of the following FDE with Caputo fractional derivative*

$$\tau \left(D^{\mu}_{0+}y\right)(\tau) = \lambda\tau^{\rho-1} \left(I^{\alpha,\beta,\sigma,\delta,\eta}_{0,\tau} 1\right)(\tau), \tag{3.13}$$

with initial condition

$$y(0) = C \tag{3.14}$$

is expressed as

$$y(\tau) = C + \lambda\Delta' \frac{\tau^{\mu+\rho+\eta-\alpha-\beta-2}}{\Gamma(\mu+\rho+\eta-\alpha-\beta-1)} + \frac{C_2\tau^{\mu-1}}{\Gamma(\mu)},$$

where Δ' is given in (3.6).

Remark : If we specialize the operator $I^{\alpha,\beta,\sigma,\delta,\eta}_{0,\tau}$ involved in the right hand side of (3.1) and (3.4) to Saigo operator. We obtain interesting unknown special cases of Theorems 3.1-3.2.

4. Conclusion

The generalized fractional operators defined by (1.1) and (1.2) are very general in nature and generalize a lot of known fractional integral operators. Further solution of Fractional differential equations associated with M-S-M fractional integral operator have been obtained in $F_{p,\nu}$ space. Therefore, it can be winded up that the solution obtained in this paper are very important with respect to physical mathematical problems and can lead to interpretation of many physical phenomena.

References

[1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1993.

[2] I. Podlubny, Fractional Differential Equations. California, USA: Academic Press, 1999.

[3] H.M. Srivastava, P. Harjule, R. Jain, A General Fractional Differential Equation Associated With an Integral Operator With the H-Function in the Kernel, Russian Journal of Mathematical Physics 22 (1) (2015) 112–126.

[4] H.M. Srivastava, D. Kumar, J. Singh, An efficient analytical technique for fractional model of vibration equation, Applied Mathematical Modelling 45 (2017) 192–204.

[5] J. Singh, D. Kumar, J.J. Nieto, A reliable algorithm for local fractional Tricomi equation arising in fractal transonic flow, Entropy 18 (6) (2016) 206.

[6] A. Choudhary, D. Kumar, J. Singh, Numerical Simulation of a fractional model of temperature distribution and heat flux in the semi infinite solid, Alexandria Engineering Journal 55 (1) (2016) 87–91.

[7] D. Zhao, J. Singh, D. Kumar, S. Rathore, X.J. Yang, An efficient computational technique for local fractional heat conduction equations in fractal media, Journal of Nonlinear Sciences and Applications 10 (2017) 1478–1486.

[8] O.I. Marichev, Volterra equation of Mellin convolution type with a Horn function in the kernel, Izvestiya Akademii Nauk BSSR. Seriya Fiziko-Matematicheskikh Nauk 1 (1974) 128–129.

[9] M. Saigo, N. Maeda, More generalization of fractional calculus, Transform Methods & Special Functions, Bulgarian Academy of Sciences, Sofia, Bulgaria, 96 (1998) 386–400.

[10] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited), John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.

[11] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives : Theory and Applications, Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania), 1993.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204 Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[13] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000.

[14] H.M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. 211 (2009) 198–210.

[15] Ž. Tomovski, R. Hilfer, H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integral Transforms Spec. Funct. 21 (2010) 797–814.

[16] A.C. McBride, A theory of fractional integrals of generalized functions, SIAM Journal on Mathematical Analysis 6 (1975) 583–599.

[17] A.C. McBride, Fractional Calculus and Integral Transforms of Generalized Functions, Res. Notes Math. 31 Pitman, San Francisco, 1979.

[18] A.C. McBride, Fractional powers of a class of ordinary differential operators, Proc. London Math. Soc. 45 (1982) 519–546.

[19] H.J. Glaeske, A.A. Kilbas, M. Saigo, A modified Bessel-type integral transform and its compositions with fractional calculus operators on spaces $F_{p,\mu}$ and $F'_{p,\mu}$, J. Comput. Appl. Math. 118 (2000) 151-168.

[20] A.A. Kilbas, M. Saigo, H.J. Glaeske, A modified transform of Bessel type and its compositions with operators of fractional integration and differentiation, Dokl. Nats. Akad. Nauk Belarusi 43 (1999) 26–30.

[21] B. Bonilla, A.A. Kilbas, M. Rivero, L. Rodriguez, J.J. Trujillo, Modified Bessel-type function and solution of differential and integral equations, Indian. J. Pure Appl. Math. 31 (2000) 93–109.

[22] H.M. Srivastava, K.C. Gupta, S.P. Goyal, The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi and Madras, 1982.

[23] L. Debnath, D. Bhatta, Integral Transforms and Their Applications, Chapman & Hall/CRC; Boca Raton, London, New York, 2007.

[24] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, 2007.