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Asymptotic Behavior of Convolution of Dependent Random Variables with Heavy-Tailed Distributions

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Abstract : In this paper, we study the asymptotic behavior of the tail of X_1+X_2 in a dependent framework; where X_1 and X_2 are two positive heavy-tailed random variables with continuous joint and common marginal distribution functions $F(x,y)$ and $F(x)$, respectively; and for some classes of heavy-tailed distributions, we obtain some bounds and convolution properties. Furthermore, we prove $P(|X_1-X_2|>x)\sim a.P(|X|>x)$ as $x\to\infty$, where a is a constant and X_1, X_2 are dependent random variables.

Keywords : Weakly Negative Dependence (WND); Negative Quadratic Dependent (NQD); Heavy-Tailed; Long-Tailed; Asymptotic behavior.
2000 Mathematics Subject Classification : 0E05; 60F99 (2000 MSC)

1 Introduction

The asymptotic tail behavior of sums of heavy-tailed random variables has been studied by many authors, Chistyakov [2], Klüppelberg [8] and Embrechts et al. [4, 5] were presented a perfect discussion in this context when the random variables are independent. In recent years, Cai and Tang [1] and Wang [9] generalized these results to multivariate cases in the deferent classes of heavy-tailed distributions. Moreover, they have started a complete research, in this topic, when the random variables are dependent. On the other hand, heavy-tailed distributions have been the focus of study of many researchers in many sciences such as Insurance and Finance in recent years. Some classes of heavy-tailed distributions have been introduced in the literatures; here we deal with long-tailed distribution functions and study asymptotic tail behavior of sums of random variables when

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the distribution functions belong to this class. A distribution function F belongs to long-tailed distributions ($F \in L$) if $\lim \overline{F}(x-u)/\overline{F}(x) = 1$ as $x \to \infty$. In this paper, we extend these results to the case that the random variables satisfy in the new definition of dependency, which will be presented below. For this propose we consider the following condition is valid for the distribution functions, which can be assumed as another class of heavy-tailed distribution functions.

$$\int_0^\infty \overline{F}(u)du < \infty. \tag{1.1}$$

Throughout this paper all distribution functions defined on $[0, \infty)$ and $f(x) \sim g(x)$ means that $\lim f(x)/g(x) = 1$ as $x \to \infty$. We denoted the tail of distribution of F by $\overline{F}(x) = 1 - F(x)$, convolution of distributions F and G by F*G and $\overline{F * G} = 1 - F * G$, and denoted n^{th} convolution of F by $F^{(n)}$ and $\overline{F}^n(x) = 1 - F^{(n)}$. All limit conditions are for $x \to \infty$ unless stated otherwise.
In following we present a new definition of dependence which is assumed in this paper.

Definition 1.1. *The random variables X_1 and X_2 are said Weakly Negatively Dependent (WND) if there exist a $C > 1$ such that, $f(x_1, x_2) \leq C.f_1(x_1).f_2(x_2)$ where $f(x_1, x_2)$, $f_1(x_1)$ and $f_2(x_2)$ are joint density and marginal densities of X_1 and X_2, respectively.*

The class of WND random variables is well defined and a large class of these random variables can be found. Some examples of this class will present in following.

Definition 1.2. *The distribution functions F and G are said to be max-sum-equivalent, written $F \sim_M G$ if*

$$\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x) \quad as \quad x \to \infty. \tag{1.2}$$

2 Examples

The following examples are evidence of WND random variables:

i) Suppose that X_1 and X_2 have half-normal distribution, then

$$f_{X_1,X_2}(x_1, x_2) = \frac{2}{\pi\sqrt{1-\rho^2}} exp\Big[-\frac{1}{2(1-\rho^2)}\{x_1^2 + x_2^2 - 2\rho x_1 x_2\}\Big]; x_1, x_2 > 0,$$

$$f_{X_i}(x_i) = \sqrt{\frac{1}{\pi}} exp\{-\frac{1}{2}x_i^2\}; i = 1, 2.$$

If $-1 < \rho \leq 0$, then X_1 and X_2 are NQD r.v.'s (Ebrahimi and Ghosh [3]). Moreover,

$$\frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_1(x_1)} f_{X_2(x_2)}} = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-\rho^2}{2(1-\rho^2)}(x_1^2 + x_2^2) + \frac{\rho}{1-\rho^2} x_1 x_2\right] \leq \frac{1}{\sqrt{1-\rho^2}}.$$

Then $f(x_1, x_2) \leq C.f_1(x_1).f_2(x_2)$, where $C = 1/\sqrt{1-\rho^2} \geq 1$. So, X_1 and X_2 are WND.

ii) Let X and Y be two random variables with joint FGM (Farlie-Gumbel-Morgenstern) distribution, we have

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \left[1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))\right].$$

On the other hand, it's obvious that

$$|1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))| \leq 1 + |\alpha|,$$

and

$$f_{X,Y}(x,y) \leq (1 + |\alpha|) f_X(x) f_Y(y).$$

Therefore, the random variables X and Y are WND with $C = 1 + |\alpha| \geq 1$. Moreover, we know if $-1 < \alpha \leq 0$, then X and Y are NQD ([3]).
In particular if $F_i(x) = 1 - (\alpha/\alpha + x)^\beta, x > 0; i = 1, 2$ ($X \sim Pareto(\alpha, \beta)$ where $\alpha > 0$ and β is positive integer), then $F_i \in C \subset D \bigcap L$. For these examples it is easy to see that the condition (1.1) holds.

3 Main results

In this section, we obtain some convolution properties of WND random variables.

Lemma 3.1. *Let X_1 and X_2 be two WND random variables with distribution functions $F_i, i = 1, 2$, then*

i) For every $x_1, x_2 \in R$ we have,

$$F_{X_1,X_2}(x_1, x_2) \leq C.F_{X_1}(x_1) F_{X_2}(x_2).$$

ii) For all positive value of x,

$$P(X_1 + X_2 > x) \leq C. \int_0^\infty \overline{F}_1(x - u) dF_2(u).$$

iii) If $h_1(.)$ and $h_2(.)$ are monotone measurable functions then $h_1(X_1)$ and $h_2(X_2)$ are WND.

Lemma 3.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers, then*

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} bi} \leq max\left\{\frac{a_i}{b_i}; i=1,...,n\right\}. \tag{3.1}$$

Lemma 3.3. *Let X_1 and X_2 be two WND random variables with distribution functions F and G, respectively, then*

i) If $F \in D$ and $G \in D$, then $H = F * G \in D$.
ii) If $F \in L$ and $G \in L$, then $H = F * G \in L$.

Proof. i) For arbitrary random variables X and Y, see Cai and Tang [1].

ii) For any $x > 0$, we have

$$\begin{aligned}\overline{H}(x) &= P(X+Y>x) \\ &= P(X+Y>x; X>x) + P(X+Y>x; Y>x) \\ &\quad + P(X+Y>x; X<x; Y<x) - P(X>x; Y>x) \\ &= P(X>x) + P(Y>x) + P(X+Y>x; X<x; Y<x) \\ &\quad - P(X>x; Y>x).\end{aligned}$$

It follows that for every $u > 0$

$$\begin{aligned}1 &\leq \frac{\overline{H}(x-u)}{\overline{H}(x)} \\ &= \frac{\overline{H}(x-u)}{P(X>x) + P(Y>x) + P(X+Y>x; X<x; Y<x) - P(X>x; Y>x)}.\end{aligned}$$

Therefore, Lemma 3.2 implies that

$$1 \leq \frac{\overline{H}(x-u)}{\overline{H}(x)} \leq \max\{d_1, d_2\},$$

where,

$$d_1 = \frac{P(X>x-u) + P(X+Y>x-u; X<x-u; Y<x-u)}{P(X>x) + P(X+Y>u; X<x; Y<x)},$$

$$d_2 = \frac{P(Y>x-u) - P(X>x-u; Y>x-u)}{P(Y>x) - P(X>x; Y>x)}.$$

Moreover,

$$d_1 \leq \frac{P(X > x-u)}{P(X > x)} + \frac{P(X+Y > x-u; X < x-u; Y < x-u)}{P(X > x)} = \frac{\overline{F}(x-u)}{\overline{F}(x)} + I_1.$$

Now the property of WND yields

$$\begin{aligned}
I_1 &= \frac{P(X+Y > x-u; X < x-u; Y < x-u)}{P(X > x)} \\
&\leq \frac{C}{P(X > x)} \int\int I_{(x-u-Y<X<x-u)} I_{(Y<x-u)} f_1(x) f_2(x) dx dy \\
&= \frac{C}{P(X > x)} \int_{Y<x-u} P(x-u-t < X < x-u) dG(t) \\
&= C. \int_0^\infty \frac{I_{(Y<x-u)}[\overline{F}(x-u-t) - \overline{F}(x-u)}{\overline{F}(x)} dG(t),
\end{aligned}$$

therefore, by $F \in L$ and *Fatou's Lemma* we get,

$$\limsup_{x\to\infty} I_1 \leq \int_0^\infty \limsup_{x\to\infty} \frac{C.I_{(Y<x-u)[\bar{F}(x-u-t)-\bar{F}(x-u)}}{\bar{F}(x)} dG(t) = 0,$$

and so,

$$\limsup_{x\to\infty} d_1 \leq \limsup_{x\to\infty} \frac{\bar{F}(x-u)}{\bar{F}(x)} + I_1 \leq 1.$$

On the other hand,

$$\begin{aligned}
d_2 &= \frac{P(Y > x-u) - P(X > x-u; Y > x-u)}{P(Y > x) - P(X > x; Y > x)} \\
&\leq \frac{P(Y > x-u)}{P(Y > x) - P(X > x; Y > x)} \\
&= \left[\frac{P(Y > x) - P(X > x; Y > x)}{P(Y > x-u)}\right]^{-1} \\
&= \left[\frac{P(Y > x)}{P(Y > x-u)} - \frac{P(Y > x; Y > x)}{P(Y > x-u)}\right]^{-1} \\
&= [I_2 - I_3]^{-1}.
\end{aligned}$$

Where $I_2 = P(Y > x)/P(Y > x-u)$ and $I_3 = P(X > x; Y > x)/P(Y > x-u)$. Since $G \in L$, hence $I_2 \to 1$ and $I_3 \to 0$ as $x \to \infty$, so $\limsup d_2 \leq 1$ as $x \to \infty$, this completes the proof. □

The following Lemma was proved by Wang and Tang [9] for ND random variables. We proved it for WND random variables.

Lemma 3.4. *Let X and Y be two WND random variables with distribution functions F and G, respectively. Then,*

$$P(X+Y>x) \geq P(X>x)+P(Y>x) \qquad as \qquad x \to \infty.$$

Proof. For any positive x we have

$$\begin{aligned} P(X+Y>x) &= P(X>x)+P(Y>x) \\ &\quad + P(X+Y>x; X<x; Y<x) - P(X>x; Y>x) \\ &\geq P(X>x)+P(Y>x) - P(X>x; Y>x) \\ &\geq P(X>x)+P(Y>x) - C.P(X>x)P(Y>x). \end{aligned}$$

The second inequality is valid by Lemma 3.1(i), then

$$\frac{P(X+Y>x)}{P(X>x)+P(Y>x)} \geq 1+o(1).$$

This completes the proof. □

Theorem 3.5. *Let X_1 and X_2 be two WND random variables with common distribution function $F \in L$, if F satisfies in condition (1.1), then*

$$2 \leq \liminf_{x \longrightarrow \infty} \frac{\overline{F}^{(2)}(x)}{\overline{F}(x)} \leq \limsup_{x \longrightarrow \infty} \frac{\overline{F}^{(2)}(x)}{\overline{F}(x)} \leq 2C.$$

Proof. For all $x > 2v > 0$, we have

$$\begin{aligned} \frac{\overline{F}^{(2)}(x)}{\overline{F}(x)} &\leq \frac{C}{\overline{F}(x)}\left[2\int_0^{v} \overline{F}(x-u)dF(u) + \int_{v}^{x-v} \overline{F}(x-u)dF(u) + \overline{F}(x)\overline{F}(x-v)\right] \\ &= K_1+K_2+K_3. \end{aligned}$$

Now, it is easy to see that, if $F \in L$, then $\lim_{v\to\infty}\lim_{x\to\infty} K_1 = 2C$ and $\lim_{x\to\infty} K_3 = 0$. Moreover, we derive $\lim_{v\to\infty}\lim_{x\to\infty} K_1 = 0$, by the following argument. For any $x > 2v$, we have

$$\int_{v}^{x-v} \bar{F}(x-u)dF(u) = \int_{v}^{x-v} \bar{F}(u)d\bar{F}(x-u) \leq \int_{v}^{x+v} \bar{F}(u)d\bar{F}(x-u) = I_1.$$

Now let,

$$I = \int_0^{\infty} \bar{F}(u)du = \sum_{n=0}^{\infty} \int_{nh}^{(n+1)h} \overline{F}(u)du.$$

we have,

$$h\sum_{n=1}^{\infty} \bar{F}(nh) = h\sum_{n=0}^{\infty} \bar{F}(nh+h) \leq I \leq h\sum_{n=0}^{\infty} \bar{F}(nh) < \infty, \tag{3.2}$$

Since $\bar{F}(nh+h+\upsilon) \le \bar{F}(u) \le \bar{F}(nh+\upsilon)$ for all $u \in [nh+\upsilon, nh+h+\upsilon]$, so we can write

$$\begin{aligned}
&\sum_{n=0}^{N_0-1} \bar{F}\left[(n+1)h+\upsilon\right]\left\{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)\right\} \\
= &\sum_{n=0}^{N_0-1} \int_{nh+\upsilon}^{(n+1)h+\upsilon} \bar{F}\left[(n+1)h+\upsilon\right] d\bar{F}(x-u) \\
\le &\sum_{n=0}^{N_0-1} \int_{nh+\upsilon}^{(n+1)h+\upsilon} \bar{F}(u) d\bar{F}(x-u) = I_1 \\
\le &\sum_{n=0}^{N_0-1} \int_{nh+\upsilon}^{(n+1)h+\upsilon} \bar{F}\left[nh_n+\upsilon\right] d\bar{F}(x-u) \\
= &\sum_{n=0}^{N_0-1} \bar{F}\left[nh+\upsilon\right]\left\{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)\right\},
\end{aligned}$$

where, $N_0 = [x/h]$. Then we get

$$\begin{aligned}
\frac{I_1}{\bar{F}(x)} &\le \sum_{n=0}^{N_0-1} \bar{F}(nh+\upsilon)\left\{\frac{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)}{\bar{F}(x)}\right\} \\
&= \sum_{n=0}^{N_0-1} \frac{\bar{F}^2(nh+\upsilon)}{\bar{F}(x)}\left\{\frac{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)}{\bar{F}(x-nh-\upsilon)}\right\}.
\end{aligned}$$

When x tend to infinity, for all value of n and υ,

$$\left\{\frac{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)}{\bar{F}(x)}\right\}$$

tends to zero. Therefore for sufficient large x and for any $\varepsilon > 0$, we have,

$$K_2 \le \frac{I_1}{\bar{F}(x)} \le \varepsilon \sum_{n=0}^{N_0-1} \bar{F}(nh+\upsilon) \le \sum_{n=0}^{\infty} \bar{F}(nh) < \varepsilon.M.$$

The final inequality valids by (3.2). This completes the proof. □

Corollary 3.6. *Let F be a distribution function that belongs to L and satisfies in condition (1.1), then F belongs to class of Subexponential distribution functions and*

$$\lim_{x\longrightarrow\infty} \int_0^x \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) = 1.$$

Theorem 3.7. *Let X_1 and X_2 be two WND random variables with common distribution function $F \in L$ which satisfies in (1.1). If the condition $m \leq \overline{F}(x)/\overline{G}(x) \leq M$ holds for some $m, M \in (0, \infty)$, then*

$$2 \leq \liminf_{x\to\infty} \frac{\overline{G}^{(2)}(x)}{\overline{G}(x)} \leq \limsup_{x\to\infty} \frac{\overline{G}^{(2)}(x)}{\overline{G}(x)} \leq 2.C.$$

as x tends to ∞. Where $G^{(2)}$ is the convolution of two WND random variables with common distribution function G.

Proof. Let Y_1 and Y_2 be two WND random variables with common distribution function G, so there exist a $C \geq 1$ such that $g(y_1, y_2) \leq C.g(y_1)g(y_2)$. Like the analogue of Theorem 3.5, for $x > 2\upsilon$ we can write,

$$\begin{aligned}
\frac{\overline{G}^{(2)}(x)}{\overline{G}(x)} &= \frac{1}{\overline{G}(x)}[P(Y_1+Y_2 > x, Y_1 < x-\upsilon) + P(Y_1+Y_2 > x, Y_2 < x-\upsilon) \\
&\quad + P(Y_1+Y_2 > x, Y_1 > x-\upsilon, Y_2 > x-\upsilon)] \\
&\leq \frac{C}{\overline{G}(x)}\left\{\int_0^{\upsilon} \overline{G}(x-u)dG(u) + \int_{\upsilon}^{x-\upsilon} \overline{G}(x-u)dG(u) + \overline{G}(x-\upsilon).\overline{G}(x)\right\} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now, $G \in L$ implies that, $\lim_{\upsilon\to\infty}\lim_{x\to\infty} I_1 = C$ and $\lim_{x\to\infty} I_3 = 0$. Moreover,

$$I_2 = \int_0^{x-\upsilon} \frac{\overline{G}(x-u)}{\overline{G}(x)}dG(u) = I_1 + \int_{x-\upsilon}^{x} \frac{\overline{G}(x-u)}{\overline{G}(x)}dG(u). \qquad (3.3)$$

On the other hand,

$$\begin{aligned}
\int_{x-\upsilon}^{x} \frac{\overline{G}(x-u)}{\overline{G}(x)}dG(u) &\leq \frac{M}{m}\int_{x-\upsilon}^{x} \frac{\overline{F}(x-u)}{\overline{F}(x)}dG(u) = \frac{M}{m}\int_{x-\upsilon}^{x} \frac{1-F(x-u)}{\overline{F}(x)}dG(u) \\
&= \frac{M}{m\overline{F}(x)}\left[P(\upsilon < Y < x-\upsilon) - \int_{\upsilon}^{x-\upsilon} F(x-u)dG(u)\right].
\end{aligned}$$

By integrating by part, we have

$$\begin{aligned}
\int_{x-\upsilon}^{x} \frac{\overline{G}(x-u)}{\overline{G}(x)}dG(u) &\leq M.\int_{x-\upsilon}^{x} \frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) \\
&\quad + \frac{M}{m\overline{F}(x)}\left[\overline{G}(\upsilon)\overline{F}(x-\upsilon) - \overline{G}(x-\upsilon)\overline{F}(\upsilon)\right]. \qquad (3.4)
\end{aligned}$$

In Theorem 3.5, we have seen that the condition (1.1) implies that

$$\lim_{\upsilon\to\infty}\lim_{x\to\infty} \int_{x-\upsilon}^{x} \frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) = 0,$$

therefore, (3.4) implies that,

$$\lim_{\upsilon\to\infty}\lim_{x\to\infty}\int_{x-\upsilon}^{x}\frac{\overline{G}(x-u)}{\overline{G}(x)}dG(u)=0. \tag{3.5}$$

Substituting (3.5) in (3.3) we get, $\lim_{\upsilon\to\infty}\lim_{x\to\infty} I_1 = C$. So $\overline{G}^{(2)}(x)/\overline{G}(x)\le 2C$ and by Lemma 3.2, we have $\overline{G}^{(2)}(x)/\overline{G}(x)\ge 2C$. This completes the proof. □

Corollary 3.8. *The Theorem 3.7 is valid if the condition* $m\le \overline{F}(x)/\overline{G}(x)\le M$ *substituted by*

$$\lim\frac{\overline{F}(x)}{\overline{G}(x)}=a; a>0 \quad as \quad x\to\infty.$$

The following Theorem is an extension of proposition 1 of the Embrechts and Goldi [4] for WND random variables.

Theorem 3.9. *Let* X *and* Y *be two WND random variables with distribution functions* F *and* G*, respectively. If* $F,G\in L$, $\overline{F}(x)=o(\overline{G}(x)$ *and* G *satisfies in condition (1.1) then* $\overline{H}(x)\sim\overline{G}(x)$ *and*

$$2\le\liminf_{x\to\infty}\frac{\overline{H}^{(2)}(x)}{\overline{H}(x)}\le\limsup_{x\to\infty}\frac{\overline{H}^{(2)}(x)}{\overline{H}(x)}\le 2C.$$

Proof. For all $x>0$, we have

$$\begin{aligned}\frac{\overline{H}(x)}{\overline{G}(x)} &= \frac{1}{\overline{G}(x)}[P(X>x)+P(Y>x)+P(X+Y>x;X<x;Y<x)\\ &\quad -P(X>x;Y>x)]\\ &\le \frac{1}{\overline{G}(x)}\left[P(X>x)+P(Y>x)+P(X+Y>x;X<x;Y<x)\right].\end{aligned}$$

By assumption, $\lim_{x\to\infty} I_1=\lim_{x\to\infty}\overline{F}(x)/\overline{G}(x)=0$, and

$$\begin{aligned}I_2 &= \frac{P(X+Y>x;X<x;Y<x)}{\overline{G}(x)}\\ &\le C.\int_0^x\frac{\overline{F}(x-u)-\overline{F}(x)}{\overline{G}(x)}dG(u)\\ &= C.\int_0^x\frac{\overline{F}(x-u)}{\overline{G}(x)}dG(u)-C.\frac{\overline{F}(x)G(x)}{\overline{G}(x)}=I_3+I_4.\end{aligned}$$

It is obvious that, $\lim I_4=0$ as $x\to\infty$. For I_3 we know for given $\varepsilon>0$ there exist $\upsilon = \upsilon(\varepsilon)$ such that for all $t > \upsilon$, $\overline{F}(t)/\overline{G}(t) \le \varepsilon$, so we have

$$\begin{aligned} I_3 &= C.\int_0^x \frac{\overline{F}(x-u)}{\overline{G}(x)} dG(u) \\ &= C.\int_0^{x-\upsilon} \frac{\overline{F}(x-u)}{\overline{G}(x)} dG(u) + C.\int_{x-\upsilon}^x \frac{\overline{F}(x-u)}{\overline{G}(x)} dG(u) \\ &\le C.\varepsilon.\int_0^{x-\upsilon} \frac{\overline{G}(x-u)}{\overline{G}(x)} dG(u) + C.\frac{\overline{G}(x-\upsilon)-\overline{G}(x)}{\overline{G}(x)}. \end{aligned}$$

Since G belongs to L and satisfies in condition (1.1), Corollary 3.6 implies that $\lim I_3 = 0$ as $x \to \infty$, therefore $\overline{H}(x)/\overline{G}(x) \le 1$. On the other hand, it is obvious that $\overline{H}(x)/\overline{G}(x) \ge 1$ holds for any nonnegative random variable, hence $\overline{H}(x) \sim \overline{G}(x)$.

The second part of Theorem follows from Theorem 3.5. □

Theorem 3.10. *Let X and Y be two WND random variables with distribution function F and G, respectively, and suppose $F, G \in L$ and F satisfy in condition (1.1), then if* $\sup_x \overline{G}(x)/\overline{F}(x) = K < \infty$ *, we have* $F \sim_M G$ *and*

$$2 \le \liminf_{x\to\infty} \frac{\overline{H}^{(2)}(x)}{\overline{H}(x)} \le \limsup_{x\to\infty} \frac{\overline{H}^{(2)}(x)}{\overline{H}(x)} \le 2C.$$

Proof. By assumption of $\sup_x \overline{G}(x)/\overline{F}(x) = K < \infty$ and Lemma 3.1, for every $\upsilon > 0$, we get

$$\begin{aligned} \overline{H}(x) &= P(X+Y>x; X<\upsilon) + P(X+Y>x; \upsilon<X<x) + P(X>x) \\ &\le C.\int_0^{\upsilon} \overline{G}(x-u)dF(u) + C.\int_{\upsilon}^x \overline{G}(x-u)dF(u) + \overline{F}(x) \\ &\le C.\overline{G}(x-u) + \overline{F}(x) \times C.K.\int_{\upsilon}^x \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) + \overline{F}(x) \\ &\le [\overline{F}(x)+\overline{G}(x)] \left\{ \left[1 + C.K.\int_{\upsilon}^x \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u)\right] \vee \left[\frac{C\overline{G}(x-\upsilon)}{\overline{G}(x)}\right] \right\}, \end{aligned}$$

where, $x \vee y = \max\{x, y\}$. So

$$\lim_{\nu\to\infty} \limsup_{x\to\infty} \frac{\overline{H}(x)}{\overline{F}(x)+\overline{G}(x)} \le 1. \tag{3.6}$$

Since $G \in L$ and F satisfies in the condition (1.1), hence Theorem 3.5 implies (3.6). So, by Lemma 3.4 we obtain $F \sim_M G$. On the other hand, for all $0 < \upsilon < x$, we have

$$\begin{aligned} 1 &\leq \frac{\overline{H}(x)}{\overline{F}(x)} \leq C.\left[\int_0^{x-\upsilon} \frac{\overline{G}(x-u)}{\overline{F}(x)} dF(u) + \int_{x-\upsilon}^{x} \frac{\overline{G}(x-u)}{\overline{F}(x)} dF(u)\right] + C \\ &\leq K.C. \int_0^{x-\upsilon} \frac{\overline{G}(x-u)}{\overline{F}(x)} dF(u) + C. \int_{x-\upsilon}^{x} \frac{1}{\overline{F}(x)} dF(u) + C. \end{aligned} \tag{3.7}$$

So, $1 \leq \limsup \overline{H}(x)/\overline{F}(x) < \infty$ as $x \to \infty$.
Now $F, G \in L$ and Lemma 3.3 imply that $H = F * G \in L$. Moreover, (3.7) and condition (1.1) for F and Theorem 3.7 yield,

$$\limsup_{x \longrightarrow \infty} \frac{\overline{H}^{(2)}(x)}{\overline{H}(x)} \leq 2C.$$

This completes the proof. □

Let X_1 and X_2 be i.i.d. random variables with distribution function F. Feller [6] has proved some symmetrization inequalities; in particular, he showed that the tail of F and the distribution function F_s of the corresponding symmetrized random variable $X_1 - X_2$ are of comparable order by using symmetrization inequalities. Moreover, Geluk [7] has studied symmetrization inequality for i.i.d random variables with the long-tailed distribution functions and proved that,

$$P(|X_1 - X_2| > x) \sim a.P(|X| > x) \qquad as \qquad x \to \infty, \tag{3.8}$$

where, a is some constant. In the following, we prove that (3.8) holds for WND random variables when $F \in L$ and satisfies in condition (1.1).

Theorem 3.11. *Let X_1 and X_2 be two WND random variables with common distribution function $F \in L$ which satisfies in condition (1.1), then*

$$2 \leq \liminf_{x \to \infty} \frac{P(|X_1 - X_2| > x)}{P(|X| > x)} \leq \limsup_{x \to \infty} \frac{P(|X_1 - X_2| > x)}{P(|X| > x)} \leq 2C.$$

Proof. It is easy to see that the distribution function of $X_1 - X_2$ is symmetric around zero, so we may assume that the support of $F(x)$ is $[0, \infty)$. Then

$$\begin{aligned} P(X_1 - X_2 > x) &\leq \int_0^{\infty} \int_0^{x - x_2} C.f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2 \\ &= C. \int_0^{\infty} P(X_1 > x - u) dF(u) \\ &= C. \int_0^{\infty} \overline{F}(x-u) dF(u), \end{aligned}$$

and so,

$$\lim_{x \to \infty} \frac{P(X_1 - X_2 > x)}{P(X_1 > x)} \leq \lim_{x \to \infty} C. \int_0^{\infty} \frac{P(X > x - u)}{P(X > x)} dF(u) = C. \tag{3.9}$$

Let $Y = -X_2$, Lemma 3.1(iii) implies that Y and X_1 are WND, and we have

$$\begin{aligned} P(X_1 - X_2 > x) &= P(X_1 + Y > x) \geq P(X_1 > x) + P(Y > x) \\ &\quad - C.P(X_1 > x)P(Y > x) \\ &\geq P(X_1 > x) - C.P(X_1 > x)P(Y > x). \end{aligned}$$

Hence,

$$\frac{P(X_1 - X_2 > x)}{P(X > x)} \geq 1 + o(1) \qquad as \qquad x \to \infty. \tag{3.10}$$

By (3.9) and (3.10) we obtain

$$1 \leq \limsup_{x \longrightarrow \infty} \frac{P(X_1 - X_2 > x)}{P(X > x)} \leq C. \tag{3.11}$$

Moreover we have $P(|X_1 - X_2| > x) = 2P(X_1 - X_2 > x)$ and $P(|X| > x) \sim P(X > x)$ as $x \to \infty$. So, (3.10) and (3.11) complete the proof. □

Remark 3.12. *Under the assumptions of Theorem 3.11, it is easy to see that the relation (3.8) holds with $a \in [2, 2C]$, $C \geq 1$. Also, if $F(x_1, x_2)$ belongs to FGM distributions, then $a \in [2, 2 + 2|\alpha|]$, $-1 < \alpha < 1$. For example, if $F_i(x) = 1 - 1/x, x > 1, i = 1, 2$, then*

$$F(x, y) = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)\left(1 + \frac{\alpha}{xy}\right); x, y > 1.$$

So, using some calculations, we obtain easily $a = 2$.

Conclusion : We generalized Theorem 3 in Geluk [7] via Theorem 3.11 in this paper. We concentrate on WND random variables and achieved the class of distributions that in i.i.d. case implies the class of Geluk [7]. Also, we obtained closure and factorization properties for class of distribution that satisfy in condition (1.1). Our achievement in independent case implies some of Embrechts et al. [4, 5] results.

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